Problem 1

Show that the redshift of the light emitted by a distant object, given in terms of the parameter $z$

\[ z = \frac{\omega_{\text{emitted}}}{\omega_{\text{received}}} - 1, \] (1)

is equal to

\[ z = \frac{a_{\text{receiver}}}{a_{\text{emitter}}} - 1 \] (2)

where $a$ is the scale factor of the universe at the time of emission and reception of the signal respectively.

Problem 2

The luminosity distance of an object is

\[ d_L = \sqrt{\frac{L}{4\pi F}} = ar(1 + z) \] (3)

with $L$ the absolute luminosity, $F$ the flux received on earth, $a$ the present size of the universe, $r$ the coordinate distance to the object and $z$ the redshift of the object. Show that

\[ d_L = H_0^{-1}(z + \frac{1}{2}(1 - q_0)z^2 + O(z^3)) \] (4)

with $H_0, q_0$ the present Hubble constant and deceleration parameter.

Problem 3

Consider the metric with a cosmological constant $\Lambda > 0$ and $k = 0$,

\[ ds^2 = -d\tau^2 + e^{2H\tau}(dx^2 + dy^2 + dz^2), \] (5)

with $H$ the Hubble parameter. A light signal is emitted at $\tau = 0$ from $x = y = z = 0$. Show that at $\tau = \infty$, the light signal travels only a finite distance in the metric $ds^2 = dx^2 + dy^2 + dz^2$, and compute this distance. Thus, light cannot access all of the space-time. This phenomenon is called the “cosmological horizon.”

Problem 4

Take the Schwarzschild metric and replace $M$ with a function $M(r)$ to get

\[ ds^2 = -(1 - \frac{2GM(r)}{r})dt^2 + (1 - \frac{2GM(r)}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \] (6)

Suppose that this is a solution of the Einstein equation for some perfect-fluid energy momentum tensor in comoving coordinates. Compute $G_{00}$ from (6) and from this $\rho(r)$. 