General Relativity
Exam December 18, 2003
10.15 – 13.15 hours

Our convention for the Riemann tensor in this exam will be:

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}.$$

Problem 1

Suppose you stand on earth and have two clocks, one in each hand, that run perfectly synchronous. You keep one hand still, while you throw the other clock upwards with the other hand. A few seconds later you catch the other clock again. Now you compare the two clocks. Which one will give the earlier and which one will give the later time?

Problem 2

In this problem we consider a family of geodesics in some curved space. That means that we have a set of points in spacetime $x^\mu(s,\lambda)$, which depends smoothly on both λ and s. In addition, for any fixed s, $x^\mu(s,\lambda)$, viewed as a function of only λ is a geodesic with affine parameter λ. Thus, s is a label that distinguishes the various geodesics.

a) Write down the equation that expresses the fact that $x^\mu(s,\lambda)$ is a geodesic for fixed s with affine parameter λ.

b) We define

$$S^\mu = \frac{\partial x^\mu(s,\lambda)}{\partial s}.$$

We also define the relative velocity

$$V^\mu = \frac{DS^\mu}{d\lambda} \equiv \frac{\partial S^\mu}{\partial \lambda} + \frac{\partial x^\rho}{\partial \lambda}\Gamma^\mu_{\rho\sigma} S^\sigma$$

and the relative acceleration

$$A^\mu = \frac{DV^\mu}{d\lambda} \equiv \frac{\partial V^\mu}{\partial \lambda} + \frac{\partial x^\rho}{\partial \lambda}\Gamma^\mu_{\rho\sigma} V^\sigma.$$

These are indeed a measure of the relative velocity and acceleration of nearby geodesics given by two different but almost identical values of s. Show that V^μ and A^μ are tensors. What is A^μ in flat space?

We now want to show that

$$A^\mu = R^\mu{}_{\nu\rho\sigma}\frac{\partial x^\nu}{\partial \lambda}\frac{\partial x^\rho}{\partial \lambda}S^\sigma. \tag{1}$$

c) Consider a fixed s and λ, say $s = s_0$ and $\lambda = \lambda_0$. Can we always find a coordinate system such that

$$\Gamma^{\mu}_{\nu\rho}(x^{\sigma}(s_0, \lambda_0)) = 0?$$

d) Given such a coordinate system, show that (1) is indeed true at the point $x^{\sigma} = x^{\sigma}(s_0, \lambda_0)$.

e) Explain why this implies that (1) is valid everywhere and in every coordinate system.

Problem 3

We consider gravitational waves propagating in the x^3 direction. The metric is $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with $h_{\mu\nu}$ small. The solution of the linearized Einstein equations is that $h_{\mu\nu}$ has to be a linear combination of solutions of the form

$$h_{\mu\nu} = C_{\mu\nu}(k) e^{ik_{\mu}x^{\mu}} \tag{2}$$

with

$$C_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & C_{11} & C_{12} & 0 \\ 0 & C_{12} & -C_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3}$$

Since we are interested in waves propagating in the x^3 direction, we will in this problem only take momenta of the form $k_{\mu} = (-k, 0, 0, k)$.

a) What is the most general form of a family of geodesics $x^{\mu}(s, \lambda)$ as defined in the previous problem in flat space? In particular,

$$\hat{x}^{\mu}(s, \lambda) = (\lambda, r\cos s, r \sin s, 0) \tag{4}$$

is a family of geodesics.

The family (4) describes how an infinitely thin ring of particles (with radius r) lying in the x^1, x^2-plane behaves in flat space: the ring does not change shape or location. We now want to investigate what happens when a gravitational wave passes through the ring. Since we assume that the gravitational wave is very weak, we will assume that the family of geodesics describing the ring is perturbed by a small amount due to the gravitational wave,

$$x^{\mu}(s, \lambda) = \hat{x}^{\mu}(s, \lambda) + \delta x^{\mu}(s, \lambda). \tag{5}$$

b) Consider the *geodesic deviation* equation (1), take for the metric a gravitational wave $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, and take for the family of geodesics (5). Expand the geodesic deviation equation (1) to first order in $h_{\mu\nu}$ and $\delta x^\mu(s,\lambda)$ (so products of these two quantities with each other are also ignored). Give the form of the geodesic deviation equation to this order and show that at this order only $R^\mu{}_{00\sigma}$ appears.

c) Verify that

$$R^\mu{}_{00\sigma} = \frac{1}{2}\partial_0\partial_0 h^\mu{}_\sigma$$

to first order in $h_{\mu\sigma}$ for $h_{\mu\nu}$ of the form as given in (2) and (3).

d) Show that the geodesic deviation equation (1) in the form as derived under b) reduces to

$$\frac{\partial^3 \delta x^\mu}{\partial\lambda^2\partial s} = \frac{1}{2}\frac{\partial \hat{x}^\mu}{\partial s}\frac{\partial^2}{\partial\lambda^2}h^\mu{}_\nu. \tag{6}$$

e) Equation (6) is solved among other by

$$\delta x^\mu = \frac{1}{2}h^\mu{}_\nu \hat{x}^\nu. \tag{7}$$

Verify this. Of all possible solutions of (6), why is this the physically relevant solution?

f) By taking a linear combination of gravitational waves, we can make one with

$$h_{11} = -h_{22} = 2C_{11}\cos k\lambda, \qquad h_{12} = h_{21} = 0$$

and also one with

$$h_{12} = h_{21} = 2C_{12}\cos k\lambda, \qquad h_{11} = h_{22} = 0.$$

(Recall that $t = \lambda$ to this order or approximation). Use (7) to describe for each of these two gravitational waves what the shape of the ring will look like as a function of time if the corresponding gravitational wave passes through it.

Problem 4

Consider the metric

$$ds^2 = -dt^2 + t^{2\alpha}dx^2 + t^{2\beta}dy^2 + t^{2\gamma}dz^2. \tag{8}$$

a) Compute the Christoffel symbols $\Gamma^\mu_{\nu\rho}$ for this metric.

b) Compute the components of the Ricci tensor $R_{\mu\nu}$ for this metric.

c) Find the conditions on α, β and γ so that (8) is a solution of the vacuum Einstein equations.

d) Examine the conditions derived under c). Argue whether or not you think that among the various solutions to the Einstein equations there are any metrics that might be candidates for a realistic cosmology.

Points per question:

problem	points
1	10
$2a$	2
$2b$	8
$2c$	4
$2d$	10
$2e$	6
$3a$	3
$3b$	3
$3c$	7
$3d$	2
$3e$	5
$3f$	10
$4a$	10
$4b$	10
$4c$	5
$4d$	5
total	100

Final grade equals $1 + 9(\text{points})/100$.