General Relativity
Take Home Set 3
Hand in on December 7, 2009, before 13.00 hours

Problem 1

In this problem we consider a family of geodesics in some curved space. That means that we have a set of points in spacetime $x^\mu(s,\lambda)$, which depends smoothly on both λ and s. In addition, for any fixed s, $x^\mu(s,\lambda)$, viewed as a function of only λ is a geodesic with affine parameter λ. Thus, s is a label that distinguishes the various geodesics.

a) Write down the equation that expresses the fact that $x^\mu(s,\lambda)$ is a geodesic for fixed s with affine parameter λ.

b) We define

$$S^\mu = \frac{\partial x^\mu(s,\lambda)}{\partial s}.$$

We also define the relative velocity

$$V^\mu = \frac{DS^\mu}{d\lambda} \equiv \frac{\partial S^\mu}{\partial \lambda} + \frac{\partial x^\rho}{\partial \lambda}\Gamma^\mu_{\rho\sigma} S^\sigma$$

and the relative acceleration

$$A^\mu = \frac{DV^\mu}{d\lambda} \equiv \frac{\partial V^\mu}{\partial \lambda} + \frac{\partial x^\rho}{\partial \lambda}\Gamma^\mu_{\rho\sigma} V^\sigma.$$

These are indeed a measure of the relative velocity and acceleration of nearby geodesics given by two different but almost identical values of s. Show that V^μ and A^μ are tensors. What is A^μ in flat space?

We now want to show that

$$A^\mu = R^\mu{}_{\nu\rho\sigma}\frac{\partial x^\nu}{\partial \lambda}\frac{\partial x^\rho}{\partial \lambda} S^\sigma. \tag{1}$$

c) Consider a fixed s and λ, say $s = s_0$ and $\lambda = \lambda_0$. Can we always find a coordinate system such that

$$\Gamma^\mu_{\nu\rho}(x^\sigma(s_0,\lambda_0)) = 0?$$

d) Given such a coordinate system, show that (1) is indeed true at the point $x^\sigma = x^\sigma(s_0,\lambda_0)$.

e) Explain why this implies that (1) is valid everywhere and in every coordinate system.

Problem 2

We consider gravitational waves propagating in the x^3 direction. The metric is $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with $h_{\mu\nu}$ small. The solution of the linearized Einstein equations is that $h_{\mu\nu}$ has to be a linear combination of solutions of the form

$$h_{\mu\nu} = C_{\mu\nu}(k) e^{ik_\mu x^\mu} \tag{2}$$

with

$$C_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & C_{11} & C_{12} & 0 \\ 0 & C_{12} & -C_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3}$$

Since we are interested in waves propagating in the x^3 direction, we will in this problem only take momenta of the form $k_\mu = (-k, 0, 0, k)$.

a) What is the most general form of a family of geodesics $x^\mu(s, \lambda)$ as defined in the previous problem in flat space? In particular,

$$\hat{x}^\mu(s, \lambda) = (\lambda, r\cos s, r\sin s, 0) \tag{4}$$

is a family of geodesics.

The family (4) describes how an infinitely thin ring of particles (with radius r) lying in the x^1, x^2-plane behaves in flat space: the ring does not change shape or location. We now want to investigate what happens when a gravitational wave passes through the ring. Since we assume that the gravitational wave is very weak, we will assume that the family of geodesics describing the ring is perturbed by a small amount due to the gravitational wave,

$$x^\mu(s, \lambda) = \hat{x}^\mu(s, \lambda) + \delta x^\mu(s, \lambda). \tag{5}$$

b) Consider the *geodesic deviation* equation (1), take for the metric a gravitational wave $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, and take for the family of geodesics (5). Expand the geodesic deviation equation (1) to first order in $h_{\mu\nu}$ and $\delta x^\mu(s, \lambda)$ (so products of these two quantities with each other are also ignored). Give the form of the geodesic deviation equation to this order and show that at this order only $R^\mu{}_{00\sigma}$ appears.

c) Verify that

$$R^\mu{}_{00\sigma} = \frac{1}{2}\partial_0\partial_0 h^\mu{}_\sigma$$

to first order in $h_{\mu\sigma}$ for $h_{\mu\nu}$ of the form as given in (2) and (3).

d) Show that the geodesic deviation equation (1) in the form as derived under b) reduces to

$$\frac{\partial^3 \delta x^\mu}{\partial\lambda^2 \partial s} = \frac{1}{2}\frac{\partial \hat{x}^\mu}{\partial s}\frac{\partial^2}{\partial\lambda^2} h^\mu{}_\nu. \tag{6}$$

e) Equation (6) is solved among other by

$$\delta x^\mu = \frac{1}{2} h^\mu{}_\nu \hat{x}^\nu. \tag{7}$$

Verify this. Of all possible solutions of (6), why is this the physically relevant solution?

f) By taking a linear combination of gravitational waves, we can make one with

$$h_{11} = -h_{22} = 2C_{11}\cos k\lambda, \qquad h_{12} = h_{21} = 0$$

and also one with

$$h_{12} = h_{21} = 2C_{12}\cos k\lambda, \qquad h_{11} = h_{22} = 0.$$

(Recall that $t = \lambda$ to this order or approximation). Use (7) to describe for each of these two gravitational waves what the shape of the ring will look like as a function of time if the corresponding gravitational wave passes through it.

Problem 3

Consider again the Schwarzschild solution. Someone is at $t = 0$ at $r = 2MG$. Show that no matter what this person does, he or she will always hit the singularity in a proper time $\Delta\tau \leq \pi MG$.

Problem 4

Take the Schwarzschild metric and replace M with a function $M(r)$ to get

$$ds^2 = -(1 - \frac{2GM(r)}{r})dt^2 + (1 - \frac{2GM(r)}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \qquad (8)$$

Suppose that this is a solution of the Einstein equation for some perfect-fluid energy momentum tensor in comoving coordinates. Compute G_{00} from (8) and from this $\rho(r)$.