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Multimatrix Models and the KP-Hierarchy

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Abstract

We analyze the critical points of multimatrix models. In particular we find the critical points of highest multicriticality of the symmetric two-matrix model with an even potential. We solve the model on the sphere and show that these critical points correspond to the (p, q) minimal models with $p + q$ =odd. Based on this experience we give a formulation of minimal models coupled to quantum gravity, in terms of differential operators, that makes the $w_{1+\infty}$-constraints very transparent. This formulation provides a natural setting to study many issues, such as flows changing both p and q.

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1 Introduction and Summary

Due to the recent dramatic progress in solving quantum gravity in two dimensions using matrix models [1, 2, 3, 4], the geometry of moduli space [5] and topological quantum field theory [6], one is for the first time able to probe the nonperturbative features of two-dimensional quantum gravity. After the proposal of Douglas [7] how multimatrix models are related to minimal matter coupled to quantum gravity, one has even been able to study all $c < 1$-minimal models coupled to quantum gravity non-perturbatively. Much effort has been spent on obtaining a precise understanding of the one-matrix model. However, the multi-matrix models turn out to be much harder to deal with. In fact, only the two-matrix model [8, 9, 10, 11] and the three-matrix model [12] have been solved completely for some simple cases.

Recently it has become clear that it is probably sufficient to consider just the two-matrix model to obtain all minimal models [9, 10, 11], in contrast to an earlier belief that the (p, q)-models can only be obtained from the matrix model with $(p-1)$ matrices. Having a better understanding of multimatrix models might shed light on their relation with KdV hierarchies, and minimal models coupled to quantum gravity. This relation still remains to be clarified.

Motivated by this, we perform in this paper a study of multimatrix models in the spherical approximation. We restrict ourselves to genus zero, because exact solutions are at present impossible to deal with, except in some simple cases [10, 11, 12], whose analysis already involves rather tedious algebra. Furthermore, for the one-matrix model we know that multicriticality in the genus-zero approximation is sufficient to guarantee multicriticality at all orders, and we expect the same thing to be valid for the multimatrix model.

In section 2, we review the genus-zero formulation of the multimatrix model. As we will show, the complete genus-zero description of multimatrix models in the continuum limit is remarkably similar to the exact description in terms of differential operators. The difference is just that we have to replace 'quantum' commutators of differential operators by 'classical' Poisson brackets, and ∂ by one of the two canonical variables that define the Poisson bracket. Thus, in a sense, including higher genus contributions amounts to quantizing the genus-zero multimatrix model. This may be useful when, for instance, one wants to extend the genus-zero results of [13] to higher genus. Finally, we analyze the continuum limit and show what fixes the order of the differential operators P and Q introduced in [7]. Directly related to the order of P and Q is the degree of multicriticality of the multimatrix model.

This analysis is applied to the case of the symmetric even two-matrix model in section 3, but the same techniques should also be useful when one wants to determine the critical potentials of other multimatrix models. In particular we determine the smallest potentials that are expected to be needed to obtain the minimal models $(p, p+1)$ and $(p, p+3)$ in the continuum limit, thereby generalizing [10, 11] where the potentials corresponding to $(4,5)$, $(5,6)$ and $(3,8)$ were found. We also compute the minimal order the potential must have to obtain an arbitrary minimal model (p, q) with $p+q=$odd.

The main disagreement between quantum gravity and multimatrix models is that multimatrix models seem to provide too many operators. Therefore it is interesting to know which operators correspond precisely to the flows in the multimatrix model generated by changes in the potentials. We show how one can very easily determine these flows in the genus-zero approximation. The flows in terms of matrices have been obtained for the two-matrix model [9, 14], and very recently for arbitrary multimatrix models in [15]; they are related to certain Toda-hierarchies. As these flows commute, they should, on the level of differential operators, be generated by fractional powers of a certain differential operator. A priori it is not clear which one that should be; there is no reason why, for instance, it should be Q rather than P. As an example, we compute some operators of the two-matrix model corresponding to the Ising model. All of these operators are fractional powers of the third order differential operator occurring in the the $[P, Q] = 1$ description of the Ising model, and not of the fourth order operator. Therefore, the Ising model is not (p, q) symmetric, and this holds presumably for other multimatrix models as well. This is the main obstruction to find the precise relation between multimatrix models and quantum gravity.

The resolution of this problem probably relies heavily on the existence of W-constraints in the multimatrix model. These constraints were first conjectured in [16, 17] and their validity was proven in [18, 19]. Motivated by the multimatrix model, we give a formulation of the W-constraints, in which they arise as the reparametrization invariance of the space $\{\sum t_{ab} Q^a P^b\}$. Alternatively, they can be seen as a consequence of the invariance under canonical transformations of the Heisenberg algebra generated by P and Q. This formulation is still completely (p, q)-symmetric. Using this formulation, we then reconsider the relation between multimatrix models and quantum gravity, and indicate possible ways to understand these apparent discrepancies. We end the paper with some remarks.

2 The p-Matrix Model

2.1 Review of the p-Matrix Model

The partition function of a general multimatrix model is given by [7]

$$Z = \int \prod_{i=1}^{p} dM_i \exp \beta\, tr \left(-\sum_{i=1}^{p} V_i(M_i) + \sum_{i=1}^{p-1} c_i M_i M_{i+1} \right) \tag{2.1}$$

where the M_i are hermitean $N \times N$ matrices. The integral over the angular parts of the M_i can be done and we are left with the following integral over the eigenvalues $\lambda_{i,n}$ of the M_i

$$Z = \text{const} \int \prod_{i=1}^{p} \prod_{n=1}^{N} d\lambda_{i,n} \triangle(\lambda_{\alpha,1}) \triangle(\lambda_{\alpha,p}) \exp \beta \left(-\sum_{i=1}^{p} \sum_{n=1}^{N} V_i(\lambda_{i,n}) + \sum_{i=1}^{p-1} \sum_{n=1}^{N} c_i \lambda_{i,n} \lambda_{i+1,n} \right) \tag{2.2}$$

where $\triangle(\lambda_{\alpha,r}) = \prod_{a<b}(\lambda_{a,r} - \lambda_{b,r})$ is a Vandermonde determinant.

Next introduce (following [20]) orthogonal polynomials of order n $A_n(x) = x^n + \dots$ and $B_n(x) = x^n + \dots$ satisfying

$$h_n \delta_{n,m} = \int \prod_{i=1}^{p} d\lambda_i A_n(\lambda_1) \exp \beta \left(-\sum_{i=1}^{p} V_i(\lambda_i) + \sum_{i=1}^{p-1} c_i \lambda_i \lambda_{i+1} \right) B_m(\lambda_p) \tag{2.3}$$

We can write $\triangle(\lambda_{\alpha,1})$ as $\det_{\alpha\beta}(A_\alpha(\lambda_{\beta,1}))$ and $\triangle(\lambda_{\alpha,p})$ as $\det_{\alpha\beta}(B_\alpha(\lambda_{\beta,p}))$. Substituting this into the partition function (2.2) and expanding the determinant yields the following well-known expression for the partition function

$$Z = \text{const} \times N! \prod_{i=0}^{N-1} h_i \tag{2.4}$$

From now on we will use orthonormal polynomials, i.e. we make redefinitions $A_n \to A_n\sqrt{h_n}$ and $B_n \to B_n\sqrt{h_n}$. For the sake of brevity, write $\exp(-\beta\mu)$ for the exponential occurring in (2.3). As usual, we define certain infinite matrices by their matrix elements with respect to the orthonormal polynomials A_n and B_m

$$Q(j)_{mn} \quad = \quad \int \prod_{i=1}^{p} d\lambda_i\, \lambda_j A_n(\lambda_1) e^{-\beta\mu} B_m(\lambda_p) \quad 1 \le j \le p \tag{2.5a}$$

$$P(1)_{mn} = \int \prod_{i=1}^{p} d\lambda_i \, A'_n(\lambda_1) e^{-\beta\mu} B_m(\lambda_p) \tag{2.5b}$$

$$P(p)_{mn} = \int \prod_{i=1}^{p} d\lambda_i \, A_n(\lambda_1) e^{-\beta\mu} B'_m(\lambda_p) \tag{2.5c}$$

In these equations, the prime denotes differentiation with respect to λ_1 and λ_p respectively. The use of indices may look a bit strange, but guarantees e.g. that the matrix corresponding to an insertion of λ_1^2 is just $Q(1)^2_{nm} \equiv \sum_r Q(1)_{nr} Q(1)_{rm}$. It is straightforward to verify the following properties of the matrices $P(i)$ and $Q(i)$

$$P(1)_{nm} = 0 \qquad m \leq n, \qquad P(1)_{m,m+1} = (m+1)\sqrt{h_m/h_{m+1}} \tag{2.6a}$$

$$P(p)_{nm} = 0 \qquad m \geq n, \qquad P(p)_{m+1,m} = (m+1)\sqrt{h_m/h_{m+1}} \tag{2.6b}$$

$$Q(1)_{nm} = 0 \quad m < n-1, \quad Q(1)_{m+1,m} = \sqrt{h_{m+1}/h_m} \tag{2.6c}$$

$$Q(p)_{nm} = 0 \quad n < m-1, \quad Q(p)_{m,m+1} = \sqrt{h_{m+1}/h_m} \tag{2.6d}$$

Another set of important identities can be obtained by considering

$$\int \prod_{i=1}^{p} d\lambda_i \, \frac{d}{d\lambda_r} \Big(A_n(\lambda_1) e^{-\beta\mu} B_m(\lambda_p) \Big) = 0 \tag{2.7}$$

for $r = 1, \ldots, p$. This gives a set of relations expressing all matrices in terms of $P(1)$ and $Q(1)$:

$$\beta^{-1} P(1) - V'_1(Q(1)) + c_1 Q(2) = 0 \tag{2.8a}$$

$$c_{r-1} Q(r-1) - V'_r(Q(r)) + c_r Q(r+1) = 0 \quad 2 \leq r \leq p-1 \tag{2.8b}$$

$$\beta^{-1} P(p) - V'_p(Q(p)) + c_{p-1} Q(p-1) = 0 \tag{2.8c}$$

Finally, the multimatrix model has a set of discrete 'string equations'. Two of them, $[P(1), Q(1)] = [Q(p), P(p)] = 1$, can be directly obtained from the definitions of P and Q (2.5), the others then follow from (2.8) and simply read

$$\beta c_r [Q(r), Q(r+1)] = 1 \tag{2.9}$$

These equations together are sufficient to determine the h_i and therefore to evaluate the partition function (2.4).

If all potentials are of finite degree, one can check that $P(1)$, $P(p)$ and $Q(r)$ are Jacobi matrices. This means that the (a,b)-matrix element is only nonvanishing if $|a-b| \leq K$ for some integer K. For instance, using (2.6) and (2.8) one finds that for

$Q(1)$ we can take $K = \prod_{r=2}^{p}(deg(V_r) - 1)$. In the continuum limit, Jacobi matrices are expected to become finite order differential operators.

To proceed, we define

$$f_i = \frac{h_i}{h_{i-1}} \tag{2.10}$$

and

$$Q(1)_{n-l,n} = \sqrt{\frac{h_{n-l}}{h_n}} R_n^{(l)} \tag{2.11}$$

for $l \geq 0$. Similar expansions can be defined for the other matrices $Q(j)$. Inserting these into (2.8) and restricting (2.8a) and (2.8c) to the matrix elements where $P(1)$ and $P(p)$ vanish, yield the usual recursion relations, which in general are very complicated.

If we take the $(m-1, m)$ matrix element of (2.8a) we find an equation, that later will turn out to be equivalent to the string equation. It reads

$$\frac{m}{\beta} = (V_1'(Q(1))_{m-1,m} - c_1 Q(2)_{m-1,m}) \sqrt{\frac{h_m}{h_{m-1}}} \tag{2.12}$$

Using the recursion relations mentioned above, we can in general eliminate all the variables like the $R_n^{(l)}$ occurring in (2.11), so that the only variables left will be the f_i defined in (2.10). Then (2.12) takes the form

$$\frac{m}{\beta} = W(f_i) \tag{2.13}$$

We now take the scaling limit in the standard way [2]. We let $N \to \infty$, $\beta/N \to 1$, and replace discrete by continuous variables: $x = m/\beta$, $\epsilon = 1/N$, $f_i \to f(x)$, $R_n^{(l)} \to R^{(l)}(x)$, etc. If in the planar limit, which will be described in just a moment, the function W occurring in (2.13) behaves as

$$W(f) \simeq W_c - (f - f_c)^k \tag{2.14}$$

we can take the double scaling limit which essentially amounts to amplifying the region around $f = f_c$. Let $\gamma = -1/k$ denote the string susceptibility, and define the lattice spacing a and the renormalized cosmological constant μ_R by

$$\lambda a^{2-\gamma} = \epsilon \tag{2.15}$$
$$W_c - a^2 \mu_R = x \tag{2.16}$$

Here λ is the parameter that controls the genus expansion of the partition function, $Z = \sum \lambda^{2g-2} Z_g$. We will assume $\lambda = 1$, which can be accomplished by a redefinition of a and μ_R. To obtain the string equation in the usual form, make expansions for f and the R's in terms of $a^{-\gamma}$,

$$f(\mu_R) = f_c + a^{-2\gamma} f^{(1)}(\mu_R) + a^{-3\gamma} f^{(2)}(\mu_R) + \dots \tag{2.17}$$

and similar for the R's. Substituting these expansions back into the recursions relations obtained from (2.8) and letting $a \to 0$ turns equation (2.13) into the string equation.

2.2 Genus-Zero Formulation

To find the critical points, i.e. the potentials that yield a behavior as in (2.14), we will now restrict ourselves to the planar limit. This means that we will neglect the dependence of f_i and $R_n^{(l)}$ on i and n, because this dependence is only relevant for higher genus as can be seen from (2.15). The matrices can now be represented as power series in the 'shift' operator

$$z = \sum_r \delta_{r-1,r} \tag{2.18}$$

As can be seen from (2.6), the expansion for $Q(1)$ reads

$$Q(1) = \frac{\sqrt{f}}{z} + \sum_{l \geq 0} R^{(l)} \left(\frac{z}{\sqrt{f}} \right)^l \tag{2.19}$$

and from (2.6a) and (2.8a) we see that

$$\frac{P^{(1)}(z)}{\beta} = V_1'(Q^{(1)}(z)) - c_1 Q^{(2)}(z) = \frac{x}{\sqrt{f}} z + \mathcal{O}(z^2) \tag{2.20}$$

Given the V_i, we can in principal determine the $P^{(i)}(z)$ and $Q^{(i)}(z)$ as functions of f. These equations are however highly nonlinear and difficult to solve. We will, therefore, follow a reverse route, and will assume that the $Q^{(i)}$ are given. One may then try to construct the potentials by using (2.8). The equations for the coefficients occurring in the potentials are linear, but do not always admit a solution. For the time being, we will restrict our attention to (2.8a). From this equation it is easy to see that, given $Q^{(1)}$ and $Q^{(2)}$, V_1 is completely and uniquely determined by requiring $V_1'(Q^{(1)}(z)) - c_1 Q^{(2)}(z)$ to be of order $\mathcal{O}(z)$. Clearly, if $Q^{(2)}(z) = az^{-n}$+higher order, V_1 will be of order $n+1$.

The requirement that $V_1'(Q^{(1)}(z)) - c_1 Q^{(2)}(z)$ is of order $\mathcal{O}(z)$ is met for every value of f. Taking for instance $u = f$, we see that there exist $Q^{(1)}(z,u)$ and $Q^{(2)}(z,u)$ labeled by one extra parameter u, such that $V_1'(Q^{(1)}(z,u)) - c_1 Q^{(2)}(z,u)$ is still of order $\mathcal{O}(z)$. The reason that we bother to introduce a new variable u here, is that we will assume that everything depends analytically on u, which need not necessarily be the case for $u = f$. To find the exact u dependence would require a knowledge of V_1. We can, however, find an equation which does not explicitly depend upon V_1, by differentiating (2.8a) with respect to both z and u, which gives two equations from which V_1'' can be eliminated. The result of this is the following equation

$$\beta^{-1}\{P^{(1)}, Q^{(1)}\}_{z,u} = c_1 \{Q^{(1)}, Q^{(2)}\}_{z,u} \tag{2.21}$$

where the 'Poisson' bracket $\{\}_{z,u}$ is defined by

$$\{A(z,u), B(z,u)\}_{z,u} = z\frac{\partial A}{\partial z}\frac{\partial B}{\partial u} - \frac{\partial A}{\partial u} z \frac{\partial B}{\partial z} \tag{2.22}$$

The extra z has been introduced for later convenience. (2.21) looks like a 'classical' analog of the equation $\beta^{-1}[P^{(1)}, Q^{(1)}] = c_1[Q^{(1)}, Q^{(2)}]$ which is valid in the original matrix model. As we will now show, the planar approximation is nothing but the replacement of 'quantum' commutators by 'classical' Poisson brackets. Consider $A = X(u)z^l$ and $B = Y(u)z^k$. On the level of matrices, this means $A = \sum_\alpha X_\alpha(u)\delta_{\alpha-l,\alpha}$ and $B = \sum_\beta Y_\beta(u)\delta_{\beta-k,\beta}$. In the planar approximation the commutator of A and B can be calculated as follows

$$\begin{aligned}
[A,B] &= \sum_\alpha (X_{\alpha-k}Y_\alpha - X_\alpha Y_{\alpha-l})\delta_{\alpha-k-l,\alpha} \\
&= \sum_\alpha \left[(X_\alpha - k\frac{\partial X_\alpha}{\partial i})Y_\alpha - X_\alpha(Y_\alpha - l\frac{\partial Y_\alpha}{\partial i})\right]\delta_{\alpha-k-l,\alpha} \\
&= \sum_\alpha \left(lX_\alpha \frac{\partial Y_\alpha}{\partial i} - k\frac{\partial X_\alpha}{\partial i}Y_\alpha\right)\delta_{\alpha-k-l,\alpha} \\
&\to \left(lX\frac{\partial Y}{\partial i} - k\frac{\partial X}{\partial i}Y\right) z^{k+l} \\
&= z\frac{\partial A}{\partial z}\frac{\partial B}{\partial i} - \frac{\partial A}{\partial i} z\frac{\partial B}{\partial z} \\
&= \frac{\partial u}{\partial i}\{A,B\}_{z,u} \\
&= \beta^{-1}\frac{\partial u}{\partial x}\{A,B\}_{z,u}
\end{aligned}$$

In particular, this gives the following version of the planar string equation

$$\{P^{(1)}(z,u), Q^{(1)}(z,u)\}_{z,u} = \beta \left(\frac{\partial u}{\partial x}\right)^{-1} \tag{2.23}$$

Indeed, taking $P^{(1)}(z,u) \equiv P(z,u) = \beta z W(f)/\sqrt{f} + \mathcal{O}(z^2)$ and $Q^{(1)}(z,u) \equiv Q(z,u) = \sqrt{f}/z + \mathcal{O}(1)$, a short calculation shows

$$\{P, Q\}_{z,u} = \beta \frac{\partial f}{\partial u} W'(f) + \mathcal{O}(z) \tag{2.24}$$

which combined with (2.23) gives $\partial W(f)/\partial x = 1$, in agreement with (2.13).

2.3 The Continuum Limit in Genus Zero

Let us now consider the differential operators that P and Q will become in the continuum limit. In that limit we have

$$z = e^{-\epsilon \partial/\partial x} = e^{a^{-\gamma} \partial/\partial \mu_R} \tag{2.25}$$

In the planar approximation $\partial/\partial\mu_R$ commutes with everything, and can be replaced by a commuting object which we will denote by ξ. Instead of z and u we can also think of P and Q as functions depending on ξ and u. Because $z\partial/\partial z = a^{\gamma}\partial/\partial\xi$ we find that

$$\{A(z,u), B(z,u)\}_{z,u} = a^{\gamma}\{A(\xi,u), B(\xi,u)\}_{\xi,u} \tag{2.26}$$

where $\{\}_{\xi,u}$ denotes the usual Poisson bracket $\{A,B\}_{\xi,u} = \partial_\xi A \partial_u B - \partial_u A \partial_\xi B$. As $\beta \sim a^{\gamma-2}$, and by using (2.16), we find the string equation in terms of ξ

$$-\frac{\partial u}{\partial \mu_R}\{P(\xi,u), Q(\xi,u)\}_{\xi,u} = 1 \tag{2.27}$$

From now on we will assume that $u = a^{2\gamma}(f - f_c)$; u will be finite if we let $a \to 0$ at fixed μ_R. Obviously, we can write $\beta^{-1}P$ as $P_0 + \sum_{n\geq p} P_n a^{-n\gamma}$ and $Q = Q_0 + \sum_{n\geq q} Q_n a^{-n\gamma}$, where P_n and Q_n are polynomials in ξ and u of degree n (ξ has degree 1 and u has degree 2). P_0 and Q_0 will not contribute to the string equation and we will ignore these constants. Naively, one would think that p and q are the orders of the differential operators that P and Q become in the double scaling limit. This is, however, not always true. If, for instance, $p = q = 2$ and $P_2 = Q_2$, it is clear that $\{P_2, Q_2\} = 0$ and that we cannot consider P_2 and Q_2 as the scaling limit of P and Q while preserving the string equation. What is essential is that (2.27) is invariant under $P \to P + \sum a_i Q^i$ and also under $Q \to Q + \sum b_i P^i$, and in some cases such redefinitions are necessary to find then true orders of P and Q. If e.g. $q \leq p$, we try to find the operator $P' = P + \sum a_i Q^i$ that is of highest order, as it is this what survives in the string equation, and not P† Actually, we only need to determine the true orders of P and Q for $u = 0$, because (2.27) can be used to find the u-dependence, without lowering the order of either P or Q. (2.27) (More precisely, we need (2.23)) together with (2.8a) for $u = 0$ imply that (2.8a) is also valid for $u \neq 0$.

If $Q = Q_0 + \sum_{n \geq q} Q_n a^{-n\gamma}$ and $\beta^{-1} P' = P'_0 + \sum_{n \geq p} P'_n a^{-n\gamma}$, we define

$$\hat{Q} = \lim_{a \to 0} a^{q\gamma}(Q - Q_0) \tag{2.28}$$

$$\hat{P} = \lim_{a \to 0} a^{p\gamma}(P'_0 - \beta^{-1} P') \tag{2.29}$$

The string equation turns into ($\mu = \mu_R$)

$$\frac{\partial u}{\partial \mu} \beta a^{-(p+q)\gamma} \{\hat{P}, \hat{Q}\}_{\xi,u} = 1 \tag{2.30}$$

To get something finite, we must have $\gamma - 2 - p\gamma - q\gamma = 0$, so that $\gamma = -2/(p + q - 1)$, coinciding with the KPZ result for a (p, q)-minimal model [22]. Furthermore, $\{\hat{P}, \hat{Q}\}_{\xi,u}$ must be independent of ξ and is therefore proportional to $u^{(p+q-3)/2}$. Then the string equation implies $u \sim \mu^{-\gamma}$.

Something interesting happens when $p + q$ =even. In this case, the above does not work, as $\{\hat{P}, \hat{Q}\}$ can never be independent of ξ (unless it vanishes). The reason that it does not work is that γ^{-1} is not an integer. Looking at (2.14) we see that the parameter upon which everything depends analytically is $\sqrt{f - f_c}$ rather that $(f - f_c)$. Indeed, we can repeat the above taking $u = a^\gamma \sqrt{f - f_c}$, and we find correctly that $\gamma = -2/(p + q - 1)$ and that $\sqrt{f - f_c} \sim u \sim \mu^{-\gamma/2}$.

One should bear this in mind when comparing the above with the spherical formalism of [23]. There, the commutator of $f\partial^a$ and $g\partial^b$ is computed in the spherical approximation by keeping only first derivatives and dropping higher ones. The result is therefore $(afg' - bgf')\partial^{a+b-1}$. If we replace ∂ by ξ and assume f and g depend on a certain variable u, then this is precisely equal to $\partial u/\partial\mu \{f\xi^a, g\xi^b\}_{\xi,u}$. Thus, our formalism is equivalent to their formalism. However, we should be careful when

†There is some ambiguity in these redefinitions, but these are irrelevant. This will become clearer when we discuss the $w_{1+\infty}$ constraints in section 5. This method has also been applied in [10, 11, 21].

using their formalism to expand everything in either u or $\sqrt{u}$ depending on whether $p+q$ is odd or even.

It is also instructive to compare this formalism with that of the topological models in [13, 24]. What they call x is our ξ and their t_0 is our μ. Therefore any result in the differential operator formalism of matrix models can be simply translated into their results by replacing commutators by a Poisson bracket with respect to x and t_0. For instance (for details see [13, 24])

$$[L_+^{i+1}, L] = \{L_+^{i+1}, L\}_{x,t_0} = \left(\frac{\partial L_+^{i+1}}{\partial x}\frac{\partial L}{\partial t_0} - \frac{\partial L_+^{i+1}}{\partial t_0}\frac{\partial L}{\partial x}\right) = \partial_x L_+^{i+1} \tag{2.31}$$

3 The Symmetric Even 2-Matrix Model

To illustrate the genus-zero formulation of sections (2.2) and (2.3), let us now consider the symmetric 2-matrix model with even potential in some more detail (see also [9, 10, 11]). In this case, on the level of matrices $P(1) = P(2)^\dagger$ and $Q(1) = Q(2)^\dagger$, or equivalently $P^{(1)}(z,u) = P^{(2)}(1/z,u)$ or $P^{(1)}(\xi,u) = P^{(2)}(-\xi,u)$. Note that now (2.19) contains only odd powers of z. Only one equation is left from (2.8), namely

$$\beta^{-1}P - V'(Q) + c_1 Q^\dagger = 0 \tag{3.1}$$

The relevant condition for criticality is now that $Q(\xi,0)$ starts at order ξ^p, that is $Q(\xi,0) = Q_0 + a\xi^p + \ldots$. Equivalently, $\partial Q/\partial\xi$ must have a zero of order $p-1$ at $\xi = 0$. Going back to z, this means that $z\partial Q(z,0)/\partial z$ must have a zero of order $(p-1)$ at $z=1$. Because $\partial Q(z,0)/\partial z$ depends only on z^2, we must have that

$$\frac{\partial Q(z,0)}{\partial z} = \frac{1}{z^2}(1-z^2)^{p-1}A(z^2) \tag{3.2}$$

for some polynomial $A(z^2)$. As $Q(z,0)$ starts of at $\sqrt{f_c}/z$ it follows that

$$Q(z,0) = \int \frac{dz}{z^2}(1-z^2)^{p-1}(-\sqrt{f_c} + z^2 B(z^2)) \tag{3.3}$$

Here the integration is such that $Q(z,0)$ contains only odd powers of z.

3.1 (p,p+1) Minimal Models

We first consider the simplest case of 3.3. The maximal power of z occurring in $Q(z,0)$ is $(degV-1)$, and therefore the minimal potential needed to realize this level of multicriticality must correspond to $B = 0$, or $degV = 2p - 2$. Taking $B = 0$ in (3.3), we find the coefficients in (2.19)

$$R_c^{(2l+1)} = (-1)^l \frac{(f_c)^{l+1}}{2l+1} \begin{pmatrix} p-1 \\ l+1 \end{pmatrix} \tag{3.4}$$

The potential V is easily determined by requiring $V'(Q(z,0)) - c_1 Q(1/z,0)$ to be an analytic function of z. Expanding the potential gives

$$\begin{aligned} V(\phi) &= \frac{(-1)^{p-2}c_1}{f_c^{p-2}(2p-2)(2p-3)}\phi^{2p-2} + \frac{(-1)^{p-3}(p-1)c_1}{(2p-5)f_c^{p-3}}\phi^{2p-4} \\ &+ \frac{(-1)^{p-4}(p-1)(p-2)(6p^2-26p+19)c_1}{3(2p-6)(2p-7)f_c^{p-4}}\phi^{2p-6} + \cdots \end{aligned} \tag{3.5}$$

The freedom to choose c_1 is related to the fact that the matrices M_i in the original matrix model can always be arbitrarily rescaled.

What is the order of $\beta^{-1}P$? It is, up to powers of Q, equal to $-c_1 Q^\dagger$. Using (2.19) and (3.4) we can substitute (2.25) to find Q as function of ξ

$$\begin{aligned} \frac{1}{\sqrt{f_c}}Q(\xi,0) &= \sum_{l=0}^{p-1}\sum_{s=0}^{\infty}(-1)^{l-1}\begin{pmatrix} p-1 \\ l \end{pmatrix}\frac{(2l-1)^{s-1}}{s!}(a^{-\gamma}\xi)^s \\ &= \frac{2^{2p-2}}{\begin{pmatrix} 2p-2 \\ p-1 \end{pmatrix}} + \frac{2^{p-1}}{p}(-a^{-\gamma}\xi)^p - \frac{2^{p-1}(p-2)}{p+1}(-a^{-\gamma}\xi)^{p+1} + \mathcal{O}(\xi^{p+2}) \end{aligned} \tag{3.6}$$

Therefore the highest order for P/β is reached by taking $P/\beta = -c_1(Q^\dagger - (-1)^p Q)$. Then P has order $p+1$, and therefore this potential is expected to correspond to the $(p, p+1)$ minimal model. However, a full proof of this would require taking the full continuum limit at all orders of the genus expansion.

3.2 (p,p+3) Minimal Models

What about the other minimal models? A simple argument shows that in general P and Q will be an adjoint and a skew-adjoint operator, and that always $p + q$ =odd, due to the $\mathbf{Z}_2$ symmetry of this model. So let us consider the next series of minimal models, the $(p, p+3)$ models. Clearly, Q must have the same form as in (3.3). If we write $Q(\xi,0)/\sqrt{f_c} = A_0 + A_p\xi^p + A_{p+1}\xi^{p+1} + \ldots$ and take for P/β again $-c_1(Q^\dagger - (-1)^p Q)$, we see that P/β is proportional to $(1 + (-1)^p)A_0 - 2A_{p+1}\xi^{p+1} -$

$2A_{p+3}\xi^{p+3} + \ldots$. Therefore, in order to reach the $(p, p+3)$ model, we must require $A_{p+1} = 0$ so that P/β starts at order $p+3$. This is only true for $p \geq 3$. If $p = 3$ we can make the order of P/β even higher by adding $\lambda(Q - A_0)^2$ to it, so as to cancel the order ξ^6-term in P/β. Then P/β will be of order 8, so that taking $A_{p+1} = 0$ will give either (3,8) or $(p, p+3)$ with $p \neq 3$. Using (3.6) it is easy to check that the Q of minimal order must be given by

$$\frac{Q(z,0)}{\sqrt{f_c}} = -\int \frac{dz}{z^2}(1-z^2)^{p-1}(1-\frac{p-2}{p}z^2) \tag{3.7}$$

The coefficients of (2.19) are

$$R_c^{(2l+1)} = (-1)^l \frac{(f_c)^{l+1}}{2l+1}\left(\begin{pmatrix} p-1 \\ l+1 \end{pmatrix}\frac{2}{p} + \begin{pmatrix} p \\ l+1 \end{pmatrix}\frac{p-2}{p}\right) \tag{3.8}$$

The potential V is of degree $2p$ and the first few terms are given by

$$\begin{aligned} V(\phi) &= \frac{(-1)^{p-1}(p-2)c_1}{2p^2(2p-1)f_c^{p-1}}\phi^{2p} + \frac{(-1)^{p-2}(p^3-2p^2-2p+6)c_1}{p^2(2p-3)f_c^{p-2}}\phi^{2p-2} \\ &\quad + \frac{(-1)^{p-3}(p-1)(6p^6-26p^5+3p^4+130p^3-132p^2-176p+240)c_1}{6p^3(2p-5)(p-2)f_c^{p-3}}\phi^{2p-4} + \cdots \end{aligned}$$

3.3 General Minimal Models

We see that in general, given r constraints on the parameters occurring in the expansion in $Q(\xi,0)/\sqrt{f_c}$, we need a potential of order ϕ^{2r} to realize a Q that satisfies these constraints. If p is even, we can repeat the above procedure to put the parameters $A_{p+1}, A_{p+3}, \ldots, A_{p+2l+1}$ equal to zero, which gives a P/β of order $p + 2l + 1$. The total number of constraints is $(p-1) + l$, so *if P is even we can realize the $(p, p+2l+1)$ model with a potential of order $2(p+l-1)$.*

If p is odd the situation is more complicated: we may have to add higher powers of $Q(\xi,0)$ to $Q(-\xi,0)$ to find the true order of P/β, as happened above for the (3,8) model. In general this adding of higher powers prevents the occurrence of models of type (p, np). To analyze this situation in some more detail, suppose that we want to construct $(p, p+2l+1)$ with p odd. Define the following (even) function $f(\xi)$

$$f(\xi) = \left(Q(\xi,0) + Q(-\xi,0) - 2A_0\sqrt{f_c}\right) + \sum_{r=1}^{[\frac{l}{p}+\frac{1}{2}]} a_{2r}(Q(\xi,0) - Q(-\xi,0))^{2r} \tag{3.9}$$

where the a_{2r} are certain parameters, and $[x]$ denotes the largest integer smaller than or equal to x. Suppose that $f(\xi)$ starts at order ξ^{2p+l+1}. Formally we can solve $f(\xi) = 0$ to find a power series for $Q(-\xi, 0)$ in terms of $Q(\xi, 0)$, say $Q(-\xi, 0) = \sum b_i(Q(\xi, 0) - A_0\sqrt{f_c})^i$. The error we make in doing so is also precisely of order ξ^{2p+l+1}. Therefore if we take

$$P/\beta = -c_1 \left(Q(-\xi, 0) - \sum b_i(Q(\xi, 0) - A_0\sqrt{f_c})^i \right) \tag{3.10}$$

P/β will have the required order. Requiring $f(\xi)$ to start at order ξ^{p+2l+1} puts l constraints on the coefficients in $Q(\xi, 0)$ and the a_{2r} together. Effectively, the number of constraints on the coefficients of Q is therefore $l - [l/p + 1/2]$. As there are also $p - 1$ constraints to make $Q(\xi, 0)$ of order p, we find that *for p odd, we can realize the $(p, p + 2l + 1)$ model with a potential of order $2(p + l - 1 - [l/p + 1/2])$.* For some lowest order potentials, the models corresponding to the points of highest multicriticality are summarized in the following table.

order V	models					
4	(2,5)	(3,4)				
6	(2,7)	(3,8)	(4,5)			
8	(2,9)	(3,10)	(4,7)	(5,6)		
10	(2,11)	(3,14)	(4,9)	(5,8)	(6,7)	
12	(2,13)	(3,16)	(4,11)	(5,12)	(6,9)	(7,8)

The Ising model [8] corresponds to (3,4), the points (3,8) and (4,5) were obtained in [10], and (5,6) in [11]. The points $(2, 2m+1)$ have been included for completeness, but in fact correspond to the case where $c_1 = 0$, so that the two matrix model essentially reduces to two copies of the one matrix model, for which these points are well known. If we do not impose the maximum number of constraints on Q, we will in general find planes of lower multicriticality, the multicritical points lying in the intersection of such planes. The (p, q)-models with $p + q$ =even can presumably also be obtained from the two-matrix model, with different potentials V_1 and V_2. For instance, in [11], the (3,5) minimal model was found in an antisymmetric even two-matrix model with 8^{th} order potential.

To illustrate some of the things said above, let us determine the constraints we have to impose for the (3,14) model. Let $Q(\xi, 0) = A_0\sqrt{f_c} + \frac{1}{2}(t_3\xi^3 + t_4\xi^4 + \cdots)$. Requiring $f(\xi)$ in (3.9) to be of order ξ^{14} gives 5 constraints

$$\begin{aligned} 0 &= t_4 \\ 0 &= t_6 + a_2 t_3^2 \\ 0 &= t_8 + 2a_3 t_3 t_5 \\ 0 &= t_{10} + 2a_2 t_3 t_7 + a_2 t_5^2 \end{aligned}$$

$$0 = t_{12} + 2a_2 t_3 t_9 + 2a_2 t_5 t_7 + a_4 t_3^4$$

From these constraints it is easy to eliminate a_2 and a_4 and we end up with the following three conditions of the t_i

$$\begin{aligned} 0 &= t_4 \\ 0 &= t_8 t_3 - 2t_5 t_6 \\ 0 &= t_{10} t_3 - 2t_6 t_7 - t_5 t_8 \end{aligned}$$

Putting $B(z^2) = b_0 + b_1 z^2 + b_2 z^4$ into (3.3), we can translate these three conditions into three conditions on the b_i. Solving these then enables you to write down a 10th order potential. The scaling limit of the corresponding matrix model should now be the (3,14) model.

4 Flows in the Multimatrix Model

4.1 Flows in Genus Zero

The flows in the multimatrix model are generated by deformations of the potentials. General arguments show that the resulting flows on the matrices $P(i)$ and $Q(i)$ will be given by commutators (for a clear explanation, see [25]). In the planar approximation the flows will then be given by certain Poisson brackets. To study these flows, consider again the equations

$$\beta^{-1} P(1) - V_1'(Q(1)) + c_1 Q(2) = 0 \tag{4.1}$$

$$c_{r-1} Q(r-1) - V_r'(Q(r)) + c_r Q(r+1) = 0 \quad 2 \le r \le p-1 \tag{4.2}$$

$$\beta^{-1} P(p) - V_p'(Q(p)) + c_{p-1} Q(p-1) = 0 \tag{4.3}$$

and assume that apart from z and u, the $P(i)$ and $Q(i)$ depend on an extra parameter λ. The potentials also depend on this deformation parameter λ. From the string equation (2.23) and the related equations (2.9) we know that we there exists a function $\psi(u)$ such that

$$\psi(u) = \beta^{-1}\{P^{(1)}, Q^{(1)}\}_{z,u} = c_r\{Q^{(r)}, Q^{(r+1)}\}_{z,u} = \beta^{-1}\{Q^{(p)}, P^{(p)}\}_{z,u} \tag{4.4}$$

The flows on the matrices P and Q are given by commutators. Translating this into Poisson brackets shows we have to write down the following set of equations describing the flows in the planar approximation

$$\left\{ \begin{aligned} \partial P^{(1)}/\partial\lambda &= \psi(u)^{-1}\{X_1, P^{(1)}\}_{z,u} \\ \partial Q^{(1)}/\partial\lambda &= \psi(u)^{-1}\{X_1, Q^{(1)}\}_{z,u} \end{aligned} \right. \tag{4.5}$$

$$\begin{cases} \partial Q^{(r)}/\partial\lambda &= \psi(u)^{-1}\{X_{r+1}, Q^{(r)}\}_{z,u} \\ \partial Q^{(r+1)}/\partial\lambda &= \psi(u)^{-1}\{X_{r+1}, Q^{(r+1)}\}_{z,u} \end{cases} \tag{4.6}$$

$$\begin{cases} \partial Q^{(p)}/\partial\lambda &= \psi(u)^{-1}\{X_{p+1}, Q^{(p)}\}_{z,u} \\ \partial P^{(p)}/\partial\lambda &= \psi(u)^{-1}\{X_{p+1}, P^{(p)}\}_{z,u} \end{cases} \tag{4.7}$$

It is straightforward to check that $\partial\{P^{(1)}, Q^{(1)}\}_{z,u}/\partial\lambda = 0$, and similar for the other cases, so that the string equations (4.4) are preserved under these flows. The reader will have noticed that we have doubly defined $\partial Q^{(r)}/\partial\lambda$, which is of course not allowed, unless both definitions give the same result. In this case this implies that $X_{r+1} - X_r$ must have vanishing Poisson bracket with $Q^{(r)}$, and in general it will be a function of $Q^{(r)}$. To determine the X_r, we differentiate (4.1) with respect to λ and z, and eliminate $V_1''(Q^{(1)})$ from these two equations. The result can be written

$$\beta^{-1}\{P^{(1)}, Q^{(1)}\}_{z,\lambda} - c_1\{Q^{(1)}, Q^{(2)}\}_{z,\lambda} + \frac{\partial V_1'}{\partial\lambda}(Q^{(1)}) z\partial_z Q^{(1)} = 0 \tag{4.8}$$

Inserting (4.5) into this gives $-z\partial_z X_1 + z\partial_z X_2 = (\partial V_1'/\partial\lambda) z\partial_z Q^{(1)}$ which can be easily integrated to $X_2 - X_1 = \partial V_1/\partial\lambda$. The integration constant is irrelevant, because the X_r are only defined up to a constant. This analysis can be repeated for the other equations (4.2) and (4.3) and the final result is

$$X_{r+1} - X_r = \frac{\partial V_r}{\partial\lambda}(Q^{(r)}) \tag{4.9}$$

which indeed has vanishing Poisson bracket with $Q^{(r)}$ as required. Adding (4.9) for all r we find $X_{p+1} - X_1 = \sum_r \partial V_r/\partial\lambda$. This equation itself does not fix the X_i completely. For that we must also demand that the structure of $Q^{(1)} = \sqrt{f}/z + \ldots$ and of $Q^{(p)} = \ldots + z\sqrt{f}$ are preserved under the flows. This completely fixes the X_r. Defining for $g(z) = \sum a_i z^i$ the plus and minus piece by $g_+(z) = a_0/2 + \sum_{i>0} a_i z^i$ and $g_-(z) = g(z) - g_+(z)$, we find that X_{p+1} must have vanishing plus part to preserve the structure of $Q^{(p)}$. Similarly, X_1 must have vanishing minus part. We can now write down the unique solution of (4.9) that is compatible with these requirements. It reads

$$X_r = \sum_{i<r}\left(\frac{\partial V_i}{\partial\lambda}(Q^{(i)})\right)_- - \sum_{i\geq r}\left(\frac{\partial V_i}{\partial\lambda}(Q^{(i)})\right)_+ \tag{4.10}$$

Using identities like $\{A_+, B_+\}_- = \{A_-, B_-\}_+ = 0$ it is possible to prove that these flows commute. It is, however, not easy to see what these flows correspond to after taking the double scaling limit. We do not know what the analog of taking the $+$ or $-$ part is on the level of differential operators. A natural approach to these flows might be to introduce singular potentials as was done in [4], see also [26]. For instance, for the $(p, p+1)$ models equation (3.6) suggests to add terms like

$$tr \left(\frac{M_1}{\sqrt{f_c}} - 2^{2p-2} / \left(\begin{array}{c} 2p-2 \\ p-1 \end{array} \right) \right)^{s/p} \tag{4.11}$$

to the matrix model potential as possible candidates of KdV-flows, but we have not worked this out in detail.

For the full flows in terms of matrices, almost the same expressions should be valid, where - now refers to upper triangular and + refers to lower triangular matrices. For the two-matrix model, this has been shown in [9, 14], and for arbitrary multimatrix models in [15].

4.2 Flows in the Ising Model

As an example, we consider the Ising model in some more detail. The Ising model has been considered previously in great detail in [8, 21, 27]. from (3.5) we read of that the critical potential is $V(\phi) = -c_1\phi^4/12f_c + 2c_1\phi^2$. By requiring $V'(Q(z,u)) - c_1 Q(1/z, u)$ to be an analytic function of z and using $u = a^{2\gamma}(f - f_c)$ we can determine the precise form of $Q(z,u)$

$$Q(z,u) = \frac{\sqrt{f_c + a^{-2\gamma}u}}{z} + 4\frac{\sqrt{f_c + a^{-2\gamma}u}}{2f_c + a^{-2\gamma}u} z - \frac{(f_c + a^{-2\gamma}u)^{3/2}}{3f_c} z^3 \tag{4.12}$$

Upon taking the continuum limit, we find that $Q(\xi, u)$ equals $\xi^3 + 3\xi u/2$. We have computed $(Q^n(z,u))_-$ for n smaller than 20, taking $f_c = 1$ for convenience. In these expressions we then replaced z by $\exp(a^{-\gamma}\xi)$, and expanded them in powers of a. Taking appropriate linear combinations of powers of Q one can construct the (spherical) differential operators that correspond to the flows (4.10). It turned out that all fractional powers $(\xi^3 + 3\xi u/2)_+^{k/3}$ with‡ $k \leq 9$ can be made in this way, either by taking only even, or either by taking only odd powers of Q. This confirms once more the doubling of the degrees of freedom in the matrix model. Furthermore, this strongly suggests that in the multimatrix model the flows are given by commutators with the fractional powers of Q, in agreement with the usual KdV picture.

‡Here the + means that we only keep nonnegative powers of ξ

5 $w_{1+\infty}$ Constraints in Multimatrix Models

What has been very important in our discussion so far, is that instead of the pair of operators (P, Q), we might as well have taken $(P + f(Q), Q)$, where f is some polynomial, to represent the string equation. We can even take $(P + f(Q), Q + g(P + f(Q)))$, which for suitable f and g is just equal to $(Q(2), Q(3))$. The set of these allowed transformations of the string equation is generated by $(P, Q) \to (P + \epsilon Q^n, Q)$ and $(P, Q) \to (P, Q + \epsilon P^m)$. We claim that invariance under these two transformations just corresponds to the $w_{1+\infty}$ constraints. To prove this, we will first formulate the KP-hierarchy in a somewhat different way, in which the so-called non-isospectral symmetries [28] of this hierarchy play an important role.

5.1 Reformulation of the KP-Hierarchy

Consider the vector space $V_L \equiv V(L, \partial/\partial L)$ of pseudo differential operators, spanned by

$$\left(\frac{\partial}{\partial L}\right)^a L^b \qquad a \geq 0, b \in \mathbf{Z} \tag{5.1}$$

where $L = \partial + \dots$ is a first order and "$\partial/\partial L = -x + \dots$" a zeroth order pseudo differential operator, satisfying $[\partial/\partial L, L] = 1$. We demand that L has no ∂^0 term, and that $\partial/\partial L$ has no ∂^{-1} term. One can simultaneously diagonalize L and $\partial/\partial L$ as $L = K\partial K^{-1}$ and $\partial/\partial L = -KxK^{-1}$. Given a pseudo differential operator A, A_+ will denote the differential operator part of A, and $A_- = A - A_+$, as usual. Note that if $A \in V_L$, it is not necessarily true that $A_+ \in V_L$ or $A_- \in V_L$.

In appearance, V_L looks very much like the Lie algebra

$$w_{1+\infty} = \{\sum t_{ab} \left(\frac{\partial}{\partial z}\right)^a z^b | a \geq 0, b \in \mathbf{Z}\} \tag{5.2}$$

and if we define a Lie bracket on V_L by taking commutators of pseudo differential operators, $w_{1+\infty}$ is isomorphic to V_L via

$$g = \sum t_{ab} \left(\frac{\partial}{\partial z}\right)^a z^b \in w_{1+\infty} \to X(g) = \sum t_{ab} \left(\frac{\partial}{\partial L}\right)^a L^b \in V_L \tag{5.3}$$

If we want to stress the L-dependence of X, we will write X_L instead of X.

Given $g \in w_{1+\infty}$, we can define a flow by defining for arbitrary $W \in V_L$[§]

$$\frac{\partial W}{\partial t_g} = [X(g)_-, W] \tag{5.4}$$

It is easy to see that this flow preserves the structure of V_L, because all the flows can be summarized by the one equation $\partial K/\partial t_g = X(g)_- K$. Let us now take $g, h \in w_{1+\infty}$ and compute the commutator of the two flows defined by g and h

$$\begin{aligned}
\left(\frac{\partial}{\partial t_g}\frac{\partial}{\partial t_h} - \frac{\partial}{\partial t_h}\frac{\partial}{\partial t_g}\right) W &= \frac{\partial}{\partial t_g}[X(h)_-, W] - \frac{\partial}{\partial t_h}[X(g)_-, W] \\
&= [[X(g)_-, X(h)]_-, W] + [X(h)_-, [X(g)_-, W]] \\
&- [[X(h)_-, X(g)]_-, W] - [X(g)_-, [X(h)_-, W]] \\
&= [[X(g)_-, X(h)]_- + [X(h)_-, X(g)_-] - [X(h)_-, X(g)]_-, W] \\
&= [[X(g), X(h)]_-, W] \\
&= [X([g,h])_-, W]
\end{aligned} \tag{5.5}$$

We see that the flows do not commute. Rather, the flows are compatible with the Lie algebra structure on $w_{1+\infty}$. This means that, at least locally, the flows can be exponentiated to an action of the *group* $\exp(w_{1+\infty})$ on the infinite dimensional manifold consisting of all vector spaces $V(L, \partial/\partial L)$. The usual KP-flows are the flows generated by the abelian subalgebra $H \subset w_{1+\infty} = \{\sum_{a>0} t_a z^a\}$.

If we consider p^{th}-reduced KP-hierarchies we impose the extra constraint $L^p_- = 0$ on L. To incorporate this into the above picture, we must study the behavior of differential operators R, satisfying $R_- = 0$, under the $w_{1+\infty}$ flows. So consider $R(L) \in V_L$ satisfying $R(L)_- = 0$. Under an infinitesimal flow $L' = \exp(\epsilon g)L$, $g \in w_{1+\infty}$, we will in general no longer have $R(L')_- = 0$. However, the remarkable thing is that we can define $R'(L') \in V_{L'}$ which still has the property $R'(L')_- = 0$. Define

$$R'(L') = R(L') - \epsilon[X_{L'}(g), R(L')] \tag{5.6}$$

then to first order in ϵ

$$\begin{aligned}
R'(L') &= R(L) + \epsilon[X_L(g)_-, R(L)] - \epsilon[X_L(g), R(L)] \\
&= R(L) + \epsilon[R(L), X_L(g)_+]
\end{aligned} \tag{5.7}$$

From (5.6) we read off that $R'(L') \in V_{L'}$, and from (5.7) that $R'(L')_- = 0$. This suggests to define the following flow on differential operators R with $R_- = 0$

$$\frac{\partial R}{\partial t_g} = [R, X_L(g)_+] \tag{5.8}$$

[§]To justify the word flow, one should think of V_L as being embedded in a much larger space of pseudo differential operators, in which V_L moves around

Note that this flow is *different* from the flow defined on V_L. Similarly as in (5.5), we find in this case that

$$\left(\frac{\partial}{\partial t_g}\frac{\partial}{\partial t_h}-\frac{\partial}{\partial t_h}\frac{\partial}{\partial t_g}\right)R=[R,X([g,h])_+] \tag{5.9}$$

and we can therefore also locally exponentiate these flows. In summary, if $R \in V_L$ and $R_- = 0$ we can consistently define $L' = e^{\epsilon g}L$ and $R' = e^{\epsilon g}R$. Although both actions are different, we still have $R'_- = 0$ and $R' \in V_{L'}$.

If we have a reduced KP-hierarchy corresponding to $L^p_- = 0$ and restrict our attention to the ordinary KP-flows only, then the 'correction' (5.6) vanishes, and this discussion does not tell us anything new. Only as soon as we start to flow outside the usual p^{th}-reduced KP-hierarchies, we get something different.

5.2 $w_{1+\infty}$-Constraints

What is the relevance of all of this to multimatrix models? These models provide us with two differential operators P and Q [7], satisfying $P_- = 0$, $Q_- = 0$ and $[P,Q] = 1$. We claim that we can always write P and Q as elements of some V_L, so that we can directly apply the above framework to this case. The flows (5.8) will preserve $P_- = 0$, $Q_- = 0$ and $[P,Q] = 1$, and the above strongly suggests that locally the space of couplings of the theory is a submanifold of the group manifold of $w_{1+\infty}$.

The $w_{1+\infty}$ constraints now take a simple form. Because $(Q^aP^b)_- = 0$, if we define $F \subset w_{1+\infty} = \{X_L^{-1}(\sum t_{ab}Q^aP^b)\}$ we see that

$$g \in F \Rightarrow \frac{\partial}{\partial t_g}L = 0 \tag{5.10}$$

As we will show in a moment, this statement is equivalent to the $w_{1+\infty}$-constraints found previously [18, 19]. It is clear from (5.5) that these constraints, when computing their commutators, form an algebra isomorphic to the positive half of the $w_{1+\infty}$-algebra: $F \simeq w^+_{1+\infty}$. Looking at the definition (5.8), we see that P and Q change under the $w_{1+\infty}$ constraints: if $X = Q^aP^b$, then under a small flow

$$\begin{aligned} Q &\rightarrow Q - \epsilon b Q^a P^{b-1} && (5.11)\\ P &\rightarrow P + \epsilon a Q^{a-1}P^b && (5.12)\end{aligned}$$

in which one recognizes the canonical transformations generated by X. Indeed, these transformations include those that were so prominently present in the multimatrix model. We can therefore conclude that regardless with which of the pairs (2.9) of matrices in the original matrix model one starts with, they all essentially give the same P and Q, and in particular the same multicriticality, everything being connected together via the $w_{1+\infty}$ constraints.

Let us compare the the constraints as written here with the $w_{1+\infty}$-constraints in [18]. Consider a critical point where P has order p and Q has order q. We must make a choice how we represent P and Q in V_L. The specific choice which corresponds to the usual formulation of the string equation is

$$Q = L^q \tag{5.13}$$

$$P = \text{"}\frac{\partial}{\partial L^q}\text{"} + L^p \equiv \frac{1}{2q}\left(L^{1-q}\frac{\partial}{\partial L} + \frac{\partial}{\partial L}L^{1-q}\right) + L^p \tag{5.14}$$

Given these representations of P and Q, L and $\partial/\partial L$ are completely and uniquely defined by the two equations $P_- = 0$ and $Q_- = 0$. As we expect from the matrix model, we can assign degree q to Q and degree p to P, so that L has degree 1 and $\partial/\partial L$ has degree $p+q-1$. As soon as we leave this critical point, it is no longer true that we can assign a well-defined degree to P and Q. To turn on the couplings we look at the flow generated by the group element

$$g = \exp(\sum_{i\geq 1} t_i L^i) \tag{5.15}$$

Performing the flow using (5.6) yields

$$Q = L^q \tag{5.16}$$

$$P = \frac{1}{2q}\left(L^{1-q}\frac{\partial}{\partial L} + \frac{\partial}{\partial L}L^{1-q}\right) + L^p + \sum_{i\geq 1}\frac{i}{q}t_i L^{i-q} \tag{5.17}$$

Given this representation of P and Q in terms of L and $\partial/\partial L$, the $w_{1+\infty}$ constraints of [18] are now easily seen to be equivalent to $(Q^a P^b)_- = 0$, showing the equivalence of the two formulations.

The formulation given here is closely related to the formulation in terms of an infinite dimensional Grassmann manifold as given in [19]. $P_- = 0$ and $Q_- = 0$ translate into two conditions on points of the Grassmann manifold, that can be expressed as the invariance of a certain vector space under two differential operators. The two differential operators together generate $w^+_{1+\infty}$, in the same way as $P_- = 0$ and $Q_- = 0$ generate $(Q^a P^b)_- = 0$.

The representation of P and Q as two elements of some V_L is not unique. If we take an arbitrary $g \in w^{-}_{1+\infty}$ where

$$w^{-}_{1+\infty} = \{\sum t_{ab} \left(\frac{\partial}{\partial z}\right)^a z^b | a \geq 0, b < 0\} \tag{5.18}$$

then $\partial P/\partial t_g = \partial Q/\partial t_g = 0$ which is clear from (5.8). However, $\partial L/\partial t_g \neq 0$, but $\partial L/\partial t_g = [X(g)_-, L] = [X(g), L]$, so that $L' = e^{\epsilon g} L$ will still be in V_L. These transformations correspond to basis transformations: $V_L = V_{L'}$.

5.3 Multimatrix Models as Models of Quantum Gravity

Let us now compare the contents of multimatrix models with that of minimal matter coupled to quantum gravity. Multimatrix models give us two operators P and Q, but they do not give us V_L. Therefore, it should not come as a surprise that it is not really important how we represent Q and P in V_L. In the standard representation (5.13) and (5.14), the physical quantities are extracted by relating the second derivative of the free energy with respect to the cosmological constant μ with L[¶] [23]

$$\frac{\partial^2 F}{\partial \mu^2} = -2res(L) \tag{5.19}$$

Under a flow with $g \in w^{-}_{1+\infty}$, $\partial res(L)/\partial t_g = -(res(X(g)))'$, which is always a polynomial in x, and therefore $res(L)$ changes at most by a term analytic in the cosmological constant. Usually, such terms are called 'nonuniversal' and neglected. Here we see they are nonuniversal in the sense that they depend on the specific basis choice for V_L.

Given P and Q in some V_L, an important issue is to find the right set of commuting flows the matrix model gives us. Such commuting flows will in general be generated by powers of a first-order differential operator O, that has a well defined degree. A priori, there is no obvious reason why we should take $O = Q^{1/q}$, and not for instance $O = P^{1/p}$ or $O = PQ^{-1}$. However the analysis of the Ising model in the previous section showed that in that case $O = Q^{1/3}$ up to O^9, and we suspect that this is the case for all multicritical points (p, q) of the multimatrix model with $p > q$.

However, the ambiguity still exists for quantum gravity coupled to minimal matter. If we take take two different possible operators O_1 and O_2, we can read off from (5.13) and (5.14) that $(O_1)^k_+ = (O_2)^k_+$ for $k < p + q - 1$, but not necessarily for

[¶]res(A) denotes the term in front of ∂^{-1} in A, as usual.

$k \geq p+q-1$. The case $k < p+q-1$ corresponds to the order parameters of the minimal model, and this is precisely the set of flows that has been analyzed in [23], so this ambiguity has not been resolved yet[‖].

An intriguing possibility is that this ambiguity is altogether irrelevant. We already saw that O_1 and O_2 can be related to each other via a basis transformation of V_L. More generally, it might be that flows corresponding to different choice of the operator O can be analytically related to each other. A clue for this is provided by the $w^+_{1+\infty}$-constraints. These correspond to non-commuting flows, but can be expressed completely in terms of the commuting flows generated by $Q^{\alpha/q}$. Recently, it has been shown that analytical redefinitions of the couplings (or, equivalently, contact terms) may be the clue to establish the precise relation between multimatrix models and minimal models coupled to quantum gravity [29]. Such redefinitions are not sufficient to relate the flows of different operators O, because these redefinitions involve higher order differential operators in the couplings, and analytical redefinitions of the couplings are only capable of producing first order operators.

The full flow structure of both multimatrix models and quantum gravity remain obscure. For multimatrix models, picking $O = Q^{1/q}$ is certainly only correct near a multicritical point. These flows never change the order of Q, which is something that must happen at some point in the multimatrix model, as it contains critical points with both different p and q. A possibility is that at the points where the order of Q changes, a singularity occurs, and that it is fundamentally impossible to interpolate smoothly between the different critical points. We already know that there is problem with such flows in the usual KdV picture of quantum gravity [30, 31].

For quantum gravity, there still is the problem that the matrix model gives too many operators. If we take $O = Q^{1/q}$, then the flows generated by O^{kq} are trivial, as they leave L invariant. The flows O^l with $l \neq 0 \bmod q$ survive. On the other hand, if we would have taken $O = P^{1/p}$, O^l with $l \neq 0 \bmod p$ survive. The problem is that in quantum gravity, both $l = 0 \bmod q$ and $l = 0 \bmod p$ do not occur. This has been shown by computing BRST-cohomology [32], see also [27, 33], and by considering the genus 1 partition function [34].

An alternative point of view is that the multimatrix models give too few operators. We can use the $w^+_{1+\infty}$-constraints to rewrite all correlators in terms of the correlators of O^l where l is *not* of the form $ap+bq, a \geq 0, b \geq 0$. There are precisely $(p-1)(q-1)/2$ such values of l. Obviously, these should correspond to the primary fields of the minimal model. Because in quantum gravity there is also one state for each null state in the minimal model, we apparently must identify O^l, $l = ap+bq, a > 0, b > 0$ with the null states, whose correlators can be expressed in

[‖]It is therefore not entirely clear that multimatrix models really describe minimal models coupled to quantum gravity terms of those of the primary fields. Finally, we must use $P^n_- = 0$ and $Q^n_- = 0$ to dispose of the flows O^l with $l = ap$ or $l = bq$. Note that in this latter picture the choice of O is not important, so in a sense it is (p, q)-symmetric.

To conclude, we note that the condition that we are dealing with a reduced hierarchy, i.e. $L^q_- = 0$, is not only preserved by the usual KdV-flows, but also by an extra Virasoro-algebra, even if we are not dealing with quantum gravity [28, 35]. Consider $P, Q \in V_L$ satisfying $[P, Q] = 1$ and assume $Q = L^q$, $Q_- = 0$, but that P is arbitrary, as is the case for an arbitrary q-reduced hierarchy. The flows that preserve $L^q_- = 0$ are determined by observing that

$$\frac{\partial L^q_-}{\partial t_g} = 0 \Rightarrow [X(g)_-, L^q] = 0 \Rightarrow [X(g)_-, Q]_- = 0 \Rightarrow [X(g), Q]_- = 0 \tag{5.20}$$

The only positive operators that are available in general, are $\sum_{i\geq 0} a_i Q^i$. Hence

$$X(g) = \sum_{i\geq 0} a_i P Q^i + \sum b_\alpha Q^{\alpha/q} \tag{5.21}$$

Therefore the 'symmetries' of a q-reduced hierarchy, that are available in this space of $w_{1+\infty}$-flows, are the KdV-flows $Q^{\alpha/q}$, plus a Virasoro algebra $\{PQ^i\}$. The quantum gravity τ-functions distinguish themselves because they are fixed points of this symmetry.

6 Remarks

In summary, we have performed a rather detailed study of multimatrix models in the spherical approximation. One of the things which we encountered was that in order to describe minimal models with $p + q =$even, we need to work with $\sqrt{f - f_c}$ rather than $(f - f_c)$ as analytic expansion parameter. In principal this approach should also be valid for $c = 1$, in which case we have an infinite chain of matrices. To make contact with the results for $c = 1$ [36, 37], one somehow needs to reach a point where the multicriticality condition (2.14) reads

$$W(f) \simeq W_c - e^{\lambda(f-f_c)} \tag{6.1}$$

As analytic expansion parameter one can take $u = \exp(\lambda(f - f_c))$, and with u we can repeat the same story as for the multimatrix model. However, now the second derivative of the free energy is not proportional to $res(L)$, but rather to $\log(res(L))$, giving precisely the logarithmic deviation from scaling behavior. We have checked that if we start with a (p,q) minimal model, and take for the second derivative of the free energy $\log(res(L))$ rather than $res(L)$, we also get correlation functions (on the sphere) that are similar to those predicted from Liouville theory [38]. It would be interesting to see whether this picture can be extended beyond genus zero.

We have not given a rigorous proof that the continuum limit really exists, as this would require a more detailed knowledge of the behavior of the matrix coefficients. It may be that one can take other continuum limits that lead to different structures. As an example, suppose that we define instead of $z^k = \sum_l \delta_{l,l+k}$ the following 'shift' operators for $\alpha = 0 \ldots s-1$

$$Z_\alpha^{(k)} = \sum_{l=\alpha \bmod s} \delta_{l,l+k} \tag{6.2}$$

In the continuum limit, we replace $Z_\alpha^{(k)}$ by $E_{\alpha,\alpha+k}e^{k\partial}$ where E_{ij} is the $s \times s$ matrix with 1 in its (i,j)-position and zeroes everywhere else, and $\alpha+k$ should be computed modulo s. If we expand Q and P in terms of these new $Z_\alpha^{(k)}$, one can obtain matrix differential operators if we tune the coefficients of the matrix model in an appropriate way. The resulting models will probably be intimately related to multi-cut matrix models as considered in [39].

A final remark concerns the $w_{1+\infty}$-constraints. If P and Q where just two coordinates, the $w_{1+\infty}$-constraints would be related to the area-preserving diffeomorphisms of the (P,Q)-plane, leaving $dP \wedge dQ$ invariant, as can be seen from (5.11) and (5.12). It is not easy to see from this what the analog of the $w_{1+\infty}$-constraints for finite matrices is. They should be related somehow to invariance under area preserving diffeomorphisms of the (M_1, M_2)-plane. Because M_2 is almost $\partial/\partial M_1$, a candidate would be to study the Ward identities obtained by inserting $M_1^a(\partial/\partial M_1)^b$ into the matrix integral, in the same way as one obtains the Virasoro constraints for the one-matrix model [40]. Very recently, this has been accomplished [41] by inserting expressions of the form $\sum_{i,n} \epsilon_i(\lambda_{i,n})^a(\partial/\partial\lambda_{i,n})^b$ in the integral (2.2).

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Note Added

After completion of this work we received [42, 43], where it is shown that (p,q)-duality holds as long as one puts the t_i with $i > p+q$ in 5.15 equal to zero. In this case, the order of P in 5.17 will not exceed p and one can perform a basis transformation of V_L, together with a transformation $P \to cf^{-1}Pf$ and $Q \to \frac{1}{c}f^{-1}Qf$ for some constant c and some function f, to put P in the form $P = L^p$. This interchanges the role of P and Q and amounts to a nonlinear transformation of the t_i in 5.15. We also received [44], in which the critical potential 3.5 for the $(p,p+1)$-model is also computed.

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