General Relativity

Take Home Set 1
Hand in on October 3

Problem 1 (Carroll 2.1)

Just because a manifold is topologically nontrivial does not necessarily mean that it cannot be covered with a single chart. In contrast to the circle S^1, show that the infinite cylinder $\mathbb{R} \times S^1$ can be covered with just one chart, by explicitly constructing the map.

Problem 2 (Carroll 2.3)

The easiest way to define the two-torus T^2 is as a square with opposite sides identified. Show that T^2 is a manifold by constructing an appropriate atlas.

Problem 3 (based on Carroll 2.6)

Consider $\mathbb{R}^3$ with standard spherical polar coordinates (r, θ, ϕ). A particle moves along a parametrized curve given by

$$x(\lambda) = \cos\lambda, \quad y(\lambda) = \sin\lambda, \quad z(\lambda) = \lambda.$$

Express the path of the curve in the (r, θ, ϕ) system. Calculate the components of the tangent vector to the curve in both the Cartesian and spherical polar coordinate system. Verify explicitly that the the components of the curve indeed transform as a tensor.

Problem 4

Consider the two-dimensional space $-x^2 + y^2 + z^2 = 1$, viewed as a subspace of three-dimensional Minkowski space $ds^2 = -dx^2 + dy^2 + dz^2$. Parametrize this space using two coordinates, and find the metric on this space.

Problem 5

Define $g \equiv \det g \equiv \det g_{\mu\nu}$ to be the determinant of the metric $g_{\mu\nu}$ viewed as a square matrix. Show that

$$\sqrt{|g|}\epsilon_{\mu_1 \ldots \mu_n} \tag{1}$$

is a tensor. Show that if $f(x^\mu)$ is a scalar function on a manifold M, the integral

$$\int dx^0 \ldots dx^{n-1} \sqrt{|g|} f(x^\mu) \tag{2}$$

gives an answer which does not depend on the choice of coordinates x^μ. In other words, this is the right way to integrate a quantity on a manifold, an operation that we would obviously like to be independent of the choice of coordinates.