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# D-Branes in Curved Space

Jan de Boer[1]

*Institute for Theoretical Physics*
*University of Amsterdam*
*Valckenierstraat 65*
*1018 XE Amsterdam, The Netherlands*

**Abstract**

We review recent work on the structure of the action for $N$ D0-branes in a curved background, with emphasis on the meaning of space-time diffeomorphism invariance. Talk given at the Francqui Colloquium, 2001, "Strings and Gravity: Tying the Forces Together."

[1]E-mail: jdeboer@science.uva.nl

Since their introduction in 1995 [1], D-branes have played a tremendously important role in string theory. It therefore does not come as a surprise that the low-energy effective action for D-branes, the Dirac-Born-Infeld action, has been studied in great detail. The effective action for a single Dp-brane in a curved (closed string) background was found in [2],

$$S \sim \int d^{p+1}\sigma e^{-\phi}\sqrt{-\det(G_{ab}+G_{i(a}\partial_{b)}X^i+G_{ij}\partial_a X^i\partial_b X^j+\alpha' F_{ab})} \tag{1}$$

where

$$G_{\mu\nu}=G_{\mu\nu}(\sigma^a,x^i(\sigma^a)) \tag{2}$$

is the induced metric on the brane. It is an exact result to all orders in $\alpha'$ for slowly varying field strengths of the world-volume gauge field. Once we start to consider multiple D-branes, the situation becomes much more complicated. Due to the Chan-Paton factors, the world-volume gauge field $A$ as well as the scalar field $X^i$ that parametrize the transverse fluctuations of the D-brane become matrix-valued. The effective action for these non-abelian degrees of freedom is given by a dimensional reduction of ten-dimensional super Yang-Mills theory [3],

$$S \sim \int d^{p+1}\sigma e^{-\phi}\mathrm{Tr}\left[\frac{1}{2}F_{ab}^2+D_aX^iD_aX^i+\frac{1}{2}[X^i,X^j]^2\right]+\ldots, \tag{3}$$

at least for a flat brane embedded in flat space, and to lowest order in $\alpha'$.

Interestingly, the generalization of (3) to curved space is unknown, even to lowest order in $\alpha'$. The problem is that the formal generalization of (2),

$$G_{\mu\nu}=G_{\mu\nu}(\sigma^a,X^i(\sigma^a)) \tag{4}$$

is ill-defined due to the fact that the right hand side of this equation has ordering problems. In other words, we need to find a metric on the space of matrix-valued fields, given the space-time metric, and there is no canonical way to do this.

There are several reasons why it would be interesting to understand the coupling of multiple D-branes to closed string background fields in more detail.

- It provides the effective action in brane world scenario's where cosmological solutions are obtained by embedding curved branes in curved backgrounds.

- New geometrical structures can appear. For example, in the presence of a closed string NS-NS B-field, the action for D-branes remains similar except that the ordinary product of fields is replaced by a non-commutative $*$-product [4, 5]. The fact that commuting coordinates are replaced by non-commuting matrices when we go from one to multiple D-branes suggests that non-commutative geometry may play a crucial role in the formulation of the coupling of other closed string background fields like the metric as well.

- In string theory there is no invariant notion of space-time geometry, rather it depends in a non-trivial way on the object that is being used to probe the geometry. In particular, closed strings see a different geometry from the one seen by open strings or the one seen by D-branes. As explained in [6], the metric seen by a single D0-brane is a well-defined quantity and is referred to by the term 'D-geometry'. The geometry that is seen by $N$ coincident D-branes provides an interesting generalization of D-geometry, and at the same time contains information about the scattering of D-branes off each other.

- By studying probe branes in the background of other source branes one can obtain low energy effective actions in gauge theory.

- There are many interesting physical phenomena for D-branes in non-trivial backgrounds, such as the enhancon [7], giant gravitons [8] and the Myers effect [9]. We would like to know whether there exist purely gravitational versions of these phenomena, for example in black hole backgrounds, which could perhaps give a natural explanation of the non-commutativity of the black hole horizon encountered in [10].

Still, one may wonder whether a local well-defined action for the transversal matrix-valued fields actually exists, given that the only quantity unambiguously available from string theory is the S-matrix. If we believe that there is a local formulation of string field theory, the action for $N$ D-branes can be computed by integrating out massive string degrees of freedom. One expects a local action as long as the masses of open strings stretched between the branes is much smaller than the masses of all other massive open string degrees of freedom. This implies that the expectation value of the scalar fields (except for the $U(1)$ part) has to be much smaller than $l_s$. In addition, the momenta of these fields (along the brane) have to be much smaller than $1/l_s$ so that we can neglect higher derivative terms, and string loop effects are suppressed as long as the string coupling $g_s \ll 1$. Our results will be applicable in this regime only. Indeed, provided the graviton momenta are chosen to scale in an appropriate way, it appears possible to find a consistent $\alpha' \to 0$ limit for multiple graviton scattering, just as one is able to do for a single graviton or other closed string modes [11, 12, 13, 14, 15].

There are several ordering problems in the theory of multiple branes. For example, the terms higher order in $\alpha'$ in (1) also suffer from an ordering problem (see e.g. [16]). We will restrict attention to the ordering problem associated with the space-time metric, and for simplicity focus on the case of $N$ D0-branes moving on a space of the form $\mathbf{R} \times \mathcal{M}_9$, where $\mathbf{R}$ represents the time direction. For such a space we can choose a static gauge where the world-volume time coincides with the time coordinate in space-time. If we restrict attention to the bosonic fields, and choose the gauge $A^0 = 0$, the action schematically (up to ordering problems) looks like

$$S \sim \int d\tau \mathrm{Tr}(G_{ij}(X)\dot{X}^i\dot{X}^j + \frac{1}{2}G_{ij}(X)G_{kl}(X)[X^i, X^k][X^j, X^l]) \tag{5}$$

where now $i, j = 1 \ldots 9$.

In the literature, there are several approaches to the ordering problems of the action (5).

- In [6, 17] axioms for "D-geometry" were formulated, and implemented on the action (5). In terms of the action (5), these axioms read (1) the action should be single trace, (2) it should be invariant under global $U(N)$ transformations, (3) the classical moduli space obtained from (5) should be $(\mathcal{M}_9)^N/S_N$, the space of $N$ unordered points on $\mathcal{M}_9$, (4) the action should have the right $U(1)$ limit, and (5) if we expand the action around diagonal matrices,

$$X^i = \begin{pmatrix} x^i_{(1)} & & 0 \\ & \ddots & \\ 0 & & x^i_{(N)} \end{pmatrix} + \begin{pmatrix} 0 & & y_{\alpha\beta} \\ & \ddots & \\ y^*_{\alpha\beta} & & 0 \end{pmatrix} \tag{6}$$

and expand the action to second order in $y_{\alpha\beta}$, the masses of these fluctuations should be equal to the geodesic distance $d(x_{(\alpha)}, x_{(\beta)})$ between $x_{(\alpha)}$ and $x_{(\beta)}$. The latter requirement follows from the observation that $y_{\alpha\beta}$ corresponds to the creation operator for an open string that starts at $x_{(\alpha)}$ and ends at $x_{(\beta)}$. The mass of this open string is proportional to the geodesic distance between the two points. In [6, 17] it was found that the above axioms can be imposed in the action but do not fix it. It was also found that once we impose supersymmetry, the axioms can only be imposed if the background is Ricci flat (for a six-dimensional transverse space). We will not run into this latter situation since we will restrict our attention to the bosonic sector only and not impose any supersymmetry.

- In [18, 19] the linear coupling to the background metric was derived using the matrix theory interpretation [20] of the action of $N$ D0-branes. The result of of this analysis is that the coupling is given by a completely symmetrized trace; this means that we expand the metric as

$$G_{ij}(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^{k_1} \dots X^{k_n} \partial_{k_1} \dots \partial_{k_n} G_{ij}(0), \tag{7}$$

replace the coordinate $x^k$ by the hermitian matrix $X^k$, substitute this into (5), completely symmetrize the resulting expression, and than take the trace. In the symmetrization, commutators are viewed as a single entity. The resulting symmetrized coupling can also be written in the following suggestive form. If we Fourier decompose the metric

$$G_{ij}(x) = \int d^9 k G_{ij}(k) e^{ikx} \tag{8}$$

the action can be rewritten as

$$S = \int d\tau d^9 k \int_0^1 d\lambda G_{ij}(k) \mathrm{Tr} \left[ e^{i\lambda k X} \dot{X}^i e^{i(1-\lambda)kX} \dot{X}^j + e^{i\lambda k X} [X^i, X^l] e^{i(1-\lambda)kX} [X^j, X^l] \right]. \tag{9}$$

This looks remarkably similar to the observables in non-commutative gauge theory introduced in [21], that consist of gauge theory operators smeared along open Wilson lines. We will make some more comments about the relation with non-commutative gauge theory later. Notice that (9) only contains the linearized coupling of the graviton; from a world-sheet point of view, this action is obtained by summing disc diagrams with a single insertion of a graviton vertex operator.

- The action (1) was derived by computing the beta-functions for open strings in a curved background. In [22] an attempt was made to generalize this to the non-abelian case. A priori this is difficult because turning on arbitrary non-diagonal vevs for the scalars $X^i$ does not have a classical geometrical interpretation (though it could have a geometrical interpretation in a suitable non-commutative geometry). In addition, turning on vevs for non-diagonal $X^i$ involves turning on massive string degrees of freedom and it is very difficult to set up a beta-function calculation that includes such massive modes (see e.g. [23]). In [22] vev's for $X^i$ were included in the world-sheet path integral via the inclusion of a term

$$P \exp \oint_{\partial\Sigma} X_\mu(y) \partial_n y^\mu ds. \tag{10}$$

Due to the structure of the divergences in this theory, one can only obtain results order by order in the graviton; at linearized order the kinetic term in (9) was reproduced.

- The linear coupling to the background metric can also be derived directly using world-sheet methods [12, 13] by computing disc diagrams with a graviton vertex operator in the interior of the disc and zero momentum scalar field vertex operators on the boundary of the disc. For the superstring, the symmetrized trace is reproduced but for the bosonic string a different answer was found [12]. As we will see later, this is consistent with the notion of diffeomorphism invariance that we will introduce.

- An alternative approach is to use the fact that non-commutative gauge theories can be obtained from commutative lower dimensional ones in the $N \to \infty$ limit, by expanding the matrix-valued fields as $X = \hat{x} + A(\hat{x})$ where the $\hat{x}$ are operators (see e.g. [24] and references therein). In this way, one finds a relation between the coupling of closed strings to commutative non-abelian theories and non-commutative abelian theories. To first order in the graviton, one again recovers the fully symmetrized answer [14]. In [25] open Wilson lines were used to find find the linearized coupling to all massive string modes in non-commutative gauge theories, but so far none of these approaches has been generalized to the non-linear level.

- One can try to use the connection with matrix theory and the known coupling of the supermembrane to background fields to determine the precise form of the coupling [26].

- One could try to use $\kappa$-symmetry as e.g. in [27], or one can consider specific curved backgrounds as in [28] in order to learn something about the coupling in the general case.

In the remainder, we will focus on a simpler question, namely what is the meaning of space-time diffeomorphism invariance for matrix-valued fields? Somehow, we expect that ordinary diffeomorphisms have to be replaced by matrix-valued diffeomorphisms, but what is the right symmetry principle?

For a single D0-brane, diffeomorphism invariance uniquely determines the kinetic term to be of the form

$$S[g,x] = \int d\tau g_{ij}(x)\dot{x}^i \dot{x}^j \tag{11}$$

and for multiple D0-branes this becomes an action of the form

$$S[g,X] = \int d\tau G_{IJ}(X)\dot{X}^I \dot{X}^J \equiv \int d\tau G_{i\alpha\beta,j\gamma\delta}(X)\dot{X}^{i\alpha\beta}\dot{X}^{j\gamma\delta}. \tag{12}$$

To distinguish the various metrics, we have called the closed string metric $g_{ij}$, and the metric appearing in the D0-brane action $G_{IJ}$. The capital indices $I$ are multi-indices $I = i\alpha\beta$, where $i = 1\ldots 9$, and $\alpha, \beta = 1\ldots N$ are the matrix indices. Thus, the multiple D0-brane action looks like the action for a single D0-brane, but with $9N^2$ instead of 9 transverse coordinates. For flat space,

$$G_{i\alpha\beta,j\gamma\delta} = \delta_{ij}\delta_{\alpha\delta}\delta_{\beta\gamma} \tag{13}$$

and the action reduces to $\int d\tau \mathrm{Tr}(\dot{X}^i \dot{X}^j)$.

Clearly, $G_{IJ}$ has to be a functional of $g_{ij}$. In other words, given a metric in 9 dimensions we have to construct a metric in $9N^2$ dimensions. What about the meaning of diffeomorphism invariance? Since physics should not depend on the choice of coordinates, given two metrics $g$ and $g'$ that are related by a diffeomorphism, we would expect that there exists a field redefinition $X \to X'$ such that

$$S[g, X] = S[g', X']. \tag{14}$$

Actually, it is rather problematic to impose (14). If we denote by DIFF the group of matrix valued diffeomorphisms $X \to X'(X)$ and by diff the group of ordinary diffeomorphisms, then there is an obvious map DIFF$\to$diff that simply forgets the ordering, but one can show that there is no group homomorphism diff$\to$DIFF [29]. Furthermore, starting from the linearized coupling of the graviton found in [18, 19] one can attempt to find higher order terms in the action using the Noether procedure. This procedure fails at second order in the graviton. In both cases, we assume that matrix diffeomorphisms are of the form $X'(X)$. However, in order to have equivalent physics in two different coordinate systems it is in principle sufficient that there exists a field redefinition $X'(X, g)$ that can also involve the metric. Thus, it is sufficient to require

$$S[g, X] = S[g'(g), X'(g, X)]. \tag{15}$$

Though it may seem rather peculiar that we need explicit metric dependence in the matrix version of diffeomorphisms, it does not contradict anything, as long as ordinary diffeomorphisms are recovered in the $U(1)$ limit. The fact that diffeomorphisms depend explicitly on the background metric is reminiscent of the Seiberg-Witten map in non-commutative gauge theory [30] where the gauge parameter of the commutative theory depends non-trivially on the non-commutative gauge field. Here the matrix-valued gauge transformation depends non-trivially on the gauge field for space-time gauge transformations, which is the space-time metric.

There is a large amount of redundancy in (15), and in order to remove some of that we will use Riemann normal coordinates and the corresponding covariant background field expansion [31, 32]. Recall that Riemann normal coordinates are obtained by choosing a basepoint $\bar{x}$, and by using the coordinates on the tangent space at $\bar{x}$ to parametrize a neighborhood of $\bar{x}$. For a tangent vector $v$ at $\bar{x}$, a point in the manifold is obtained by following a geodesic $x(t)$ with $x(0) = \bar{x}$ and $\dot{x}(0) = v$ for unit time. In Riemann normal coordinates and in the covariant background field expansion, only space-time tensors appear. For example, the expansion of the metric through order $x^4$ reads

$$\begin{aligned} g_{ij}(x) &= \delta_{ij} + \frac{1}{3} R_{iklj}(\bar{x}) x^k x^l + \frac{1}{6} \nabla_m R_{iklj}(\bar{x}) x^k x^l x^m \\ &\quad + \frac{2}{45} R_{iklp}(\bar{x}) R^p{}_{mnj}(\bar{x}) x^k x^l x^m x^n + \frac{1}{20} R_{iklj;mn}(\bar{x}) x^k x^l x^m x^n + O(x^5). \end{aligned} \tag{16}$$

Since the action (12) involves a $9N^2$ dimensional metric, we can apply Riemann normal coordinates to matrix space as well. Although we could expand around any fixed set of matrices, we will expand around matrices proportional to the identity matrix

$$X^i = \bar{x}^i \mathbb{1} \tag{17}$$

which corresponds to a stack of coincident D-branes located at $\bar{x}^i$. The expansion of the metric $G_{IJ}$ is similar to that in (17),

$$\begin{aligned} G_{IJ}(X) &= \delta_{IJ} + \frac{1}{3} R_{IKLJ}(\bar{x}) X^K X^L + \frac{1}{6} \nabla_M R_{IKLJ}(\bar{x}) X^K X^L X^M \\ &+ \frac{2}{45} R_{IKLP}(\bar{x}) R^P{}_{MNJ}(\bar{x}) X^K X^L X^M X^N + \frac{1}{20} R_{IKLJ;MN}(\bar{x}) X^K X^L X^M X^N + O(X^5). \end{aligned} \tag{18}$$

Therefore, in these coordinates it is sufficient to find the tensors $R_{IJKL}(\bar{x})$, $\nabla_M R_{IJKL}(\bar{x})$, etc. Since the metric $G_{IJ}(X)$ was constructed out of $g_{ij}(x)$, the corresponding curvature tensors in matrix space have to be constructed out of $g_{ij}(x)$ as well. One can use string theory to argue [29] that matrix space tensors evaluated at $X^i = \bar{x}^i \mathbb{1}$ have to be expressed in terms of space-time tensors evaluated at $\bar{x}^i$. For instance, $R_{IJKL}(\bar{x}) \equiv R_{i\alpha_1\alpha_2, j\beta_1\beta_2, k\gamma_1\gamma_2, l\delta_1\delta_2}(\bar{x})$ will typically involve terms of the form

$$R_{ijkl}(\bar{x})(\delta_{\alpha_2\beta_1}\delta_{\beta_2\gamma_1}\delta_{\gamma_2\delta_1}\delta_{\delta_2\alpha_1}) + \ldots \tag{19}$$

Once we construct our matrix space tensors in terms of space-time tensors, we have effectively achieved diffeomorphism invariance for all diffeomorphisms that leave $\bar{x}$ fixed, since they simply act as $\bar{x}$-dependent rotations of $x^i$ and $X^i$.

The remaining diffeomorphisms are the ones that map normal coordinates based at $\bar{x}$ to normal coordinates based at a different point $\bar{z}$. Invariance under these diffeomorphisms means that the form of the action should remain exactly identical under the corresponding change of matrix normal coordinates, except that $\bar{x}$ is replaced everywhere by $\bar{z}$. Indeed, if we imagine computing the action directly using world-sheet techniques, the amplitudes obtained when the branes are located at a point $\bar{x}$ are identical to those obtained when the branes are located at $\bar{z}$, except that all quantities are evaluated at $\bar{z}$ rather than $\bar{x}$. This base-point independence significantly constrains the possible form of $R_{IJKL}, \nabla_M R_{IJKL}$, etc. A change of base-point is not merely a change of coordinates from the world-sheet point of view, it also involves a condensation of the scalar fields that parametrize the transverse fluctuations of the branes, since the location of the branes has to be moved as well.

We are now ready to formulate a set of axioms that the curvatures $R_{IJKL}$ and their covariant derivatives have to obey, in order that they give rise to a diffeomorphism invariant action for multiple D0 branes.

($a$) The curvature tensor $R_{IJKL}$ has to have to usual symmetries, meaning antisymmetric in the first and second pair of indices, and symmetric under exchange of the first and second pair.

($b$) It should have cyclic symmetry.

($c$) Covariant derivatives $\nabla_I$ should obey the Bianchi identity, $\nabla_{[M} R_{IJ]KL} = 0$.

($d$) Multiple covariant derivatives should, when anti-symmetrized, yield the usual rules, e.g. $[\nabla_I, \nabla_J] R_{KLMN} = R_{IJK}{}^P R_{PLMN} + 3$ more terms.

($e$) The $U(N)$ indices should be contracted in a single trace style, as the action contains a single trace.

($f$) To have the right behavior under change of basepoint, we require that

$$\delta^{\alpha\beta}\nabla_{i\alpha\beta}(\text{anything}) = \nabla_i(\text{anything}) \tag{20}$$

($g$) To first order in the curvature, the symmetrized trace prescription, found in [18, 19] should emerge.

($h$) It should have the right $U(1)$ limit for diagonal matrices.

Conditions ($a$) through ($d$) guarantee that the curvature tensors and their derivatives all come from a single metric $g_{IJ}(X)$. If we would just write down some random tensors that would violate one of the conditions ($a$) through ($d$), they could never correspond to the curvature tensors of some metric $g_{IJ}(X)$.

Condition ($e$) implies that the curvatures and its covariant derivatives can be written as sums of ordinary tensors with indices $i, j, k, l, \ldots$ times tensors $\Delta_{\alpha\beta\gamma\delta}$, where the latter are defined as

$$\Delta_{\alpha\beta\gamma\delta} = \delta_{\alpha_2\beta_1}\delta_{\beta_2\gamma_1}\delta_{\gamma_2\delta_1}\delta_{\delta_2\alpha_1} \tag{21}$$

and similarly with more indices. The tensors $\Delta$ are cyclically invariant, and yield single trace expressions when contracted with matrix-valued coordinates.

Condition ($f$) is crucial, and expresses the constraint of base-point independence. In the way we have set things up, it is guaranteed that under an infinitesimal change of base point all tensors transform as $T \to \epsilon^i \delta^{\alpha_1\alpha_2}\nabla_{i\alpha_1\alpha_2}T$. However, we want this to be equivalent to a change of the base point in all ordinary tensors that appear in $T$, without affecting the matrix structure. Therefore, we demand that $\epsilon^i\delta^{\alpha_1\alpha_2}\nabla_{i\alpha_1\alpha_2}T = \epsilon^i\nabla_i T$ for all $T$.

Finally, conditions ($g$) and ($h$) are obvious. Condition ($g$) is not implied by the others, and if we drop it, more general solutions for the matrix curvatures $R_{IJKL}$ can be found. This is in agreement with the non-symmetrized trace answer found for the bosonic string in [12]. Since this latter result was derived using world-sheet calculations, it should nevertheless be diffeomorphism invariant.

Conditions ($a$)-($h$) are the "axioms" of the matrix geometry. Once the curvature tensors have been specified, the action in normal coordinates follows directly by applying the expansion (17) to (12).

We have analyzed these conditions for terms up to order $X^6$, for which we need the curvature tensor and its first two derivatives. The curvature tensor and its first covariant derivative are given by a fully symmetrized answer, but the second derivative schematically looks like

$$\nabla_M\nabla_N R_{IJKL} = \text{symmetrized}(\nabla_m\nabla_n R_{ijkl}) + \text{terms quadratic in } R. \tag{22}$$

There are 120 different terms quadratic in $R$ that can appear in (22), and after imposing ($a$)-($h$) a 32-parameter family of terms survives. The full result is very lengthy, a simpler two-parameter family is given explicitly in [29]. The action therefore contains many terms that are of the form $\mathrm{Tr}(\dot{X}XXX\dot{X}XX)$ times something quadratic in $R$. Interestingly, the action always contains terms that do not appear in the $U(1)$ limit. Therefore, the action cannot be obtained by starting with the $U(1)$ result and ordering the terms that appear there in a suitable way.

So far, we have focused exclusively on the kinetic terms in the action. Of course, the complete non-abelian DBI action has many more terms; it has a potential term, fermionic terms, Wess-Zumino terms and higher order derivative terms. Obviously, these terms all have to be separately covariant. For those terms that only involve the transversal scalar fields $X$, the notion of covariance is identical to that for the kinetic terms. In other words, the terms should admit an expansion in terms of space-time tensors evaluated at $\bar{x}$, and they should be base-point independent. Suppose that we have determined the kinetic term up to a certain order, then we know the transformations for $X^I$ that implement a change of base-point up to that order. The same transformation should also yield a change of base-point for all other terms in the action, which are thereby severely constrained.

Interestingly, it is possible to write down a potential in curved space in terms of the matrix valued metric $G_{IJ}(X)$ that is completely covariant, and that reduces to the usual potential in flat space. To write the potential, we introduce a matrix version of the vielbein by writing

$$G_{IJ}(X) = \sum_A E^A_{I^t} E^A_{J^t}, \qquad G^{IJ} E^A_{I^t} E^B_{J^t} = \delta^{AB}, \tag{23}$$

where $A, B = 1 \ldots dN^2$. Here $I^t$ denotes $i\beta\alpha$ for $I = i\alpha\beta$. The base point independence of the action $G_{IJ}\dot{X}^I\dot{X}^J$ implies a simple transformation rule for $E^A_{I^t}\dot{X}^I$ under a change of basepoint, namely it rotates by an $SO(dN^2)$ transformation. Notice that $E^A_I \equiv E^A_{i\alpha\beta}$ is a matrix, which we will denote by $E^A_i$. Thus it is $\mathrm{Tr}(E^A_i \dot{X}^i)$ that transforms nicely. Now observe that taking the time derivative acts as a derivation on the algebra of matrices (i.e. it satisfies the Leibnitz rule), but so does the operation $V : X \to [X, V]$ for fixed $V$. Therefore, $\mathrm{Tr}(E^A_i[X^i, V]) = \mathrm{Tr}([E^A_i, X^i]V)$ will also transform nicely under a change of base-point, if we keep $V$ inert. We claim that a covariant version of the potential is

$$\frac{1}{2}\mathrm{Tr}([X^i, X^j]^2) \to \sum_{A,B} \frac{1}{4}\mathrm{Tr}([E^A_i, X^i][E^B_j, X^j][E^A_k, X^k][E^B_l, X^l]). \tag{24}$$

We just explained that $\mathrm{Tr}[E^A_i, X^i]$ transforms under a base-point changing transformation in a simply way, via an $SO(dN^2)$ transformation. Eq. (24) is therefore automatically covariant. One can also verify that in flat space, it reduces to the usual answer. Moreover, we know the full linearized coupling of the potential term to the metric [18, 19, 12]; it is given by a fully symmetrized expression. A somewhat tedious calculation shows that this is correctly reproduced by (24). This is strong evidence that eq. (24) is the correct curved-space version of the potential. One can even show that the masses of the off-diagonal fluctuations are given by exactly the geodesic distance [29], which was one of the axioms of D-geometry of [6].

Because of the covariance and base-point independence of the metric, the result that the masses of off-diagonal fluctuations equals the geodesic length will remain valid in any coordinate system. This leads to the interesting conclusion that the geodesic distance in space-time can be directly read off from $G_{IJ}(X)$, evaluated on diagonal $X$. There is no need to integrate a line element along a geodesic. More precisely, if we write the kinetic term in the form

$$\sum_t \mathrm{Tr}(P^t_{ij}(X)\dot{X}^i Q^t_{ji}(X)\dot{X}^j) \tag{25}$$

with some set $P^t_{ij}, Q^t_{ji}$, then the geodesic distance $d(x,y)$ satisfies

$$d(x,y)^2 = \sum_t P^t_{ij}(x) Q^t_{ji}(y)(x-y)^i(x-y)^j. \tag{26}$$

It would be interesting to find similar covariant expressions for the other terms in the action, such as the higher derivative terms. As long as these only depend on the space-time metric $g_{ij}$, one may hope that they admit an expression in terms of the vielbein so that covariance can be made manifest. This is presently under investigation. Inclusion of fermions presumably requires some new ingredients.

One can try to use a similar strategy to study the couplings to other closed string background fields. If we work again in matrix Riemann normal coordinates, that would involve finding a matrix version of the closed string background field and its covariant derivatives. For instance, if the closed string field is a scalar function $F$, we would need to determine $F(\bar{x}), \nabla_I F(\bar{x}), \nabla_I \nabla_J F(\bar{x})$ etc in terms of $F(\bar{x})$, $\nabla_i F(\bar{x})$, and the curvature and its covariant derivatives. Again, these matrix covariant derivatives have to obey consistency conditions, such as $\nabla_{[I}\nabla_{J]}\nabla_K F = R_{IJK}{}^L \nabla_L F$, as well as the requirement of base point independence $\delta^{\alpha\beta}\nabla_{i\alpha\beta}(\text{anything}) = \nabla_i(\text{anything})$. It would be interesting to understand whether there is some unifying principle at work here, perhaps one involving non-commutative geometry.

A role for non-commutative geometry is of course suggested by the fact that for multiple D-branes, the transverse coordinates are replaced by matrices. In non-commutative geometry, the space of functions is replaced by a non-commutative algebra, and the obvious candidate here would be to consider the algebra

$$\mathcal{A} = C^\infty(M) \otimes M_N(C). \tag{27}$$

This algebra does not yet carry any metric information. From the representation theoretical point of view, it is very close to the original algebra $C^\infty(M)$ (they are Morita equivalent). Following Connes, the construction of a Riemannian structure requires a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ which in addition to $\mathcal{A}$ also contains a Hilbert space $\mathcal{H}$ and a self-adjoint operator $D$ obeying certain properties [33]. It would be interesting to find triples $(\mathcal{A}, \mathcal{H}, D)$ that describe in a natural way metrics relevant for multiple D-branes, and that incorporate the notion of covariance. The form of the action and the potential suggest that the vielbein $E^A_I$ introduced in (23) will play an important role in such a construction.

As we explained before, the linearized coupling to the graviton can also be derived using the connection between non-commutative abelian gauge theories and commutative non-abelian gauge theories. It would be very interesting to see whether a similar connection can be made at higher order. The following remarks are suggestive that such connections may exist.

1 There is a close relation between the world-sheet calculations involving gravitons for non-commutative gauge theories and D-branes [13, 12]. Both lead to a result where certain operators are smeared along some kind of Wilson line (see (9)) . This structure becomes more complicated when more than one graviton vertex operator is involved [34], and the precise geometry underlying such calculations has not been uncovered.

2 Non-commutative spaces are constructed in deformation quantization from commutative spaces equipped with a closed two-form, and this is also how they arise in string

theory. The diffeomorphisms of the commutative space that preserve the two-form become gauge transformations of the non-commutative space (they are just canonical transformations). Thus, there should also be a relation between the coupling of a gauge field in non-commutative gauge theory and the coupling of the graviton in non abelian gauge theory.

3 There is a one-to-one correspondence between single trace expressions one can write down in terms of matrix valued coordinates $X^i$, and expressions involving ordinary coordinates and a closed two-form $B^{IJ}$. Open string amplitudes depend only on the combination $\mathcal{F} = B + F$ and T-duality maps $F \leftrightarrow [X, X]$. This map extends to

$$\begin{aligned} X^i X^j + X^j X^i &\leftrightarrow 2x^i x^j \\ [X^i, X^j] &\leftrightarrow B^{ij} \\ [X^k, [X^i, X^j]] &\leftrightarrow B^{kl}\partial_l B^{ij} \end{aligned} \tag{28}$$

etcetera. (Due to the Jacobi relation, the $B$-field must obey certain constraints. These are equivalent to the integrability constraints in deformation quantization [35].) This suggests there should be a relation (some kind of Seiberg-Witten map) between a single D brane in a transversal B-field, and multiple D-branes without a transversal B-field.

Altogether, we have uncovered a glimpse of an intricate geometrical structure that encodes the behavior of multiple D-branes in curved space. The precise mathematical structure underlying this geometry, and the corresponding stringy fuzziness of space-time, are still waiting to be uncovered.

Nevertheless, there are several directions in which the results here can be extended. A covariant formulation of the coupling to other closed string background fields, and an investigation of analogs of the Myers effect in curved backgrounds are two such issues, which are presently being investigated.

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