General Relativity
Take Home Set 3
Hand in on December 7

Problem 1

The Friedmann equations with a cosmological constant read

$$\begin{aligned}\frac{\ddot{a}}{a} &= \frac{\Lambda}{3} - \frac{4\pi G}{3}(\rho + 3p) \\ \left(\frac{\dot{a}}{a}\right)^2 &= \frac{\Lambda}{3} + \frac{8\pi G}{3}\rho - \frac{k}{a^2}.\end{aligned} \tag{1}$$

Consider (1) and assume that ρ and p describe conventional matter (i.e. both are nonnegative). When are there static solutions to these equations? If a static solution has $p = 0$, what is then the relation between a and ρ? Are these static solutions stable against small perturbations?

Problem 2

Consider the metric in n dimensions

$$ds^2 = -dt^2 + \sum_{i=1}^{n-1} t^{2\alpha_i} dx_i^2. \tag{2}$$

Find the conditions of the $n-1$ constants α_i so that this metric is a solution of the vacuum Einstein equations. Are these reasonable candidates for a realistic cosmology?

Problem 3

Consider the metric that solves the Friedmann equations with a cosmological constant $\Lambda > 0$ and $k = 0$,

$$ds^2 = -d\tau^2 + e^{2H\tau}(dx^2 + dy^2 + dz^2), \tag{3}$$

with H the Hubble parameter. A light signal is emitted at $\tau = 0$ from $x = y = z = 0$. Show that at $\tau = \infty$, the light signal travels only a finite distance in the metric $ds^2 = dx^2 + dy^2 + dz^2$, and compute this distance. Thus, light cannot access all of the space-time. This phenomenon is called the "cosmological horizon."

Problem 4

Carroll exercise 8.5. (page 375)

Problem 5

Take the Schwarzschild metric and replace M with a function $M(r)$ to get

$$ds^2 = -(1 - \frac{2GM(r)}{r})dt^2 + (1 - \frac{2GM(r)}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{4}$$

Suppose that this is a solution of the Einstein equation for some perfect-fluid energy momentum tensor in comoving coordinates. Compute G_{00} from (4) and from this $\rho(r)$.