general relativity – october 5, 2006

material discussed in class

Roughly 3.1-3.3 in the book.

exercises

- Show that the requirement that $\nabla_\mu A^\nu$ is a tensor implies the transformation rule

$$\Gamma^{\nu'}_{\mu'\lambda'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\lambda}{\partial x^{\lambda'}}\frac{\partial x^{\nu'}}{\partial x^\nu}\Gamma^\nu_{\mu\lambda} - \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\lambda}{\partial x^{\lambda'}}\frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\lambda}. \tag{1}$$

- Compute the eight Christoffel symbols $\Gamma^\lambda_{\mu\nu}$ for the two-sphere of unit radius.

- Consider the unit two-sphere, and consider the vector A_0^μ which is the unit vector in the θ-direction, at the point p_0 which is $\theta = \pi/2$, $\phi = 0$ in polar coordinates. What happens to the vector if we parallel transport it once around the equator, i.e. along the path $(\theta(\lambda), \phi(\lambda)) = (\pi/2, \lambda)$ with $0 \leq \lambda \leq 2\pi$? Next consider a curve which consists of four separate pieces,

$$\begin{aligned}
\gamma_1(\lambda) &= (\pi/2, \lambda) \quad \text{for} \quad 0 \leq \lambda \leq \lambda_1 \\
\gamma_2(\lambda) &= (\pi/2 - \lambda, \lambda_1) \quad \text{for} \quad 0 \leq \lambda \leq \lambda_2 \\
\gamma_3(\lambda) &= (\pi/2 - \lambda_2, \lambda_1 - \lambda) \quad \text{for} \quad 0 \leq \lambda \leq \lambda_1 \\
\gamma_4(\lambda) &= (\pi/2 - \lambda_2 + \lambda, 0) \quad \text{for} \quad 0 \leq \lambda \leq \lambda_2
\end{aligned}$$

  with $0 \leq \lambda_1 \leq 2\pi$ and $0 \leq \lambda_2 \leq \pi/2$. Find out what happens to the vector A_0^μ once we parallel transport it around this closed path. What is the area of the two-sphere enclosed by this closed curve? Is this area somehow related to the result of the parallel transport?

- Consider the length of a path which is everywhere timelike,

$$\Delta\tau = \int_{\lambda_0}^{\lambda_1} d\lambda \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}.$$

  Work out the variation of this expression, keeping in mind that now $g_{\mu\nu}$ is also a function of x. After computing the variation, change the parameter λ to proper time τ, so that

$$d\tau = \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}d\lambda.$$

  Show that demanding that the variation vanishes leads to the geodesic equation with the parameter λ replaced by τ.