ITP-SB-95-33

Finite W-algebras and applications

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Talk presented by T.Tjin at the Meeting on W-symmetry, Marseille, Juli 3-7, 1995.

Let Q_α be the conserved quantities (charges) of a physical system. There is no physical principle that says that the commutator (or Poisson bracket) of two of these charges must again be a linear combination of charges. The only thing that one can show, using the Jacobi identity, is that the commutator of two charges is again conserved and therefore some combination of the charges Q_α. In other words

$$[Q_\alpha, Q_\beta] = F_{\alpha\beta}(Q) \tag{1}$$

where for simplicity we denoted $Q \equiv \{Q_\gamma\}$. The space of states of the system will carry a unitary representation of the algebra (1) which is irreducible on the energy eigenspaces. As such (1) describes the degeneracy in the spectrum of the Hamiltonian.

Non-linear symmetries were taken seriously for the first time in the context of conformal field theory (CFT). From the above argument it is however clear that non-linear symmetry algebras are not restricted to CFT, or even field theory as a whole. They appear in the simplest and in the most sophisticated models of physics. Here we shall illustrate this by giving some examples.

Consider a 2-dimensional isotropic harmonic oscillator. In terms of raising and lowering operators its Hamiltonian can simply be written as $H = a^\dagger a + b^\dagger b + 1$, where $[a, a^\dagger] = [b, b^\dagger] = 1$ and all other commutators vanish. The space of states is now obtained by repeated action of $a^\dagger$ and $b^\dagger$ on the ground state $|0\rangle$. The complete set of conserved quantities is given by H, $S_+ = ab^\dagger$, $S_- = a^\dagger b$ and $S_0 = b^\dagger b - a^\dagger a$. The algebra these quantities form is nothing but $sl(2)$: $[S_0, S_\pm] = \pm 2S_\pm$ and $[S_+, S_-] = S_0$. Obviously this is a linear algebra. The Hilbert space decomposes into a direct sum of the irreducible unitary finite dimensional representations of sl_2 which all appear with multiplicity 1.

Now, consider the same system in the anisotropic case [1, 2]. That is, consider for example the Hamiltonian $H = \omega_1(a^\dagger a + \frac{1}{2}) + \omega_2(b^\dagger b + \frac{1}{2})$. This Hamiltonian will only have a degenerate energy spectrum if there exists positive integers p, q such that $p\omega_1 = q\omega_2$. In all other cases the Hamiltonian is essentially the only conserved quantity in the system. Now, consider for example the case $2\omega_1 = \omega_2$. The operators S_a no longer commute with the Hamiltonian. A complete set of conserved quantities is however given by $j_+ = \frac{1}{\sqrt{3}} a^2 b^\dagger$,

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$j_- = \frac{1}{\sqrt{3}}(a^\dagger)^2 b$ and $j_0 = \frac{2}{3}(b^\dagger b - a^\dagger a)$. The commutation relations among them are given by

$$[j_0, j_\pm] = \pm 2 j_\pm \qquad [j_+, j_-] = j_0^2 + C \tag{2}$$

where $C = \frac{1}{4} - \frac{4}{9}H^2$. Obviously this algebra is non-linear and it provides a very simple example of a non-linear symmetry algebra appearing in basic physics. The Hilbert space of the anisotropic oscillator decomposes into a direct sum of irreducible unitary finite dimensional representations of the algebra (2), which are nothing but the energy eigenspaces of the Hamiltonian H. Note that the situation is completely analogous to the isotropic case and that the linear algebra one obtains there is really a very special case.

Another system where the symmetry turns out to be non-linear is a point mass moving in a $\frac{1}{r}$ potential. The Hamiltonian of such a system is given by: $H = \frac{\vec{p}^2}{2m} - \frac{\mu}{r}$, where μ is a fixed constant. Planetary motion and the motion of a charged particle in a Coulomb field (for example a hydrogen atom) are of this type. Due to the obvious rotation symmetry the angular momentum $\vec{L} = \vec{p} \times \vec{r}$ is conserved. Now, it is well known that in the hydrogen atom states are characterized by 3 quantum numbers: l, m and n, which are the angular momentum and energy quantum numbers respectively. The energy of the state $|l, m, n\rangle$ depends only on n: $E_n \sim \frac{1}{n^2}$. The energy is independent of m for any rotationally symmetric potential. However, the fact that the energy does not depend on the magnetic quantum number l is special to the $\frac{1}{r}$ potential and points to the existence of another conserved quantity (=symmetry). This conserved quantity is the so called 'Runge-Lenz vector': $\vec{R} = \vec{L} \times \vec{p} + \mu\frac{\vec{r}}{r}$. Putting the system on a large 3-sphere with radius $R = \frac{1}{\lambda}$ the commutation relations between the conserved quantities are given by

$$\begin{aligned}
[L_i, L_j] &= \epsilon_{ijk} L_k \\
[R_i, L_j] &= \epsilon_{ijk} R_k \\
[R_i, R_j] &= \epsilon_{ijk}(\lambda \vec{L}^2 - 2H) L_k
\end{aligned} \tag{3}$$

Obviously this algebra is highly non-linear. In the Euclidean limit $R \to \infty$ (3) reduces to the well known 'hidden $SO(4)$' symmetry of the hydrogen atom [3]. It should be noted that even in that case the algebra is non-linear due to the term HL_k. However, on each energy eigenspace the Hamiltonian is just a multiple of unity and therefore the action of the algebra (3) on one particular eigenspace is equivalent to that of $SO(4)$.

There are many more interesting examples of important physical systems with non-linear symmetries. For example, it was recently discovered that that non-linear symmetries play a role in intermediate statistics [4]. Furthermore, it can be shown that non-linear algebras often play the role of spectrum generating algebra for chains of Schrödinger operators [5, 10].

¿From the above it follows that in order to give a complete theory of symmetry in physics it is not enough to look at group theory. The representation theory on Hilbert spaces of non-linear algebras is however much more complicated than the representation theory of Lie algebras. It is therefore instructive to first consider non-linear algebras that are in some way related to Lie algebras. An important class of such algebras are the finite W-algebras [6, 7].

Finite W-algebras are finitely generated non-linear symmetry algebras that can be obtained from Lie algebras by reduction. We shall briefly explain what is meant by 'reduction', first classically, then quantum mechanically.

Let g be a Lie algebra and $C^{\infty}(g^*)$ the set of differentiable functions on its dual. Let $\{\phi_\alpha\} \subset C^{\infty}(g^*)$ by a set of functions such that $\{\phi_\alpha, \phi_\beta\} = f^\gamma \phi_\gamma$, where $f^\gamma \in C^{\infty}(g^*)$ and $\{.,.\}$ is the Kirillov-Kostant Poisson structure on g^*. The functions ϕ_α are called 'first class constraints'. The set I of elements of the form $f^\gamma \phi_\gamma$, with $f^\gamma \in C^{\infty}(g^*)$ is a multiplicative ideal in $C^{\infty}(g^*)$, for if $g \in C^{\infty}(g^*)$, then $g(f^\gamma) = (gf^\gamma)\phi_\gamma \equiv h^\gamma \phi_\gamma \in I$. Therefore we can define the algebra

$$A = C^{\infty}(g^*)/I. \tag{4}$$

However, this algebra is not a Poisson algebra because I is in general not a Poisson ideal. Consider therefore the idealizer of I, that is the largest subalgebra of $C^{\infty}(g^*)$ in which I *is* a Poisson ideal. This set is given by $\{f \in C^{\infty}(g^*) \mid \{f, \phi_\alpha\} \in I, \text{ for all } \alpha\} \equiv N(I)$. The algebra

$$A_{red} = N(I)/I \tag{5}$$

is a Poisson algebra and its Poisson structure $\{.,.\}$ is given by $\{[f],[g]\}^* = [\{f,g\}]$, where $[f] = f + I$. The Poisson algebra $(A_{red}, \{.,.\}^*)$ is called a classical 'finite W-algebra'. Note that, given a certain Lie algebra g there are many choices one can make for the set of first class constraints. The class of finite W-algebras is therefore very large.

In general finite W-algebras are highly non-linear. Furthermore, both (2) and (3) are examples of finite W-algebras. To be precise, (2) is a reduction of $sl(3)$ while (3) is a reduction of $sl(4)$ [6, 8].

Finite W-algebras can be quantized by BRST methods [7]. We will not go into this here however. Suffice it to say that a large class of finite W-algebras, those associated with sl_2 embeddings [9], has been quantized explicitly [7]. For some special cases not related to sl_2 embeddings the BRST cohomology has also been calculated, but there are still large classes of finite W-algebras that have not been quantized.

The fact that finite W-algebras are reductions of Lie algebras is significant especially when one tries to construct their representation theory or when trying to construct the 'W-coadjoint orbits' (the latter problem has been solved for finite W-algebras associated with arbitrary $sl(2)$ embeddings [10]). In principle any representation of the Lie algebra descends to a representation of the finite W-algebra in question. It has not been proven that this functorial relationship is surjective, but there are several indications that this is indeed the case. Recently general formulas for the Kac-determinant and characters of finite W-algebras associated to arbitrary $sl(2)$ embeddings were conjectured [8] and it was shown that complex finite W-algebras admit 'compact' and 'non-compact' real forms that respectively do and don't admit unitary representations [10].

Finite W-algebras often contain linear subalgebras. For example it can be shown that there exist non-linear extensions of $SU(3) \times SU(2) \times U(1)$, the gauge group of the standard model [10]. It is of course tempting to speculate as to the physical consequences of the non-linear gauge symmetry, for in principle such a theory might be a new candidate for a grand unification theory. In order to investigate this it is necessary to understand how to construct (gauge) field theories with finite W-symmetry At this point in time it is not known how to do this in the general case, even though some partial results have been obtained [10]. If one relaxes the requirement that the conserved charges in the field theory must form an algebra of the form $\{Q_\alpha, Q_\beta\} = P_{\alpha\beta}(Q)$ to include the case where $\{Q_\alpha, Q_\beta\} = P^\gamma_{\alpha\beta}(T_\delta)Q_\gamma$, where T_δ are extra scalar fields, one for each generator of the finite W-algebra and $P^{\gamma}_{\alpha\beta}(Q_\delta) = P_{\alpha\beta}(Q_\delta)$, then it is straight forward to write down topologically invariant theories with this internal symmetry [11, 10]. In particular it is possible to write down non-linear generalizations of topological BF-theories, which are well known (in the linear case) to be related to Reidemeister-Ray-Singer torsion. It would be extremely interesting to see whether these non-linear topological theories actually describe generalizations of such geometrical structures. This would require a much better understanding of the notion of global W-transformations and the concept of W-fiber bundles, that is, fibre bundles with a structure W-algebra instead of a structure group.

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