THU-92/26
ITFA-92/24
9/92

W algebras, W Gravities and Their Moduli Spaces

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Abstract

By generalizing the Drinfel'd–Sokolov reduction we construct a large class of W algebras as reductions of Kac-Moody algebras. Furthermore we construct actions, invariant under local left and right W transformations, which are the classical covariant induced actions for W gravity.

Talk presented by T. Tjin at the 'Workshop on Superstrings and Related Topics', Trieste, July 1992.

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1. Introduction

W algebras were introduced by Zamolodchikov [1] as extensions of the Virasoro algebra which plays a central role in Conformal Field theory. Since W algebras are extensions of the Virasoro algebra a Hilbert space which is a direct sum of infinitely many irreducible Virasoro representations may be just a finite direct sum of irreducible W representations. Ever since the original paper by Zamolodchikov there has been an outpouring of papers on the subject (for a recent review see [2]). It is now clear that these W algebras play a central role in many areas of two-dimensional physics, most notably Toda theories [3], gauged WZW models [4], reductions of the KP hierarchy [5] and in the matrix model formulation of two-dimensional quantum gravity [6].

Many W-algebras were constructed by what has become known as the 'direct method' which amounts to solving the Jacobi identities for an algebra in which part of the commutation relations are fixed by imposing that all generators are primary fields. Although this method does give a lot of explicit information, it is extremely cumbersome. Therefore, different methods were developed which construct W-algebras as reductions of others. One of those methods is the so called 'coset construction', see [7, 8]. Another method, called quantum Drinfel'd–Sokolov reduction [9, 10, 11, 12], constructs the aforementioned 'standard' W algebras as Hamiltonian reductions of Kac–Moody algebras. This method has the important advantage that, due to general theorems of Hamiltonian reduction [13], closure of the algebra and Jacobi identities are ensured.

In [15, 16] it was shown that apart from the Drinfel'd–Sokolov reduction of the $SL_3(\mathbb{R})$ Kac–Moody algebra, leading to the 'standard' W_3 algebra, there exists another reduction leading to an algebra closely resembling (a bosonic version of the) $N = 2$ super conformal algebra. This algebra was named $W_3^{(2)}$. Shortly after, it was shown that one can associate a reduction to every embedding of sl_2 into the finite Lie algebra underlying the Kac–Moody algebra [17, 18]. These reductions generalize the Drinfel'd–Sokolov reductions, and contain among others the aforementioned ones.

Given this large set of non-linear W algebras it has been an open problem what their geometrical interpretation is. Whereas we know that the Virasoro algebra arises after gauge fixing the two-dimensional diffeomorphism invariance, a similar mechanism for the W algebras is unknown. Many groups attempted to get a better understanding of W geometry by the construction of an action invariant under *local* left and right W transformations. Hull [19] was the first to construct an action invariant under *chiral* W_3 transformations. Schoutens *et al.* extended these results to the non-chiral case [20], and also discussed some of the quantum properties of these theories.

In [21] a different point of view was taken and it was shown that the non-linear W transformations are just homotopy contractions of ordinary gauge tranformations. This result was used to construct a left, right invariant action for arbitrary W algebras, whose properties show it can be seen as the covariant induced action for W gravity. It was shown that the action is simply a Legendre transform of the WZW action based on the group underlying the Kac–Moody algebra. The moduli space for W gravity was shown to be the quotient of a subspace of the space of flat $SL_n(\mathbb{C})$ connections by the modular group. For $n = 2$ this is the usual moduli space of Riemann surfaces.

More details on the work described here can be found in [17, 21].

2. Generalized Drinfel'd–Sokolov Reductions

A classical W-algebra with conformal weights $\{\Delta_i\}_{i\in I} \subset \frac{1}{2}\mathbb{N}$ is a Poisson algebra generated by fields $T(z), \{A_i(z)\}_{i\in I}$ such that

1. $\{T(z), T(w)\} = T'(w)\delta(z-w) + 2T(w)\delta'(z-w) + \frac{c}{12}\delta'''(z-w)$
2. $\{T(z), A_i(w)\} = \Delta_i A_i(w)\delta'(z-w) + A_i'(w)\delta(z-w)$
3. Jacobi identities are satisfied.

Finding relations $\{A_i(z), A_j(w)\}$ such that part 3 of this defintion is satisfied is in general rather complicated. We shall therefore construct W-algebras as Poisson reductions of the Kirillov Poisson algebra on an sl_n Kac Moody algebra. This algebra is generated by fields $\{J^a(z)\}$ satisfying

$$\{J^a(z), J^b(w)\} = f^{ab}_c J^c(w)\delta(z-w) - kg^{ab}\delta'(z-w) \tag{2.1}$$

where g^{ab} is the inverse of the matrix $g_{ab} = Tr(t_a t_b)$, $\{t_a\}$ is a basis of sl_n and f^c_{ab} are the structure constants in this basis. This algebra, which is well known to be the current algebra of the SL_n WZW model, is itself a W-algebra with $T(z) = g_{ab}J^a(z)J^b(z)$ the so called Sugawara stress energy tensor.

Let $i : sl_2 \hookrightarrow sl_n$ be an sl_2 embedding into sl_n. Under the adjoint action of sl_2 the algebra sl_n (seen as a representation of sl_2) branches into a direct sum of p irreducible sl_2 multiplets. Let $\{t_{k,m}\}_{m=-j_k}^{j_k}$ be a basis of the k^{th} multiplet where j_k is the highest weight of this multiplet. The numbering is chosen such that $t_{1,\pm 1} = t_\pm$ and $t_{1,0} = t_3$ where $\{t_3, t_\pm\}$ is $i(sl_2)$. An arbitrary map $J : S^1 \to sl_n$ can then be written as

$$J(z) = \sum_{k=1}^{p} \sum_{m=-j_k}^{j_k} J^{k,m}(z) t_{k,m} \tag{2.2}$$

Impose now the constraints $\phi^{1,1}(z) \equiv J^{1,1}(z) - 1 = 0$ and $\phi^{k,m}(z) \equiv J^{k,m}(z) = 0$ for $m > 0, k \neq 1$. The constraints $\{\phi^{k,m}(z)\}_{m\geq 1}$ turn out to be first class which means that they generate gauge invariance. This gauge invariance can be completely fixed by gauging away the fields $\{J^{k,m}(z)\}_{m>-j_k}$. After constraining and gauge fixing the currents look like

$$J_{fix}(z) = \sum_{k=1}^{p} J^{k,-j_k}(z) t_{k,-j_k} + t_+ \tag{2.3}$$

The Poisson bracket (2.1) on the set of elements (2.2) induces a Poisson bracket $\{.,.\}^*$ on the set of elements (2.3) which is nothing but the Dirac bracket. One now has to calculate the Dirac brackets between the fields $\{J^{k,-j_k}\}$. It is one of our main results that the fields $T(z) = \mathrm{Tr}(J^2_{fix}(z))$ and $\{J^{k,-j_k}(z)\}_{k>1}$ generate a W algebra with conformal weights $\{\Delta_k = j_k + 1\}_{k>1}$ w.r.t. the Dirac bracket.

3. A General Expression for W Algebras

It is possible to give a closed expression for the Poisson bracket of a general W algebra. For this we need a linear operator $L : sl_n \to sl_n$ which is the inverse of ad_{t_+}. In terms of the basis $\{t_{k,m}\}_{m=-j_k}^{j_k}$ of sl_n it is defined by requiring (i) $L(\mathrm{ad}_{t_+}(t_{k,m})) = t_{k,m}$ for $m < j_k$ and (ii) $L(t_{k,-j_k}) = 0$. Let us also define a field $W(z)$ by

$$W(z) = J_{fix}(z) - t_+ = \sum_{k=1}^{p} J^{k,-j_k} t_{k,-j_k}. \tag{3.1}$$

The Poisson bracket of two functionials Q_1 and Q_2 of the fields $\{J^{k,-j_k}(z)\}_{k>1}$ is given by

$$\{Q_1, Q_2\}^* = \int dz \mathrm{Tr}\left(\frac{\delta Q_2}{\delta W(z)} (k\partial + \mathrm{ad}_{W(z)}) \frac{1}{1 + L(k\partial + \mathrm{ad}_{W(z)})} \frac{\delta Q_1}{\delta W(z)} \right), \tag{3.2}$$

where

$$\frac{\delta Q}{\delta W(z)} = \sum_{k=1}^{p} \frac{\delta Q}{\delta J^{k,-j_k}(z)} t^{k,-j_k} \tag{3.3}$$

and $\{t^{k,m}\}$ is the dual basis of $\{t_{k,m}\}$, so that $\mathrm{Tr}(t^{k,m} t_{k',m'}) = \delta^k_{k'} \delta^m_{m'}$. In (3.2), $1/(1 + L(\partial + \mathrm{ad}_{W(z)}))$ denotes the series $\sum_{i \geq 0} (-L(\partial + \mathrm{ad}_{W(z)}))^i$. This series always truncates after a finite number of steps: $(-L(\partial + \mathrm{ad}_{W(z)}))^i = 0$ for $i \geq 2\Delta_{max}$, where Δ_{max} is the largest conformal weight of the W algebra under consideration.

4. Classical W Gravity

Introduce a new constrained current $\overline{J}_{fix}$ of the form $\overline{J}_{fix}(z) = \sum_{k=1}^{p} \overline{J}^{k,-j_k}(z) \bar{t}_{k,-j_k} + \bar{t}_+$, where $\bar{t}_{k,m}$ is the transpose of $t_{k,m}$. For this constrained current one can write down a Poisson bracket analogous to (3.2), showing that the $\{\overline{J}^{k,-j_k}\}$ generate a $\overline{W}$ algebra isomorphic to the W algebra generated by $J_{fix}(z)$. One might wonder whether the $W, \overline{W}$ transformations as following from the previously discussed Poisson brackets, *e.g.* (3.2), have an interpretation as symmetries of a field theory. Such a theory would then commonly be called 'a theory for W gravity'. It can be shown that such an interpretation of the above algebras indeed exists: an action invariant under these $W, \overline{W}$ transformations can be constructed.

Consider the following definition of the *chiral* induced action $\Gamma[\mu_i]$ for W gravity:

$$e^{-\Gamma[\mu_i]} = \left\langle e^{-\int d^2z \sum_i \mu_i W_i} \right\rangle. \tag{4.1}$$

In this formula the μ_i are generalizations of the Beltrami differential $\mu = \mu_2$, and the W_i are matter currents forming a W algebra. The induced action can be calculated by expanding the exponent and making use of the operator product algebra of the W_i fields, which in the classical limit we consider here reduces to the Poisson algebra as given in (3.2). Doing so, one derives classical Ward identities for the induced action whose solution is given by: $\Gamma[\mu_i] = kS_{wzw}(hf)$. Here S_{wzw} is the well known WZW action, $f = \exp(-zt_+)$, and $h \in SL_n(\mathrm{I\!R})$ is such that

$$h^{-1}\bar{\partial}h = \frac{1}{1 + L(k\partial + \mathrm{ad}_{W(z)})} F(\mu_i), \tag{4.2}$$

where $F(\mu_i) \in \ker(t_+)$, whose independent components are labeled by the μ_i.

The chiral action is *not* invariant under the W transformations, but rather has an anomalous transformation behaviour. The precise form of this anomaly is directly related to the anomalous terms in the Poisson brackets of the W fields, *e.g.* the $c/12\,\delta'''(z-w)$ term of the Virasoro algebra.

In a similar way we can construct a chiral action $\Gamma[\bar{\mu}_i]$ starting from matter currents $\overline{W}_i$ which form a $\overline{W}$ algebra. To obtain from these two chiral actions an action which is invariant under $W, \overline{W}$ transformations, we should construct a local counterterm $\Delta\Gamma$ whose anomalous behaviour under $W, \overline{W}$ transformations cancels that of the chiral actions. So schematically

$$S = \Gamma[\mu_i] + \Delta\Gamma + \Gamma[\bar{\mu}_i]. \tag{4.3}$$

This counterterm can be constructed as follows: Let $g, \bar{g}$ be elements of $SL_n(\mathrm{I\!R})$ such that $g^{-1}\partial g = J_{fix}$ and $\bar{g}^{-1}\bar{\partial}\bar{g} = \overline{J}_{fix}$, and let $\bar{h}$ be the 'conjugate' (by which we mean taking the transpose and putting bars) of h as defined in (4.2). Then the final result for the covariant induced action of arbitrary W gravity reads

$$S(\mu_i, \bar{\mu}_i, G) = -k \min_{W_i, \overline{W}_i} \left(S_{wzw}(gG\bar{g}^{-1}) - S_{wzw}(gh^{-1}) - S_{wzw}(\bar{g}^{-1}\bar{h}) \right), \tag{4.4}$$

i.e. it is the Legendre transform w.r.t. the fields $W_i, \overline{W}_i$ of the WZW action. Recall that $W_i, \overline{W}_i$ label the independent components of J_{fix} and $\overline{J}_{fix}$, respectively, and are dual to the parameters $\mu_i, \bar{\mu}_i$. In (4.4) G is an extra group element of $SL_n(\mathrm{I\!R})$ needed to make the action invariant. One can work out the Legendre transform quite easily since the $W_i, \overline{W}_i$ appear in an algebraic and at most quadratic way in (4.4). Specializing to a particular W algebra one finds in general that some of the components of G appear only algebraically, making it possible to integrate them out. For instance if one considers the case of $SL_2(\mathrm{I\!R})$, for which the corresponding W algebra is simply the Virasoro algebra, (4.4) reduces to the well known Polyakov action for two-dimensional gravity [14]

$$S_{pol} \sim \int d^2z\, R \frac{1}{\Box} R, \tag{4.5}$$

once we intergrate out the auxiliary fields.

5. Moduli Spaces for W Algebras

In the same way as the moduli space of Riemann surfaces plays an important role for conformal field theories coupled to gravity, on expects a generalized moduli space to play an important role for conformal field theories coupled to W gravity. Such a generalized W moduli space can be defined as the space of W fields on a Riemann surface modulo W transformations. Let us sketch some of the steps involved in the computation.

First, one has to construct W algebras on arbitrary Riemann surfaces. Because currents transform as a connection one-form with values in sl_n, the constraint $J^{1,1} = 1$ cannot be imposed globally. The resolution of this problem is to twist the trivial sl_n bundle for which J is a connection into a non-trivial vector bundle E, such that the $(J - J_{ref})^{1,1} - 1 = 0$ can be imposed globally. The fixed reference connection J_{ref} is needed, because 0 is no longer a globally well-defined connection for E. It turns out that E is isomorphic to the direct sum of line bundles $\oplus_k \oplus_{m=-j_k}^{j_k} K^m$, where K is the holomorphic cotangent bundle of the Riemann surface. Using E, one can define W algebras on arbitrary Riemann surfaces using a generalization of (3.2).

Next, one shows that the space of W fields modulo W transformations is the same as the space of holomorphic W fields modulo holomorphic W transformations. Because these spaces are finite dimensional, they are easy to investigate. The W moduli spaces computed in this way are actually only W Teichmüller spaces; to obtain W moduli space one has to take the quotient by the action of the modular group. To prove that no moduli are lost when passing to holomorphic W fields, a geometrical construction of W algebras is used: W algebras can be obtained as a homotopy contraction (see e.g. [22]) of ordinary gauge transformations. The homotopy operator is precisely the operator L defined in the previous section, as it can be used to defined a map from the space of $(1,0)$ forms with values in E to the space of sections of E.

For genus $g > 1$, the W moduli spaces are $(g-1)(n^2-1)$ dimensional subspaces of the $2(g-1)(n^2-1)$ dimensional space of irreducible flat $SL_n(\mathbb{C})$ over a Riemann surface. Furthermore, they are related in a natural way to so-called Higgs bundles [23, 24, 25, 26, 27]. This correspondence can be used to show that for the principal sl_2 embeddings in sl_n, the W moduli space is the moduli space of flat $SL_n(\mathbb{R})$ bundles. For other embeddings, the interpretation of the W moduli spaces is unclear.

References

[1] A. B. Zamolodchikov, Theor. Math. Phys. **65** (1985) 1205; V. A. Fateev and A. B. Zamolodchikov, Nucl. Phys. **B280** (1987) 644.

[2] P. Bouwknegt and K. Schoutens, '$\mathcal{W}$-Symmetry in Conformal Field Theory', CERN-TH.6583/92, ITP-SB-92-23, to be published in Physics Reports.

[3] A. Bilal and G. Gervais, Phys. Lett. **B206** (1988) 412; P. Mansfield and B. Spence, Nucl. Phys. **B362** (1991) 294; L. Fehér, L. O'Raifeartaigh, P. Ruelle and I. Tsutsui, 'Generalized Toda theories and W algebras associated with integral gradings,' ETH-TH/91-16, DIAS-STP-91-17; A. N. Leznov and M. V. Saveliev, Lett. Math. Phys. **6** (1982) 505; Comm. Math. Phys. **89** (1983) 59.

[4] J. Balog, L. Fehér, P. Forgacs, L. O'Raifeartaigh and A. Wipf, Phys. Lett. **B227** (1989) 214.

[5] I. M. Gelfand and L. A. Dikii, Russ. Math. Surv. **30** (1975) 77; Funct. Anal. Appl. **10** (1976) 4; ibid. **11** (1977) 93; B. A. Kupershmidt and G. Wilson, Inv. Math. **62** (1981) 403.

[6] J. Goeree, Nucl. Phys. **B358** (1991) 737; M. Fukuma, H. Kawai and R. Nakayama, Int. J. Mod. Phys. **A6** (1991) 439, Tokyo/KEK preprint UT-582 KEK 91-37; J. de Boer, Nucl. Phys. **B366** (1991) 602.

[7] P. Goddard, A. Kent and D. Olive, Phys. Lett. **B152** (1985) 88.

[8] F. Bais, P. Bouwknegt, M. Surridge and K. Schoutens, Nucl. Phys. **B304** (1988) 348.

[9] V. G. Drinfel'd and V. V. Sokolov, J. Sov. Math. **30** (1985) 1975, Sov. Math. Doklady **3** (1981) 457.

[10] B. L. Feigin and E. Frenkel, Phys. Lett. **B246** (1990) 75; E. Frenkel, 'W Algebras and Langlands-Drinfel'd Correspondence', Harvard preprint 1991.

[11] V. Fateev and S. Lukyanov, Int. Journal of Mod. Phys. **A3** (1988) 507.

[12] M. Bershadsky and H. Ooguri, Comm. Math. Phys. **126** (1989) 49.

[13] R. Abraham and J. E. Marsden, 'Foundations of Mechanics', the Benjamin/Cummings Publishing Company inc.

[14] A. M. Polyakov, Phys. Lett. **B103** (1981) 207.

[15] A. M. Polyakov, Int. J. Mod. Phys. **A5** (1990) 833.

[16] M. Bershadsky, Commun. Math. Phys. **139** (1991) 71.

[17] F. A. Bais, T. Tjin and P. van Driel, Nucl. Phys. **B357** (1991) 632.

[18] J. Balog, L. Fehér, P. Forgacs, L. O'Raifeartaigh and A. Wipf, Phys. Lett. **B227** (1989) 214; Phys. Lett. **B244** (1990) 435; Ann. Phys. (N. Y.) **203** (1990) 76.

[19] C. M. Hull, Phys. Lett. **B240** (1990) 110; ‘Classical and Quantum W Gravity’, to appear in Proc. of ‘Strings and Symmetries 1991’, Stony Brook, May 1991.

[20] K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, Nucl. Phys. **B349** (1991) 791, Phys. Lett. **B243** (1991) 248; Nucl. Phys. **B364** (1991) 584; Nucl. Phys. **B371** (1992) 315; ‘On the Effective Action of Chiral W_3 Gravity’, Stony Brook preprint ITP-SB-91/21; ‘Example of a quantum field theory based on a non-linear Lie algebra’, contribution to the *Festscrift on the occasion of M. Veltman's 60th birthday*, CERN-TH.6361/92, ITP-SB-91-34, LBL-31382, UCB-PTH-91/52, May 1991; H. Ooguri, K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, ‘The Induced Action of W_3 Gravity’, ITP-SB-91/16, RIMS-764 (June 1991).

[21] J. de Boer and J. Goeree, Phys. Lett. **B274** (1992) 289; Nucl. Phys. **B381** (1992) 329; ‘Covariant W Gravity and Its Moduli Space from Gauge Theory’, THU-92/14.

[22] R. Bott, L.W. Tu, Differential Forms in Algebraic Topology, Springer Graduate Texts in Mathematics 82, Springer-Verlag, New York, 1982.

[23] N.J. Hitchin, Proc. London Math. Soc. (3) **55** (1987) 59.

[24] N.J. Hitchin, In ‘Global Geometry and Mathematical Physics’, Eds. M. Francaviglia, F. Gherardelli, Lecture Notes in Mathematics 1451, Springer-Verlag, Berlin, 1990.

[25] N.J. Hitchin, Topology **31** (1992) 449.

[26] C.T. Simpson, ‘Higgs Bundles and Local Systems’, Princeton University Preprint.

[27] C.T. Simpson, Journal of the A.M.S. **1** (1988) 867.