General Relativity
Take Home Set 3
Hand in on November 29

Problem 1

Show that the redshift of the light emitted by a distant object, given in terms of the parameter z

$$z = \frac{\omega_{\text{emitted}}}{\omega_{\text{received}}} - 1, \tag{1}$$

is equal to

$$z = \frac{a_{\text{receiver}}}{a_{\text{emitter}}} - 1 \tag{2}$$

where a is the scale factor of the universe at the time of emission and reception of the signal respectively.

Problem 2

The luminosity distance of an object is

$$d_L = \sqrt{\frac{L}{4\pi F}} = ar(1+z) \tag{3}$$

with L the absolute luminosity, F the flux received on earth, a the present size of the universe, r the coordinate distance to the object and z the redshift of the object. Show that

$$d_L = H_0^{-1}(z + \frac{1}{2}(1-q_0)z^2 + \mathcal{O}(z^3)) \tag{4}$$

with H_0, q_0 the present Hubble constant and deceleration parameter.

Problem 3

Consider the metric with a cosmological constant $\Lambda > 0$ and $k = 0$,

$$ds^2 = -d\tau^2 + e^{2H\tau}(dx^2 + dy^2 + dz^2), \tag{5}$$

with H the Hubble parameter. A light signal is emitted at $\tau = 0$ from $x = y = z = 0$. Show that at $\tau = \infty$, the light signal travels only a finite distance in the metric $ds^2 = dx^2 + dy^2 + dz^2$, and compute this distance. Thus, light cannot access all of the space-time. This phenomenon is called the "cosmological horizon."

Problem 4

Take the Schwarzschild metric and replace M with a function $M(r)$ to get

$$ds^2 = -(1 - \frac{2GM(r)}{r})dt^2 + (1 - \frac{2GM(r)}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{6}$$

Suppose that this is a solution of the Einstein equation for some perfect-fluid energy momentum tensor in comoving coordinates. Compute G_{00} from (6) and from this $\rho(r)$.