General Relativity
Take Home Set 2
Hand in on October 31

Problem 1

Consider the following metric on $\mathbb{R}^2$

$$ds^2 = e^{\phi(x,y)}(dx^2 + dy^2).$$

a) Compute the Christoffel symbols and Riemann curvature tensor for this metric.

b) Next we take the metric on the plane induced by the stereographic projection, i.e.

$$e^{\phi(x,y)} = \left(\frac{4}{4 + x^2 + y^2}\right)^2.$$

Compute $\int\int_{\mathbb{R}^2} \sqrt{g}\, dxdy$ and $\int\int_{\mathbb{R}^2} \sqrt{g}R\, dxdy$.

c) Suppose we add to ϕ a function of x, y that decays very rapidly at infinity. Would either of the two above integrals change? Explain your answer.

d) Consider geodesics in the above metric. Show that if a geodesic is a straight line, that straight line points everywhere in the direction of the gradient of $\phi(x, y)$.

Problem 2

a) In four dimensions ($n = 4$), the Einstein equations imply

$$R_{\mu\nu} = \kappa(T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu}). \tag{1}$$

Here, $T = g^{\mu\nu}T_{\mu\nu}$. What is $R_{\mu\nu}$ in terms of $T_{\mu\nu}$ if $n \neq 4$ and $n \neq 2$?.

b) Show that in two dimenstions the Einstein equations are equivalent to $T_{\mu\nu} = 0$. To do this, first write down the most general Riemann tensor in two dimensions that is compatible with the symmetries that the Riemann tensor should have. From this infer the most general form of the Ricci tensor.

Problem 3

a) Suppose we make a change of coordinates of the form

$$x^{\mu'} = x^\mu + \epsilon K^\mu(x^\nu) \tag{2}$$

where ϵ is a very small parameter. Express $g_{\mu'\nu'}(x^{\lambda'})$ in terms of K^μ and $g_{\mu\nu}(x^\lambda)$ to first order in ϵ.

b) Show that the requirement that $g_{\mu'\nu'}(x^{\lambda'}) = g_{\mu\nu}(x^{\lambda})$ to first order in ϵ is equivalent to the Killing vector condition

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0. \tag{3}$$

Problem 4 (Carroll 3.10)

Define the Weyl tensor $C_{\rho\sigma\mu\nu}$ in terms of the Riemann curvature tensor by

$$\begin{aligned} C_{\rho\sigma\mu\nu} &= R_{\rho\sigma\mu\nu} - \frac{2}{(n-2)}\left(g_{\rho[\mu}R_{\nu]\sigma} - g_{\sigma[\mu}R_{\nu]\rho}\right) \\ &\quad + \frac{2}{(n-1)(n-2)} g_{\rho[\mu}g_{\nu]\sigma}R. \end{aligned} \tag{4}$$

Recall that the notation $[\mu\nu]$ indicates antisymmetrization in the two indices, as e.g. in $F_{[\mu\nu]} \equiv \frac{1}{2}(F_{\mu\nu} - F_{\nu\mu})$. Show that in dimension $n \geq 4$,

$$\nabla_\rho C^\rho{}_{\sigma\mu\nu} = 2\frac{(n-3)}{(n-2)}\left(\nabla_{[\mu}R_{\nu]\sigma} + \frac{1}{2(n-1)} g_{\sigma[\mu}\nabla_{\nu]}R\right). \tag{5}$$

(OPTIONAL) Problem 5 (Carroll 3.9)

Consider an arbitrary metric tensor $g^{(1)}_{\mu\nu}$ and another metric $g^{(2)}_{\mu\nu}$ that is related to $g^{(1)}$ by an overall rescaling, $g^{(2)}_{\mu\nu} = \omega^2(x) g^{(1)}_{\mu\nu}$. Notice that $\omega^2(x)$ does not carry any indices. In this case we say that the two metrics $g^{(1)}$ and $g^{(2)}$ are conformally equivalent. Show that the Weyl tensor for the metric $g^{(1)}$ is identical to that for $g^{(2)}$.