general relativity – September 24, 2009

material discussed in class

Roughly 3.1-3.3 in the book.

exercises

• Show that the requirement that $\nabla_{\mu} A^\nu$ is a tensor implies the transformation rule

$$\Gamma_{\mu'\lambda'}^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \Gamma_{\mu\lambda}^{\nu} - \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial^2 x^{\nu'}}{\partial x^{\nu} \partial x^{\lambda'}}. \quad (1)$$

• Compute the eight Christoffel symbols $\Gamma_{\mu\nu}^{\lambda}$ for the two-sphere of unit radius.

• Consider the unit two-sphere, and consider the vector $A_0^\mu$ which is the unit vector in the $\theta$-direction, at the point $p_0$ which is $\theta = \pi/2$, $\phi = 0$ in polar coordinates. What happens to the vector if we parallel transport it once around the equator, i.e. along the path $(\theta(\lambda), \phi(\lambda)) = (\pi/2, \lambda)$ with $0 \leq \lambda \leq 2\pi$? Next consider a curve which consists of four separate pieces,

$$\begin{align*}
\gamma_1(\lambda) &= (\pi/2, \lambda) \quad \text{for } 0 \leq \lambda \leq \lambda_1 \\
\gamma_2(\lambda) &= (\pi/2 - \lambda, \lambda_1) \quad \text{for } 0 \leq \lambda \leq \lambda_2 \\
\gamma_3(\lambda) &= (\pi/2 - \lambda_2, \lambda_1 - \lambda) \quad \text{for } 0 \leq \lambda \leq \lambda_1 \\
\gamma_4(\lambda) &= (\pi/2 - \lambda_2 + \lambda, 0) \quad \text{for } 0 \leq \lambda \leq \lambda_2
\end{align*}$$

with $0 \leq \lambda_1 \leq 2\pi$ and $0 \leq \lambda_2 \leq \pi/2$. Find out what happens to the vector $A_0^\mu$ once we parallel transport it around this closed path. What is the area of the two-sphere enclosed by this closed curve? Is this area somehow related to the result of the parallel transport?

• Consider the length of a path which is everywhere timelike,

$$\Delta \tau = \int_{\lambda_0}^{\lambda_1} d\lambda \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}.$$ 

Work out the variation of this expression, keeping in mind that now $g_{\mu\nu}$ is also a function of $x$. After computing the variation, change the parameter $\lambda$ to proper time $\tau$, so that

$$d\tau = \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda.$$ 

Show that demanding that the variation vanishes leads to the geodesic equation with the parameter $\lambda$ replaced by $\tau$. 

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