ITP-SB-95-34

Some geometrical aspects of W-algebras in string theory[1]

Jan de Boer[2]
Institute for Theoretical Physics, State University of Stony Brook, Stony Brook, NY 11794-3840, USA

In this talk, the focus will mainly be on geometrical apects of W algebras as they appear in string theories, rather than in integrable systems and so on. In string theory, W algebras can play a role in two ways. Either they can appear as a 'global' symmetry algebra in ordinary string theory, or they can be used as a local symmetry algebra to construct generalizations of string theory usually called W-strings. In the first case the physical states are given by the BRST cohomology of a BRST operator involving only the Virasoro algebra, in the second case of a BRST operator involving the complete W algebra. We will first look briefly at the first possibility, and then in somewhat more detail at the second. Several important open problems will be mentioned; the resolution of some of these seems to be necessary before serious new progress in the field can be made.

To see how W algebras can appear as global symmetry algebras and what consequences their presence can have, we consider a non-linear supersymmetric sigma model

$$S = \int d^2z d^2\theta (G_{ij} + B_{ij}) D\Phi^i \bar{D}\Phi^j \tag{1}$$

and look for higher spin conserved quantities of the form

$$W = \Omega_{i_1 \ldots i_n} D\Phi^{i_1} \cdots D\Phi^{i_n}. \tag{2}$$

(such a situation has been studied e.g. in [1]). Conservation $\bar{D}W = 0$ of W implies that

$$\nabla_k^{(+)} \Omega_{i_1 \ldots i_n} = 0, \tag{3}$$

where $\nabla_k^{(+)}$ is the covariant derivative with torsion $H = dB$. The existence of solutions of (3), i.e. the existence of covariantly closed differential forms, imposes certain restrictions on the metric and anti-symmetric tensor field, and solutions will therefore typically only appear at special points in the moduli space of the sigma model. The easiest way to derive a condition on the target space geometry, given a solution to (3), is to work out $[\nabla_k^+, \nabla_l^+]\Omega_{i_1 \ldots i_n} = 0$, yielding

$$R^m{}_{i_1 kl} \Omega_{m i_2 \ldots i_n} + \cdots R^m{}_{i_n kl} \Omega_{i_1 \ldots i_{n-1} m} = 0. \tag{4}$$

This is a linear equation for the curvature, and imposing constraints on the curvature usually implies that the holonomy of the target space is reduced. This suggest a close relation between points in the moduli space where enhanced world-sheet symmetries occur and points where the target space has reduced holonomy. The first open problem we would like to mention is to make this correspondence more precise: find consistent d-dimensional string backgrounds with holonomy $H \subset SO(d)$, and their corresponding enhanced symmetry algebras.

[1] Talk presented at W'95, Marseille, July 3-7, 1995
[2] email: deboer@insti.physics.sunysb.edu

We can only indicate some partial results here. If $H = 1$, then the target space is flat and the (chiral part of the) enhanced symmetry algebra is generated by $\{\partial\phi^i, \psi^i\}_{i=1\ldots d}$. For more general $H \subset SO(d)$ we expect the algebra A to satisfy

$$\frac{\{\partial\phi^i, \psi^i\}}{H} \supset A \supset \frac{\hat{SO}(d)_1}{\hat{H}_1} \tag{5}$$

where $\hat{H}_k$ denotes the affine Lie algebra based on H of level k. The right hand side of this inclusion is a subalgebra of the left hand, which can be seen by realizing $\hat{SO}(d)_1$ in terms of d free fermions.

Three examples where A is known (and it is non-linear) are:

(i) Calabi-Yau manifolds. Here, $H = SU(d/2)$, and A is an $N = 2$ superalgebra, containing an $N = 2$ super Virasoro algebra, and a chiral and an anti-chiral $N = 2$ superfield of dimension $d/2$. The latter are related to the existence of holomorphic and anti-holomorphic d-forms on a Calabi-Yau manifold. For an application to $d = 6$ Calabi-Yau manifolds, see e.g. [2].

(ii) Seven dimensional real manifolds with $H = G_2 \subset SO(7)$. The algebra is an $N = 1$ superalgebra, that contains besides the $N = 1$ super Virasoro algebra two superfields of dimension 3/2 and 2. The presence of these generators can be traced back to the existence of a covariantly closed three-form whose dual four-form is also covariantly closed. This algebra was first constructed in this context in [3, 4] and is actually isomorphic to a Hamiltonian reduction of $Osp(2|4)$ [5]. These manifolds are interesting for string compactifications because they admit a covariantly constant killing spinor and hence lead to target space supersymmetric string theories in three dimensions. Compact manifolds with this holonomy were only recently constructed [6].

(iii) Eight dimensional real manifolds with holonomy $spin(7) \subset SO(8)$. These manifolds have a self-dual covariantly closed four form which gives rise to an $N = 1$ superalgebra containing besides the $N = 1$ super Virasoro algebra a superfield of dimension 2. Compactifying strings on such manifolds leads to supersymmetric string theories in two dimensions.

It is well known that string theories compactified on Calabi-Yau manifolds have a beautiful property called mirror symmetry. It turns out that a generalized version of mirror symmetry also applies to the other two cases [3], although it no longer identifies just two manifolds with each other but whole families of manifolds. It seems that the smaller the symmetry algebra is, the larger the mirror symmetry becomes, because smaller algebras have a smaller number of quantum numbers, which can be used to distinguish different target spaces from each other. In addition, although the algebras in cases (ii) and (iii) are just $N = 1$ algebras, they still admit a version of spectral flow and topological twisting, in which the distinguished subalgebra $\hat{SO}(d)_1/\hat{H}_1$ plays an important role. It would be interesting to generalize these properties to arbitrary H.

Finally, it is important to keep in mind that we assumed that the algebra is realized locally here. This is definitely to weak, since it is known that using duality symmetries one can obtain new sigma models that are equivalent to the old one, but where the symmetry algebra is realized possibly in a non-local way [7, 8]. Conversely, given a symmetry algebra A it is definitely to strong to require that the target space manifold must have holonomy H, but to find the most general criterium for a sigma model to have a symmetry algebra A is an interesting and unsolved problem (cf. [9]).

This concludes the discussion of W algebras as a global symmetry algebra. For the remainder we will focus on W algebras as local symmetry algebras, i.e. W strings. Most of the discussion here is based on [10, 11]. W strings are obtained by coupling W matter to W gravity. In practise one does this as follows. One starts with a conformal field theory whose chiral algebra is a W algebra. Next, one gauges this W algebra, i.e. one couples the conformal field theory to gauge fields in such a way as to make the theory invariant under W transformations with arbitrary rather than holomorphic parameters. If ϕ denotes the matter fields, and $W_\alpha(\phi)$ the generators of the W algebra, then the gauged action looks like

$$S = S_{cft}(\phi) + \int d^2z \mu_\alpha W_\alpha(\phi) + \int d^2z \bar{\mu}_\alpha \bar{W}_\alpha(\phi) + \dots \tag{6}$$

where the dots denote terms of higher order in the gauge fields $\mu_\alpha, \bar{\mu}_\alpha$. In general there will be terms of infinite order in the gauge fields in order to close the algebra, but one can in all known cases write down a finite simple expression for the full gauged action involving some auxiliary fields. Consider for example one free scalar field with as W algebra just the Virasoro algebra. In that case one can introduce two auxiliary fields $A, \bar{A}$, and the gauged action simply reads

$$S = \int d^2z (\frac{1}{2}\partial\phi\bar{\partial}\phi - (A-\partial\phi)(\bar{A}-\bar{\partial}\phi) - \frac{1}{2}\mu A^2 - \frac{1}{2}\bar{\mu}\bar{A}^2) \tag{7}$$

Integrating out the auxiliary fields we find that the action becomes $\frac{1}{2}\int d^2z \sqrt{g} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$, with the metric given by the line element $ds^2 = |dz + \mu d\bar{z}|^2$. This shows that gauging the Virasoro algebra is the same as coupling to gravity, and furthermore that the gauge fields can be identified with the Beltrami differentials. Note that the terms in (7) of second order in $A, \bar{A}$ are of the form $M_{ij}A^iA^j$, and that the determinant of M vanishes exactly when $|\mu| = 1$. This is related to the fact that only those Beltrami differentials are allowed that satisfy $|\mu| < 1$.

For W-strings, the μ_α form a set of generalized Beltrami differentials that are part of a generalized 'W-metric'. In the same way as the Polyakov path integral for string theory involves an integration over the matter fields as well as over the metric, the path integral for W strings involves an integral over the matter fields and the generalized Beltrami differentials.

In ordinary string theory, an extremely important role is played by the moduli space of Riemann surfaces, which is the set of metrics modulo diffeomorphisms and Weyl rescalings. In the path integral, after gauge fixing the diffeomorphism symmetries, one is still left with an integral over the moduli space of Riemann surfaces, which one has to do by hand in order to compute for example correlation functions. In addition, this moduli space plays an important role in the beautiful geometrical structures that have been shown to underly pure gravity and gravity coupled to minimal models. In order to compute correlation functions for W strings, to examine claims about embeddings of W strings and about the equivalence of W strings to ordinary strings, as well as to find similar geometrical interpretations for pure W gravity and W gravity coupled to W minimal matter, it is unavoidable to have a good understanding of the structure of the moduli space of W strings. It is not necessary to cook up any artificial definition of this moduli space, the definition follows immediately from the path integral. In the case of ordinary strings, the moduli space is the space of metrics modulo diffeomorphisms and Weyl rescalings, or equivalently, as one sees from the path integral, it is the set of Beltrami differentials modulo Virasoro transformations. In the same way, the path integral defines for us the moduli space of W strings to be given by

$$\mathcal{M}_W = \frac{\{W\text{-gauge fields } \mu_\alpha\}}{W\text{-transformations}}. \tag{8}$$

Restricting for simplicity our attention to chiral W gravity from now on (i.e. $\bar{\mu}_\alpha = 0$), we can define an induced action $\Gamma(\mu_\alpha)$ for pure W gravity by integrating out the matter fields from (6)

$$e^{-\Gamma(\mu_\alpha)} = \int \mathcal{D}\phi e^{-S_{cft}(\phi) + \int \mu_\alpha W_\alpha(\phi)} \tag{9}$$

Taking $\Gamma(\mu_\alpha)$ as the definition of the action of pure W gravity, we can also quantize this theory, leading to an effective action $\Gamma(W_\alpha)$ for pure W gravity

$$e^{-\Gamma(W_\alpha)} = \int \mathcal{D}\mu_\alpha e^{\int \mu_\alpha W_\alpha - \Gamma(\mu_\alpha)} \tag{10}$$

The fields W_α here are independent fields and have nothing to do with the original generators $W_\alpha(\phi)$ of the W algebra. If h_α denotes the spin of $W_\alpha(\phi)$, then this is also the spin of W_α, and these fields are naturally paired via $\int \mu_\alpha W_\alpha$ to the Beltrami differentials, which are fields of spin $(1-h_\alpha, h_\alpha)$. We will find it more convenient to use as definition for the moduli space of W gravity the dual version of (8), namely

$$\mathcal{M}_W = \frac{\{W\text{-fields } W_\alpha\}}{W\text{-transformations}}. \tag{11}$$

From here on we will assume that the W algebra has been obtained by Drinfeld-Sokolov reduction of an affine Lie algebra $\hat{g}$ with respect to some sl_2 subalgebra $\{t_-, t_0, t_+\}$ of g. In that case, the effective action $\Gamma(W_\alpha)$ can be shown, modulo normalizations, to be equal to a constrained WZNW model,

$$\Gamma(W_\alpha) = S_{WZNW}(g)|_{g^{-1}\partial g = t_+ + W} \tag{12}$$

where $t_+ + W$ is the standard form of a Drinfeld-Sokolov constrained current, e.g. for the W_3 algebra it reads

$$\begin{pmatrix} 0 & 1 & 0 \\ T & 0 & 1 \\ W & T & 0 \end{pmatrix} \tag{13}$$

Furthermore, one can show that W transformations act in this context as those gauge transformations of the constrained connection $\partial + t_+ + W$ that preserve its form. Hence

$$\mathcal{M}_W = \frac{\{\text{DS-constrained connections}\}}{\text{special gauge transformations}}. \tag{14}$$

If $t_+ + W$ could really always be written in the form $g^{-1}\partial g$, it would always be pure gauge, and the moduli space $\mathcal{M}$ would be trivial. We have, however, been very careless in defining everthing so far, and basically assumed we were working on the complex plane. If one wants to define the same objects on an arbitrary Riemann surface, one has to be more careful.

What does it actually mean to do a Hamiltonian reduction on an arbitrary Riemann surface Σ? In the process of doing a Hamiltonian reduction, we impose certain constraints on the components of the currents of a WZNW model. In the WZNW model, the currents transform as spin one fields on Σ. In order to put some of them equal to a non-zero constant, which transform as a spin zero field, we first have to change the conformal weight of the relevant components of the current to zero. Algebraically this is implemented by improving (or sometimes called twisting) the energy-momentum tensor, i.e. by adding the total derivative of some components of the current that live in the Cartan subalgebra to the energy-momentum tensor,

$$T \sim (J^a J^a) \qquad \rightarrow \qquad T \sim (J^a J^a) + h_a \partial J^a. \tag{15}$$

Algebraically, this is a rather trivial operation, but geometrically the consequences are quite drastic. Namely, it implies that $t^+ + W$ is no longer part of a connection on a trivial bundle, but on a non-trivial one. Thus the algebraic procedure of twisting the energy-momentum tensor corresponds geometrically to a genuine twisting procedure. For example, for W_3, the components of the currents change their spins as follows

$$\begin{pmatrix} K^1 & K^1 & K^1 \\ K^1 & K^1 & K^1 \\ K^1 & K^1 & K^1 \end{pmatrix} \qquad \to \qquad \begin{pmatrix} K^1 & K^0 & K^{-1} \\ K^2 & K^1 & K^0 \\ K^3 & K^2 & K^1 \end{pmatrix}. \tag{16}$$

where K^h denotes the line bundle of holomorphic h-differentials, i.e. sections of the h^{th} power of the holomorphic cotangent line bundle. The current or connection on the right hand side of (16) is actually a connection on the rank three vector bundle $V = K^{-1} \oplus K^0 \oplus K^{+1}$, as one may easily check. More generally, if $t_0 = \text{diag}(d_1, \ldots, d_n)$ then the twisted current is part of a connection on the vector bundle

$$V = K^{-d_1} \oplus K^{-d_2} \oplus \cdots \oplus K^{-d_n} \tag{17}$$

A second problem that arises on non-trivial Riemann surfaces is that $A = t_+ + W$ transforms as a connection when going from one co-ordinate patch to another, and its transformation rule contains an inhomogeneous term which would spoil its specific constrained form. On a trivial surface, one co-ordinate patch suffices, and one does not have to worry about this problem. On a non-trivial surface, one is forced to introduce a background connection B, and one has to impose the constraints on $A - B$ rather than A. Since B also has an inhomogeneous term in its transformation rule, $A - B$ does not, and it transforms as an $\text{End}(V)$ valued one-form, where V is the vector bundle in (17). In the case at hand, the structure group of V can be reduced to $GL(1)$, and if we equip Σ with a background metric $ds^2 = \rho dz d\bar{z}$, a choice for B is

$$B = \text{diag}(d_1 \partial \log \rho, \ldots, d_n \partial \log \rho) \tag{18}$$

We can now give the correct definition of W algebras on an arbitrary Riemann surface Σ: W-transformations are those gauge transformations that preserve the form of the connection

$$D_W = \partial + t_+ + W + B. \tag{19}$$

The appearance of the somewhat arbitrary background connection B may seem unnatural, but it actually already appeared in the work of Quillen [12] on determinants of Cauchy-Riemann operators. More physically, it can be understood as a regularization ambiguity of the path integral

$$\det(\partial + A_z) \sim \int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{-\int \bar{\psi}(\partial + A_z)\psi} \tag{20}$$

We can now easily write down actions for W gravity on an arbitrary Riemann surface, using the generalized WZNW action for non-trivial bundles

$$kS_{wznw}(A;B) = -\frac{k}{8\pi}\int \text{Tr}((A-B)\wedge *(A-B)) - \frac{ik}{12\pi}\int \text{Tr}(A-B)^3 \tag{21}$$

which shares many properties with the usual WZNW action, like the following version of the Polyakov-Wiegmann identity

$$S(A;B) = S(A;C) + S(C;B) - \frac{k}{\pi}\int \text{Tr}((A-C)_z (C-B)_{\bar{z}}). \tag{22}$$

Using this generalized WZNW action one can for example reproduce the actions for induced gravity on an arbitrary Riemann surface as found in [13]. in which the background connection B appears as a reference projective structure.

Coming back to what we were after, namely a description of the moduli space of W gravity, we have so far established it is given by

$$\mathcal{M}_W = \frac{\{\partial + B + t_+ + W\}}{W\text{-transformations}} \tag{23}$$

The dimension of the tangent space to the moduli space can now be found from the cohomology of the complex

$$0 \overset{D_W}{\to} \{\text{allowed parameters}\} \overset{D_W}{\to} \delta W \overset{D_W}{\to} 0. \tag{24}$$

where the allowed parameters are those parameters for infinitesimal gauge transformations that preserve the form of D_W. This cohomolgy seems difficult to compute, that it is nevertheless possible is due to the following observation. Denote by L the inverse of $\mathrm{ad}(t_+)$, extended by 0 on the cokernel of $\mathrm{ad}(t_+)$. There is a method to map one complex into another in such a way that the cohomology of the complex does not change, by means of a so-called homotopy contraction. For this one needs a homotopy operator, and it turns out that L is precisely such a homotopy operator. If we start with a complex as in (24), but now with arbtirary parameters and arbitrary variations δW that may change the form of D_W,

$$0 \overset{D_W}{\to} \Omega^0(\Sigma; sl(V)) \overset{D_W}{\to} \Omega^1(\Sigma; sl(V)) \overset{D_W}{\to} 0, \tag{25}$$

and perform repeatedly a homotopy contraction of this complex, we end up, after a finite number of steps, with precisely the complex (24)! (for details, see [10, 11]). In other words, W transformations are a homotopy contraction of ordinary gauge transformations. One of the nice consequences of this observation is that one can obtain a simple explicit formula for arbitrary W transformations and W algebras in terms of L. Furthermore, it implies that to compute the cohomolgy of (24) we only need to compute the cohomology of (25). From the Riemann-Roch theorem, it follows that the cohomology of (24) satisfies

$$\dim H^1 - \dim H^0 = (g-1)\dim(G). \tag{26}$$

Comparing dimensions one can now show that

$$\mathcal{M}_W = \frac{\{\partial + B + t_+ + W | \bar{\partial} W = 0\}}{\{\text{gauge trafos that preserve this form}\}} \tag{27}$$

The denominator contains only those gauge transformations with a constant parameter with values in the centralizer of the embedded sl_2, which is e.g. zero dimensional for the W_N algebras, and one-dimensional for $W_3^{(2)}$. Thus, (27) provides us with a simple finite dimensional model of the moduli space!

The next question which arises, is whether these moduli spaces have a natural interpretation. The theory developed by Narasimhan and Seshadri [14] associates flat G bundles to moduli spaces of connections of the type we are considering here. Unfortunately, the anti-holomorphic structure that appears in our case is not stable in their sense, and we have to go beyond their theory. It turns out that the proper generalization exists already in the literature and goes under the name of Higgs bundles [15]. A Higgs bundle is a pair (V, θ) with V a holomorphic vector bundle and θ a holomorphic section $\theta \in H^0(\Sigma; \mathrm{End}(V) \otimes K)$, and their is an obvious map from $\mathcal{M}_W$ to the space of Higgs bundles, with V as in (17), and θ equal to $t_+ + W$, with $\bar{\partial} W = 0$. Using the results in [15] this implies that $\mathcal{M}_W$ is a subset of the moduli space of flat $G_{\mathbf{C}}$ bundles. For the W_N algebras, one can be more precise and, using a certain symmetric bilinear form, prove that $\mathcal{M}_{W_N}$ is actually a component of the moduli space of flat $sl(N, \mathbf{R})$ bundles.

An open problem is to find whether there is a corresponding interpretation for W-algebras obtained from non-principally embedded sl_2's. For example, for $W_3^{(2)}$,

$$\mathcal{M}_{W_3^{(2)}} = \frac{\{J, G^+, G^-, T, \text{holomorphic} | G^+ \neq 0 \text{ or } G^- \neq 0\}}{\{\mathbf{C} \setminus \{0\}\}} \tag{28}$$

where $\lambda \in \mathbf{C}$ acts via $G^+ \to \lambda G^+$ and $G^- \to \lambda^{-1} G^-$.

In the case of the W_N algebra, to actually construct the flat $sl(N, \mathbf{R})$ connection corresponding to a Higgs bundle (V, θ), one needs the so-called Hermitian-Yang-Mills metric on the Higgs bundle, which is given by a Hermitian object Ω transforming in a suitable way such that the inner product of two section s_1, s_2 of V is given by $\int \rho d^2 z (s_1^\dagger \Omega s_2)$. In our case the condition that the metric given by Ω is Hermitian-Yang-Mills equations reads

$$-\bar{\partial}(\Omega^{-1} \partial \Omega) + [\theta, \Omega^{-1} \theta^\dagger \Omega] = 0. \tag{29}$$

This is a very interesting equation; for $\theta = t_+$ it is precisely the Toda equation for $sl(N)$, but for other θ it describes a Toda equation in some kind of W background. In the case of sl_2, it can be explicitly solved for any θ and one recovers the usual expression for constant curvature metrics in terms of holomorphic quadratic differentials. In addition, one finds that positive definiteness of the metric implies that $|\mu| < 1$. All this suggests that Ω could play the same role for W_N as the usual metric plays for the Virasoro algebra, and it is an interesting open problem to see if W gravity can be reformulated as a theory of metrics on V. Furthermore it would be worthwile to further explore the solutions of (29) and to see whether this leads for example to a suitable generalization of the condition $|\mu| < 1$ to W_N. A different but related connection between Higgs bundles and Toda theory has been studied in [16]

An explicit expression for the full flat connection is only known in general if $\theta = t_+$. It reads

$$D = \partial + B + t_+ + \bar{\partial} - L(R_{z\bar{z}}), \tag{30}$$

where a constant curvature reference metric with curvature $R_{z\bar{z}}$ is assumed.

Coming back to applications to W-strings, the moduli space plays an important role if we want to compute correlation functions. However, if there are operators present on the surface, what we are really interested in the moduli space for W-gravity for surfaces with punctures (or marked points). Unfortunately, this moduli space is much less well understood than the case without punctures.

Let us review how one computes in practise in gravity correlation functions in genus zero. Rather than writing down a suitable set of Beltrami differentials that parametrize the moduli space of the punctured sphere, one takes one fixed sphere and integrates over the locations of the punctures. The reason that this is possible is that starting with some Beltrami differential, one can always make a co-ordinate transformation so as to get rid of the Beltrami differential, but at the cost of moving the punctures around. Thus, the moduli space of the N-punctured sphere is simply $(S^2)^N / SL(2)$. The SL_2 can be used to fix three of the punctures at prefered points, and what remains is an integration over the location of the remaining $N-3$ punctures.

For W_N gravity, we have only very limited knowledge of the moduli space in the presence of punctures, and this is one of the main unsolved problems in the field. Without proper understanding of this moduli space, we can never compute general correlation functions in W gravity. For ordinary gravity, each extra puncture introduces one extra modulus and the dimension of the moduli space of the N-puntured sphere is $N-3$. One can also see this from the fact that the operators one puts at the location of the puncture have ghost number one. In W_M gravity, the standard tachyonic operators have ghost number $M(M-1)/2$, and thus the dimension of the moduli space of the N-puntured sphere in the case of W_M gravity should be $NM(M-1)/2 - (M^2-1)$. This immediately suggests a candidate for this moduli space as the space of flat $sl(n, \mathbf{R})$ bundles with a reduction of the structure group to a Borel subgroup of $sl(n, \mathbf{R})$ at the marked points. This has the correct dimension, but we do not have a good parametrization of the generalized Beltrami differentials for this case at our disposal. Even if we would have such a description it would, as in gravity, be highly preferable to have a description analogous to $(S^2)^N/SL(2)$. Purely on dimensional grounds, this should be something like $X^N/SL(M)$, with X a $M(M-1)/2$-dimensional complex manifold, and $X \supset S^2$. Clearly, besides the location of the puncture, X contains many new 'hidden' co-ordinates. These new co-ordinates, together with the old one, constitute what is presumably a W-superspace. This is another object whose existence has not been established yet, but which would be extremely useful. On this W-superspace, W-transformations should be realized geometrically, and it should be possible to extend every field from one on S^2 to one on this space X. For example, for W_3, one would like to define a field depending on three co-ordinates, so a natural guess would be something like[17]

$$\phi(z_1, z_2, z_3) \sim \exp(z_1 L_{-1}) \exp(z_2 W_{-1}) \exp(z_3 W_{-2}) \phi(0,0,0) \tag{31}$$

Unfortunately, this naive definition makes no sense as it stands here, which can be seen e.g. from the commutation relations of the W_3 algebra. In our opinion, finding the correct definition and interpretation of the space X and fields on it is the central theme in the quest for the solution to the ill-posed problem 'what is W-geometry?'.

Yet another open problem is to find the correct geometrical interpretation for the states with a different ghost number that exist in W-strings. Is the corresponding moduli space the moduli space of flat $sl(n, \mathbf{R})$ bundles with reductions to different Borel subgroups at the punctures? Despite all these problems, there are some correlation functions in W-strings that can be computed, namely those for which the moduli space is exactly zero-dimensional. However, it is important to keep in mind that these correlation functions constitute only a small subset of all correlation functions, and claims in the literature regarding W-strings where either moduli are ignored or only correlation functions corresponding to zero-dimensional moduli spaces are computed, should be regarded with a certain amount of criticism.

There are several other approaches to W-geometry (see e.g. [18]), based upon generalized projective structures, jet bundles and special embeddings of Riemann surfaces in target spaces. It is important to understand moduli also in these frameworks, how to build a field-theoretical description for W-strings based upon them, and how to interpret (from a string point of view) the genus dependence of the target spaces in which the Riemann surfaces are embedded in some of the approaches.

Some further open problems are:

-does topological W-gravity describe intersection theory on the moduli space of flat $sl_N(\mathbf{R})$ bundles?

-is is possible to compute correlation functions for non-critical W strings by viewing them as some topological field theory, e.g. a topological Landau-Ginzburg theory?

-what is the modular group for W-gravity? The moduli space we have been considering here is more like the Teichmüller space for W-gravity, since we have no good description of global W transformations. Also, what is the generalization of the condition $|\mu| < 1$ to the case of W gravity?

-what is the metric and the analytic structure of the moduli space?

-Do W strings count the number of 'W-holomorphic' maps into some target space?

-Compute the generalized β-functions for W-strings, and examine their phenomenological potential.

Finally, we would like to mention some recent developments in mathematics[3] that may be relevant for some of these issues. In [19] Beilinson and Drinfeld introduce the notion of a g-oper, which is closely related to Hamiltonian reduction on an arbitrary surface, and to the moduli space of W-gravity. For example, an sl_n-oper on Σ is a rank n holomorphic vector bundle V with a filtration $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$, and a connection $D : V \rightarrow V \otimes K$ such that $D(V_i) \subset V_{i+1} \otimes K$ and D induces an isomorphism $V_i/V_{i-1} \xrightarrow{\sim} (V_{i+1}/V_i) \otimes K$. Therefore, V can be identified with the vector bundle given in (17), if we use the principal embedding of sl_2 in sl_n, and the filtration is $0 = (t_-)^n(V) \subset (t_-)^{n-1}(V) \subset \cdots t_-(V) \subset V$. In addition, each of the connections in (27) satisfies the properties required to call V together with this connection a g-oper. Thus, there is a natural one-to-one correspondence between the moduli space for W-gravity and the set of isomorphism classes of analytic g-opers, called $Op(g)_\Sigma$. This space is further analyzed in [19], in particular it is shown that is the quotient of the space of projective structures on Σ times the vector space of holomorphic fields W (as in (27)), modulo an equivalence relation where one changes both the projective structure and the holomorphic quadratic differential in the set $\{W\}$ by another holomorphic quadratic differential. Furthermore, the space of opers with singularities, which is relevant for the moduli space of puntured surfaces, has been studied in [19]. Unfortunately, it is not yet clear whether the results obtained in [19] will be helpful to obtain a useful description of the moduli spaces for W-gravity, and opers have been defined for principal embeddings only, but this is certainly a connection worth exploring further. Opers also appear in a geometric version of Langlands duality (see e.g. [20]). If g^L denotes the Langlands dual of g, then there is a correspondence between the set of g^L-opers, and certain $\mathcal{D}$-modules on the moduli space of principal G-bundles on Σ. These $\mathcal{D}$-modules describe certain integrable systems, which can be worked out explicitly in the case Σ is a sphere with marked points. Whether this relation can lead to further useful insights in the structure of the moduli space of W-gravity remains to be understood.

To summarize, starting from the path integral description of W-strings, one can derive, without the need to make any ad-hoc hypothesis, what one should consider as being the moduli space of W-strings. The main problem which has not been solved so far is to find a useful description of the W-moduli space of surfaces with punctures, in particular a W-superspace description of this moduli space. It seems hard to make a lot of further progress in W-strings before these issues are better understood.

References

[1] P.S. Howe and G. Papadopoulos, Commun. Math. Phys. **151** (1993) 467.

[2] S. Odake, Mod. Phys. Lett. **A4** (1989) 557; T. Eguchi, H. Ooguri, A. Taormina and S.-K. Yang, Nucl. Phys. **B315** (1989) 193.

[3] S.L. Shatashvili and C. Vafa, 'Superstrings and manifolds of exceptional holonomy', HUTP-94-A016, hep-th/9407025.

[4] J. de Boer and M. Rocek, unpublished.

[5] K. Ito, J.O. Madsen and J.L. Petersen, Phys. Lett. **B292** (1992) 298.

[6] D.D. Joyce, 'Compact 7-manifolds with holonomy G_2',I,II, IAS preprints 1994.

[7] B. Kim, Phys. Lett. **B335** (1994) 51.

[3] I would like to thank J. Harnad for drawing my attention to these papers.

[8] E. Kiritsis, Nucl. Phys. **B405** (1993) 109.

[9] F.S. Hassan 'T-dualities and non-local supersymmtries', hep-th/9504048; E. Alvarez, L. Alvarez-Gaumé and I. Bakas, 'T-duality and space-time supersymmetry', hep-th/9507112.

[10] J. de Boer and J. Goeree, Nucl. Phys. **B401** (1993) 369.

[11] J. de Boer, 'Extended conformal symmetry in non-critical string theory', PhD-thesis, Utrecht, June 1993.

[12] D. Quillen, Funct. Anal. Appl. **19** (1985) 31.

[13] M. Knecht, S. Lazzarini and F. Thuillier, Phys. Lett. **B251** (1990) 279; M. Knecht, S. Lazzarini and R. Stora, Phys. Lett. **B262** (1991) 25; Phys. Lett. **B273** (1991) 63; K. Yoshida, Mod. Phys. Lett. **A4** (1989) 71; Int. J. Mod. Phys. **A5** (1990) 4559; R. Zucchini, Phys. Lett. **B260** (1990) 296; Commun. Math. Phys. **152** (1993) 269.

[14] M.S. Narasimhan and C.S. Seshadri, Ann. Math. **82** (1965) 540.

[15] N.J. Hitchin, in 'Global Geometry and Mathematical Physics', Eds. M. Francaviglia, F. Gherardelli, Lecture Notes in Mathematics 1451, Springer-Verlag, Berlin, 1990; Proc. London Math. Soc. (3) **55** (1987) 59; Progr. Theor. Phys. Supp. **102** (1990) 159; Topology **31** (1992) 449; C.T. Simpson, Publ. Math. I.H.E.S. **75** (1992) 5; Journal of the A.M.S. **1** (1988) 867.

[16] E. Aldrovando and G. Falqui, 'Geometry of Higgs and Toda fields on Riemann surfaces', hep-th/9312093; 'Toda field theory as a clue to the geometry of W(N) gravity', hep-th/9411184.

[17] M. Bershadsky, W. Lerche, D. Nemeschansky and N.P. Warner, Nucl. Phys. **B401** (1993) 304.

[18] A. Gerasimov, A. Levin and A. Marshakov, Nucl. Phys. **B360** (1991) 791; A. Bilal, V. Fock and I. Kogan, Nucl. Phys. **B359** (1991) 635; J.-L. Gervais and Y. Matsuo, Commun. Math. Phys. **152** (1993) 317; G. Sotkov and M. Stanishkov, Nucl. Phys. **B356** (1991) 439; G. Sotkov, M. Stanishkov and C.J. Zhu, Nucl. Phys. **B356** (1991) 245; C.M. Hull, Nucl. Phys. **B413** (1993) 296; R. Zucchini, Class. Quant. Grav. **10** (1993) 253; Class. Quant. Grav. **11** (1994) 1697; Phys. Lett. **323** (1994) 322; J. Geom. Phys. **16** (1995) 237; 'Drinfeld-Sokolov gravity', hep-th/9508054. S. Govindarajan and T. Jayaraman, Phys. Lett. **B345** (1995) 211.

[19] A. Beilinson and V. Drinfeld, 'Opers', preprint, 1993.

[20] E. Frenkel, 'Affine algebras, Langlands duality and Bethe ansatz', to appear in Proceedings of the International Congress on Mathematical Physics, Paris, 1994.