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Jan de Boer[‡] and Jacob Goeree[§]

Institute for Theoretical Physics, University of Utrecht
Princetonplein 5, P.O. Box 80.006
3508 TA Utrecht, the Netherlands

Abstract

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[‡]e-mail: deboer@ruunts.fys.ruu.nl
[§]e-mail: goeree@ruunts.fys.ruu.nl

1. Introduction

2. W Algebras

2.1. Soldering

We start with the $(1,0)$ part of a connection on a (necessarily trivial) Lie algebra bundle with Lie algebra $\mathbf{g}$ on the complex plane. This $(1,0)$ part can be written as $D = \partial + \mathrm{ad} A_z$. Locally, A_z is a Lie algebra valued one form. Thus, under co-ordinate transformations, A_z transforms as $\delta_\epsilon A_z = A_z \partial\epsilon + \partial A_z \epsilon$. These transformations can also be seen as certain field dependent gauge transformations, namely those with parameter ϵA_z, $\delta_\epsilon A_z = (\partial + \mathrm{ad} A_z)(A_z\epsilon)$. This fact could have been anticipated from the Sugawara form of the stress-energy tensor, $T = \frac{1}{2}\mathrm{Tr}(A_z A_z)$, and is in fact true in an arbitrary number of dimensions. Now we want to ask ourselves the following question: if we impose certain constraints on the connection A_z, is it still possible to view co-ordinate transformations as field dependent gauge transformations? If we simply put some components equal to zero, the answer is yes, because we can use the same transformations as before. The situation is more interesting if we put some components of A_z equal to nonzero constants or functions of z. In that case the requirement that these nonzero components transform properly under co-ordinate transformations forces us to add extra terms to $A_z\epsilon$. This procedure is sometimes called soldering [5], as the new co-ordinate transformations are obtained by combining the old co-ordinate transformations with gauge transformations. The new co-ordinate transformations are in general such that it is no longer possible to view A_z as a Lie-algebra valued one-form. The different components of A_z transform with different spins, and the Lie algebra bundle is twisted correspondingly. Although this does not affect the global topology of the bundle on the complex plane, it does so when considering surfaces with non-trivial topology.

To study the different constraints that can be imposed, consider a constrained connection $D = \partial + C + W$, where C are the constraints that are imposed on certain components of A_z, and W contains the unconstrained components of A_z. Decompose W in components, $W = W_i T^i$. To simplify the analysis, we first perform a gauge transformation with $g = \exp\tilde{C}$, where $\tilde{C}$ is such that $\partial \exp\tilde{C} = (\exp\tilde{C})C$. This maps $\partial + C + W$ into $\partial + \tilde{W} = \partial + W_i S^i$ with $S^i = g T^i g^{-1}$. If C is a constant, one can take $\tilde{C} = zC$, but if C is z-dependent, $\tilde{C}$ can be more complicated. Suppose that infinitesimal co-ordinate transformations correspond to gauge transformations with parameter $X(\epsilon)$, where ϵ is the parameter of the infinitesimal co-ordinate transformation. The requirement that the W_i transform as fields with a well-defined spin h_i gives

$$\partial X(\epsilon) + \sum_i W_i [S^i, X(\epsilon)] = \sum_i (\frac{c}{12}\delta_{i,0}\partial^3\epsilon + h_i W_i \partial\epsilon + \partial W_i \epsilon) S^i, \tag{2.1}$$

where we left open the possibility that there is one distinguished spin 2 field (the energy momentum tensor) $T = W_0$ whose transformation rule contains a central term $c\partial^3\epsilon/12$. We can write $X(\epsilon) = X^{(0)}(\epsilon) + X^{(1)}(\epsilon) + X^{(2)}(\epsilon) + \ldots$, where $X^{(t)}(\epsilon)$ is of order t in the W_i. The restriction of (2.1) to terms independent of W_i gives

$$\partial X(\epsilon) = \frac{c}{12}\partial^3\epsilon S^0. \tag{2.2}$$

Taking the $\int dx$ of this equation we immediately deduce $\partial^3 S^0 = 0$, and

$$X^{(0)}(\epsilon) = \frac{c}{12}(\partial^2\epsilon S^0 - \partial\epsilon\partial S^0 + \epsilon\partial^2 S^0). \tag{2.3}$$

The part of (2.1) that is of first order in W_i reads

$$\partial X^{(1)}(\epsilon) + \sum_i W_i[S^i, X^{(0)}(\epsilon)] = \sum_i (h_i W_i \partial\epsilon + \partial W_i \epsilon) S^i, \tag{2.4}$$

and taking the $\int dx$ of this equation we find

$$\int dx \sum_i W_i([S^i, X^{(0)}(\epsilon)] - h_i\partial\epsilon S^i + \partial(\epsilon S^i)) = 0, \tag{2.5}$$

and thus $X^{(1)} = \epsilon\sum_i W_i S^i = \epsilon\tilde{W}$, which was precisely the result for an unconstrained connection. Furthermore, the restriction of (2.1) to terms of order > 1 yields $X^{(t)} = 0$ for $t > 1$. Altogether this shows that in addition to the standard co-ordinate transformations, there is an extra gauge transformation with parameter $X^{(0)}(\epsilon)$ given by (2.3). Furthermore, S^0 must satisfy the following equations, as follows from (2.2) and (2.4)

$$\begin{aligned}
\partial^3 S^0 &= 0, \\
[S^i, \frac{c}{12}S^0] &= 0, \\
[S^i, -\frac{c}{12}\partial S^0] &= (h_i - 1)S^i, \\
[S^i, \frac{c}{12}\partial^2 S^0] &= -\partial S^i.
\end{aligned} \tag{2.6}$$

If $c = 0$, these equations tell us that $h_i = 1$ and $S^i = \text{const}$, and as in this case $X^{(0)} = 0$, we just reproduce the standard behavior of A_z under co-ordinate transformations, and we have not soldered anything. Thus, from now on we assume $c \neq 0$. Because $\partial^3 S^0 = 0$, we can write S^0 as

$$S^0 = \frac{12}{c}\left(-\Lambda^- - z\Lambda^0 + \frac{1}{2}z^2\Lambda^+\right), \tag{2.7}$$

where $\Lambda^{-,0,+}$ are certain elements of $\mathbf{g}$. Taking $i = 0$ in (2.6) gives the following commutation relations for Λ

$$\begin{aligned}
[\Lambda^0, \Lambda^+] &= \Lambda^+, \\
[\Lambda^0, \Lambda^-] &= -\Lambda^-, \\
[\Lambda^+, \Lambda^-] &= \Lambda^0,
\end{aligned} \tag{2.8}$$

which is precisely an sl_2 algebra. To simplify the remaining equations, we perform a gauge transformation with $G = \exp(+z\Lambda^+)$, *i.e.* we introduce $R^i = G^{-1}S^iG$. The conditions (2.6) can with the help of the relations $R^0 = G^{-1}S^0G = -\frac{12}{c}\Lambda^-$, $G^{-1}\partial S^0 G = -\frac{12}{c}\Lambda^0$ and $G^{-1}\partial^2 S^0 G = \frac{12}{c}\Lambda^+$ be rewritten as

$$\begin{aligned} [R^i, \Lambda^-] &= 0, \\ [R^i, \Lambda^0] &= (h_i - 1)R^i, \\ \partial R^i &= 0. \end{aligned} \tag{2.9}$$

What this implies for the connection $A_z = W_i S^i$ is seen most clearly if we also perform a gauge transformation $A_z \to G^{-1}A_zG + G^{-1}\partial G = W_iR^i - \Lambda^+$. Here, the R^i are certain elements of $\mathbf{g}$. This shows that we need to satisfy the following requirements for a non-trivial soldering: we need an sl_2 embedding in $\mathbf{g}$, and if we denote the three generators of sl_2 by $\{\Lambda^-, \Lambda^0, \Lambda^+\}$, the unconstrained components of A_z must be lowest weights of sl_2 under the adjoint action, and their spins are given by $h_i = 1 - \text{weight}$. The constrained components of A_z must be equal to $-\Lambda^+$. One of the unconstrained components, namely $-12T\Lambda^-/c$, behaves as a stress-energy tensor under co-ordinate transformations, with central charge c.

In the case where the unconstrained components of A_z contain all sl_2 lowest weights, *i.e.* all of $\ker \mathrm{ad}(\Lambda^-)$, there are many more gauge transformations that preserve the form of A_z. These together constitute the W algebra corresponding to the sl_2 embedding. In the next section we explicitly describe this W algebra. W algebras from general sl_2 embeddings were first studied in [1], and later more extensively; see e.g. [2] and references therein.

2.2. The structure of the W algebra

Associated to an embedding of sl_2 in $\mathbf{g}$ is a decomposition $\mathbf{g} = \mathbf{g}_- \oplus \mathbf{g}_0 \oplus \mathbf{g}_+$. If $\mathrm{ad}\Lambda^0$ has only integral eigenvalues, then the decomposition is with respect to the sign of the eigenvalue. If $\mathrm{ad}\Lambda^0$ has no nonintegral eigenvalues, then it is with respect to the sign of the eigenvalue of $\mathrm{ad}\delta$. If $\mathrm{ad}\Lambda^0$ has halfintegral eigenvalues, then the situation is somewhat more complicated. Let $\mathbf{g}_{\frac{1}{2}}$ be the subspace of $\mathbf{g}$ of Λ^0-eigenvalue $+\frac{1}{2}$. On $\mathbf{g}_{\frac{1}{2}}$ there is a non-degenerate skew-form $\omega(X,Y) = \mathrm{Tr}(\Lambda^-[X,Y])$. Thus we can decompose $\mathbf{g}_{\frac{1}{2}} = \mathcal{I}_+ \oplus \mathcal{I}_-$ into two maximally isotropic subspaces. Now there is a gradation of $\mathbf{g}$ such that $\mathcal{I}_\pm$ has degree $\pm\frac{1}{2}$, and $\Lambda^{-,0,+}$ has degree 0. The sum of this gradation and the gradation given by Λ^0 defines a new gradation of $\mathbf{g}$ which is integral, and the decomposition $\mathbf{g} = \mathbf{g}_- \oplus \mathbf{g}_0 \oplus \mathbf{g}_+$ is with respect to this new gradation, which we denote by δ.

The W algebra is the result of imposing certain constraints on A_z. If we decompose $A_z = A_z^i t_i$ and $\Lambda^+ = l^i t_i$ in terms of a basis t_i of $\mathbf{g}$, the constraints are

$$A_z^i = l^i, t_i \in \mathbf{g}_+. \tag{2.10}$$

Here we assume that the basis $\{t_i\}$ is such that every basis element has a well-defined degree with respect to δ and Λ^0. The constraints (2.10) are first-class. If we would have taken $\mathbf{g}_+$ to be the positive degree part of $\mathbf{g}$ with respect to the Λ^0 gradation, (2.10) will not necessarily be first-class, and that is the reason why we introduced a new gradation. First-class constraints generate gauge invariance, and in this case these are the gauge transformations with parameter in $\mathbf{g}_-$ that preserve (2.10). These gauge transformations can be used to put A_z in the form

$$A_z = \Lambda^+ + W, W \in \ker \mathrm{ad}(\Lambda^-). \tag{2.11}$$

At this stage there are two equivalent ways to compute the structure of the W algebra. One can compute the Poisson brackets of the gauge invariant polynomials on the reduced phase space defined by (2.10) (hamiltonian reduction), or one can impose (2.11) directly as a set of second class constraints and use the Dirac bracket to compute the structure of the W-algebra. The Poisson brackets before imposing the constraints reads

$$\{A_z^a(z), A_z^b(w)\} = \mathrm{Tr}(t^a(\partial_w + \mathrm{ad}(A_z(w))t^b\delta(z-w)\}, \tag{2.12}$$

which is nothing but an affine Lie algebra with level $k=1$. Indices are raised and lowered using the metric $g_{ab} = \mathrm{Tr}(t_a t_b)$ and its inverse. The Poisson brackets for the W algebra can be written in a remarkably simple form. For this purpose introduce a linear operator $L : \mathbf{g} \to \mathbf{g}$, which is the inverse of $\mathrm{ad}(\Lambda^+)$ viewed as a linear operator $\mathrm{imad}(\Lambda^-) \to \mathrm{imad}(\Lambda^+)$, and L is extended by 0 to the rest of $\mathbf{g}$. The Poisson brackets of the W algebra read [3, 4]

$$\{W^a(z), W^b(w)\} = \mathrm{Tr}(t^a(\partial_w + \mathrm{ad}(W(w))\frac{1}{1+L(\partial_w + \mathrm{ad}(W(w)))}t^b\delta(z-w)), \tag{2.13}$$

where $W = W^a t_a$ is the part of A_z in (2.11) that survives the reduction. This W algebra always has a fixed central charge, but by some approriate rescalings W algebras with arbitrary central charges can be constructed.

To quantize the W algebra, it is most convenient to work with the formulation with first-class constraints and to use standard BRST quantization. Because the constraints are first-class, there is no need to introduce any auxiliary fields. In this way it can be proven that all W algebras associated to sl_2 embeddings can be quantized.

There is also a generalization of the Miura map for these generalized W algebras. For the standard W algebras, related to principal sl_2 embeddings, the Miura map expresses the W fields in terms of free fields. In the general case, the Miura map expresses the W fields in terms of the affine Lie algebra based on $\mathbf{g}_0$. If the latter is abelian, this is just an algebra of free scalar fields, but in general it is the direct sum of affine Lie algebras. The Miura map can be constructed by finding for every W^a the associated gauge invariant polynomial on the reduced phase space subject to (2.10), and then by restricting this gauge invariant polynomial to $\mathbf{g}_0$. It can be shown that this gives a homomorphism of Poisson algebras. Free field realizations for arbitrary W algebras can be found by first applying the Miura transformation, and subsequently by replacing the $\mathbf{g}_0$ currents by expressions in terms of free fields. For quantum W algebras the same procedure works, one only has to associate to every W^a an element in the cohomology of the BRST operator rather than a gauge invariant polynomial.

To illustrate some of these things, we will now discuss the example of the Polyakov-Bershadsky $W_3^{(2)}$ algebra [5, 7].

2.3. The $W_3^{(2)}$ Algebra

Consider the following basis of sl_3

$$A_z^i t_i = \left(\begin{array}{ccc} \frac{A^4}{6} + \frac{A^5}{2} & A^2 & A^1 \\ A^6 & -\frac{A^4}{3} & A^3 \\ A^8 & A^7 & \frac{A^6}{6} - \frac{A^5}{2} \end{array} \right). \tag{2.14}$$

The sl_2 embedding that gives the $W_3^{(2)}$ algebra is given by $\Lambda^+ = t_1$, $\Lambda^0 = t_5$ and $\Lambda^- = t_8/2$. This is an example where the grading given by Λ^0 is non-integral, and $\mathbf{g}_{\frac{1}{2}}$ is spanned by t_2 and t_3. To get an integral gradation, we add to Λ^0 the gradation given by $-t_4$, which assigns degree $-\frac{1}{2}$ to t_2 and degree $+\frac{1}{2}$ to t_3. Thus, we get the following gradation

$$\left(\begin{array}{ccc} 0 & \frac{1}{2} & 1 \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ -1 & -\frac{1}{2} & 0 \end{array} \right) + \left(\begin{array}{ccc} 0 & -\frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{array} \right) = \left(\begin{array}{ccc} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{array} \right), \tag{2.15}$$

which also shows what the decomposition $\mathbf{g} = \mathbf{g}_- \oplus \mathbf{g}_0 \oplus \mathbf{g}_+$ looks like. The first class constraints are $A^1 = 1$ and $A^3 = 0$, and using a gauge transformation A_z can be put in the form $\Lambda^+ + W$,

$$A_z = \left(\begin{array}{ccc} -J/2 & 0 & 1 \\ G^+ & J & 0 \\ T & G^- & -J/2 \end{array} \right). \tag{2.16}$$

To find the gauge invariant polynomials associated to T, G^+, G^-, J, we parametrize an arbitrary $\mathbf{g}_-$ gauge transformation by

$$n = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 1 \end{array} \right) \tag{2.17}$$

and solve a, b from

$$\partial n n^{-1} + n \left(\begin{array}{ccc} \frac{A^4}{6} + \frac{A^5}{2} & A^2 & 1 \\ A^6 & -\frac{A^4}{3} & 0 \\ A^8 & A^7 & \frac{A^6}{6} - \frac{A^5}{2} \end{array} \right) n^{-1} = \left(\begin{array}{ccc} -J/2 & 0 & 1 \\ G^+ & J & 0 \\ T & G^- & -J/2 \end{array} \right). \tag{2.18}$$

This gives $a = A^5/2$, $b = A^2$, and if we substitute these back into (2.18) we can directly read of the gauge invariant polynomials

$$J = -\frac{1}{3}A^4,$$

$$
\begin{aligned}
G^{+} &= A^{6}, \\
G^{-} &= A^{7}+\frac{1}{2}A^{2}A^{5}-\frac{1}{2}A^{2}A^{4}+\partial A^{2}, \\
T &= A^{8}+A^{2}A^{6}+\frac{1}{4}A^{5}A^{5}+\frac{1}{2}\partial A^{5}.
\end{aligned} \tag{2.19}
$$

The structure of the classical $W_3^{(2)}$ algebra can now be found by computing the Poisson brackets of (2.19) using the brackets (2.12). If we restore the dependence on the level k of the affine sl_3 by multiplying all derivatives in (2.12) and (2.19), and if we introduce

$$
\mathcal{T}=\frac{1}{k}(T+\frac{3}{4}JJ), \tag{2.20}
$$

then we find the following structure of the classical $W_3^{(2)}$ algebra

$$
\begin{aligned}
J(z)J(w) &\sim \frac{\frac{2k}{3}}{(z-w)^{2}}, \\
J(z)G^{\pm}(w) &\sim \frac{\pm G^{\pm}(w)}{(z-w)}, \\
\mathcal{T}(z)J(w) &\sim \frac{J(w)}{(z-w)^{2}}+\frac{\partial J(w)}{(z-w)}, \\
\mathcal{T}(z)G^{\pm}(w) &\sim \frac{\frac{3}{2}G^{\pm}(w)}{(z-w)^{2}}+\frac{\partial G^{\pm}(w)}{(z-w)}, \\
\mathcal{T}(z)\mathcal{T}(w) &\sim \frac{-3k}{(z-w)^{4}}+\frac{2\mathcal{T}(w)}{(z-w)^{2}}+\frac{\partial\mathcal{T}(w)}{(z-w)}, \\
G^{\pm}(z)G^{\pm}(w) &\sim 0, \\
G^{\pm}(z)G^{\mp}(w) &\sim \frac{\pm 2k^{2}}{(z-w)^{3}}+\frac{3kJ(w)}{(z-w)^{2}}+\frac{\pm k\mathcal{T}(w)\mp 3J(w)J(w)+\frac{3}{2}k\partial J(w)}{(z-w)},
\end{aligned} \tag{2.21}
$$

where we used the correspondence $\frac{(n-1)!}{(z-w)^{n}} \leftrightarrow \partial_w^{n-1}\delta(z-w)$ between classical OPE's and Poisson brackets. The same formulas can also be derived using the formula for the Dirac bracket, or the expression (2.13).

Let us now discuss free field realizations of the $W_3^{(2)}$ algebra. First we give the Miura transformation, which expresses the generators of $W_3^{(2)}$ in terms of affine $\mathbf{g}_0$ and is obtained from (2.19) by simply restricting everything to $\mathbf{g}_0$, *i.e.* we put $A^7=A^8=0$. The Miura transform reads explicitly

$$
\begin{aligned}
J &= -\frac{1}{3}A^{4}, \\
G^{+} &= A^{6}, \\
G^{-} &= \frac{1}{2}A^{2}A^{5}-\frac{1}{2}A^{2}A^{4}+k\partial A^{2}, \\
\mathcal{T} &= \frac{1}{k}(A^{2}A^{6}+\frac{1}{4}A^{5}A^{5}+\frac{1}{12}A^{4}A^{4})+\frac{1}{2}\partial A^{5}.
\end{aligned} \tag{2.22}
$$

The following basis transformation

$$
A^{4} = 3J^{0}+U,
$$

$$
\begin{aligned}
A^5 &= J^0 - U, \\
A^2 &= J^+\sqrt{2}, \\
A^6 &= J^-\sqrt{2},
\end{aligned}
\tag{2.23}
$$

makes explicit that $\mathbf{g}_0 = sl_2 \oplus \mathbb{R}$, as the J^i and U have the following OPE's

$$
\begin{aligned}
U(z)U(w) &\sim \frac{\frac{3k}{2}}{(z-w)^2}, \\
J^0(z)J^\pm(w) &\sim \frac{\pm J^\pm(w)}{(z-w)}, \\
J^0(z)J^0(w) &\sim \frac{\frac{k}{2}}{(z-w)^2}, \\
J^+(z)J^-(w) &\sim \frac{k}{(z-w)^2} + \frac{J^0(w)}{(z-w)}.
\end{aligned}
\tag{2.24}
$$

Using the Wakimoto construction [6] it is straightforward to express U and the J^i in terms of two free scalar fields ϕ_1, ϕ_2 and a bosonic β, γ system with OPE's

$$
\begin{aligned}
\partial\phi_i(z)\partial\phi_j(w) &\sim \frac{\delta_{ij}}{(z-w)^2}, \\
\beta(z)\gamma(w) &\sim \frac{1}{(z-w)}.
\end{aligned}
\tag{2.25}
$$

One finds the following expressions for U and J^i

$$
\begin{aligned}
J^+ &= \frac{1}{\sqrt{2}}\beta, \\
J^0 &= \sqrt{\frac{k}{2}}\partial\phi_1 - \beta\gamma, \\
J^- &= \frac{-1}{\sqrt{2}}\beta\gamma^2 + \sqrt{\frac{k}{2}}\gamma\partial\phi_1 + \frac{k}{\sqrt{2}}\partial\gamma, \\
U &= \sqrt{\frac{3k}{2}}\partial\phi_2.
\end{aligned}
\tag{2.26}
$$

The standard free field realization of $W_3^{(2)}$ as given in [7] is recovered by substituting (2.26) back into (2.22)

$$
\begin{aligned}
J &= \beta\gamma - \sqrt{\frac{k}{2}}\partial\phi_1 - \sqrt{\frac{k}{6}}\partial\phi_2, \\
G^+ &= -\beta\gamma^2 + \sqrt{2k}\gamma\partial\phi_1 + k\partial\gamma, \\
G^- &= \beta^2\gamma - \sqrt{\frac{k}{2}}\beta\partial\phi_1 - \sqrt{\frac{3k}{2}}\beta\partial\phi_2 + k\partial\beta, \\
\mathcal{T} &= \frac{1}{2}\partial\phi_1\partial\phi_1 + \frac{1}{2}\partial\phi_2\partial\phi_2 + \frac{1}{2}\beta\partial\gamma - \frac{1}{2}\partial\beta\gamma + \sqrt{\frac{k}{8}}\partial^2\phi_1 - \sqrt{\frac{3k}{8}}\partial^2\phi_2.
\end{aligned}
\tag{2.27}
$$

However, we can also construct a different free field realization, by applying the automorphism $J^+ \leftrightarrow J^-$ and $J^0 \leftrightarrow -J^0$ to (2.26) before substituting these expressions into (2.22). This leads to a completely different free field realization

$$\begin{aligned}
J &= -\beta\gamma + \sqrt{\frac{k}{2}}\partial\phi_1 - \sqrt{\frac{k}{6}}\partial\phi_2, \\
G^+ &= \beta, \\
G^- &= \beta^2\gamma^3 - 3k\beta\gamma\partial\gamma - k\partial\beta\gamma^2 + k^2\partial^2\gamma - 3\sqrt{\frac{k}{2}}\beta\gamma^2\partial\phi_1 + \sqrt{\frac{3k}{2}}\beta\gamma^2\partial\phi_2 + 3k\sqrt{\frac{k}{2}}\partial\gamma\partial\phi_1 \\
&\quad -k\sqrt{\frac{3k}{2}}\partial\gamma\partial\phi_2 + k\sqrt{2k}\gamma\partial^2\phi_1 + k\gamma\partial\phi_1\partial\phi_1 - k\sqrt{3}\gamma\partial\phi_1\partial\phi_2, \\
\mathcal{T} &= \frac{1}{2}\partial\phi_1\partial\phi_1 + \frac{1}{2}\partial\phi_2\partial\phi_2 + \frac{3}{2}\beta\partial\gamma + \frac{1}{2}\partial\beta\gamma - \sqrt{\frac{k}{8}}\partial^2\phi_1 - \sqrt{\frac{3k}{8}}\partial^2\phi_2. \qquad (2.28)
\end{aligned}$$

From the expression for $\mathcal{T}$ we see that in (2.28) the β, γ system has a different spin than in (2.27). The $W_3^{(2)}$ algebra is a bosonic version of the $N = 2$ superconformal algebra, and the two different free field realizations given here are closely related to those for the $N = 2$ superconformal algebra given in [8]. We see both can easily be obtained from the general framework given here.

So far the discussion has been purely classical. To quantize the $W_3^{(2)}$ algebra, one introduces a b, c system for every generator of $\mathbf{g}_+$, and writes down the BRST operator that imposes the constraints $A^1 = 1$ and $A^3 = 0$. It is simply $Q = \int(c_1(A^1 - 1) + c_3A^3)$. Then one computes the BRST cohomology on the space of normal ordered expressions containing A^i, b^1, c_1, b^3, c_3. This gives the quantum versions of (2.19), where the expressions now also contain ghosts. The Miura transformation gets simply replaced by a quantum Miura transformation in the same way, and the quantum free field realizations can then be obtained by using the quantum version of the Wakimoto construction. The expressions are identical to the ones given here, up to renormalizations. For more details on the BRST cohomology, see [9].

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