Relativistic notation

The theories we will consider are consistent with special relativity. To make explicit how the different quantities transform under Lorentz transformations it will be convenient to use four-vector notation. A point in space-time is described by a four component vector $x^\mu = (x^0, x^1, x^2, x^3) = (t, x, y, z)$, where we chose units such that $c = 1$. According to special relativity the inner product

$$(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = x^\mu \eta_{\mu\nu} x^\nu$$

is the same for observers in any inertial frame. Here the Greek indices run from 0 to 3, and $\eta_{\mu\nu}$ is the matrix $\mathrm{diag}(1, -1, -1, -1)$. Furthermore we used the summation convention: indices that appear twice are summed over. Since this inner product is preserved under Lorentz transformations,

$$x^\mu \to L^\mu{}_\nu x^\nu,$$

we conclude that these transformations should satisfy the rule

$$L^T \eta L = \eta \quad \text{or} \quad L^\rho{}_\mu L^\sigma{}_\nu \eta_{\rho\sigma} = \eta_{\mu\nu}.$$

Note that the three dimensional rotations are contained in the Lorentz group. In order to avoid having to write numerous η's, four-vectors with lower indices are introduced:

$$x_\mu = \eta_{\mu\nu} x^\nu \quad \text{or} \quad (x_0, x_1, x_2, x_3) = (t, -x, -y, -z).$$

If one wants to construct a Lorentz invariant expression, one only has to make sure that all upper indices are contracted with lower indices. Apart from the coordinates there are other quantities that transform as four-vectors, e.g. the energy and momentum $p^\mu = (E, p_x, p_y, p_z)$. Furthermore we can consider Lorentz tensors of arbitrary rank, $T^{\mu_1 \cdots \mu_r}$, which by definition transform under Lorentz transformations as

$$T^{\mu_1 \cdots \mu_r} \to L^{\mu_1}{}_{\nu_1} \ldots L^{\mu_r}{}_{\nu_r} T^{\nu_1 \cdots \nu_r}.$$

Indices can be lowered using η, e.g.

$$T_{\mu_1}{}^{\mu_2 \cdots \mu_r} = \eta_{\mu_1 \nu} T^{\nu \cdots \mu_r}.$$

Again invariant quantities can be formed by contracting all upper indices with an equal number of lower indices.
The reader should verify that the derivative with respect to x^μ transforms as a vector with a lower index:

$$\frac{\partial}{\partial x^\mu} = \partial_\mu.$$

(Write $x'^\mu = L^\mu{}_\nu x^\nu$ and use the composition rule for derivatives).

Fields

A field ϕ assigns to each point x in spacetime a quantity $\phi(x)$. This quantity may be a real or complex number, vector, tensor, etc. As examples one may think of temperature, velocity of flow in a liquid, stress in a solid body, or the electric and magnetic fields known from Maxwell theory. The dynamics of the field theory are described by a set of differential

equations the fields satisfy (e.g. the Maxwell equations). After quantization a large class of relativistic field theories describe particles.
Under Lorentz transformations $x \to x'(x)$, the fields transform as discussed in the previous section, but also their arguments are transformed. The relations between the transformed field, which we will here denote, for the sake of clarity, by a tilde, and the original field are

$$\tilde{T}^{\mu_1\cdots}{}_{\alpha_1\cdots}(x') = L^{\mu_1}{}_{\nu_1}\ldots L_{\alpha_1}{}^{\beta_1}\ldots T^{\nu_1\cdots}{}_{\beta_1\cdots}(x(x')).$$

Note that to obtain the values of the transformed fields the original fields should be evaluated in the same point, independent of the choice of coordinates. As an example consider a scalar field $\phi(x)$, which itself does not transform under a transformation, in a one-dimensional space. Let the transformation be the constant translation $x' = x - a$. The value of the transformed field $\tilde{\phi}$ in the point x' will be equal to the original field ϕ in the point $x' + a$. (Draw a picture).

Problem 1 **Relativistic scalar field theory**

We will set up a field theory for a scalar field $\phi(x)$ in the Lagrange formalism. We assume that the action $S[\phi]$ of the field $\phi(x)$ is given by

$$S[\phi] = \int \mathrm{d}^4x\, \mathcal{L}(\phi, \partial_\mu\phi),$$

where $\mathcal{L}$ is the Lagrangian density. This means that $\mathcal{L}$ is a local function depending only on ϕ and its first derivatives. In analogy to classical mechanics, the variational principle requires that the action is stationary with respect to infinitesimal variations in the field. In order to derive the equation of motion for the field from this condition, consider an infinitesimal variation $\delta\phi(x)$, which vanishes at infinity, of the field $\phi(x)$. This gives rise to a variation of the action:

$$\delta S = \int \mathrm{d}^4x\, \delta\mathcal{L}.$$

(a) Show that requiring δS to be stationary for any variation $\delta\phi$ gives the Euler-Lagrange equation

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = 0.$$

Use partial integration and the fact that $\delta\phi$ vanishes at space-time infinity.

Now consider the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2.$$

Since ϕ is a scalar field this Lagrangian is Lorentz invariant, and it gives rise to a relativistic field theory.

(b) Determine the Euler-Lagrange equation for ϕ. This equation is the *Klein–Gordon* equation. Upon quantization the field ϕ describes free scalar particles of mass m.

If an action is invariant under a continuous group of transformations, Noether's theorem states that there exists a conserved quantity associated to this symmetry. We can calculate this quantity as follows. Let ξ be a constant infinitesimal transformation parameter.

Invariance of the action implies that the Lagrangian density can only change by a total derivative, so

$$\delta\mathcal{L} = \xi\partial_\mu\Omega^\mu,$$

for some vector Ω. Let us now take ξ to be x-dependent, where we again require $\xi(x)$ to vanish at space-time infinity. The variation of the Lagrangian density will now in general acquire an extra term proportional to the derivative $\partial_\mu\xi$:

$$\begin{aligned}\delta\mathcal{L} &= \xi\partial_\mu\Omega^\mu + \partial_\mu\xi N^\mu \\ &= \xi\partial_\mu J^\mu + \partial_\mu\left(\xi N^\mu\right),\end{aligned}$$

where $J^\mu = \Omega^\mu - N^\mu$. Since the second term is a total derivative, and ξ vanishes at the boundary at infinity, we get for the variation of the action

$$\delta S = \int \mathrm{d}^4x\xi\partial_\mu J^\mu.$$

Recall now that under any infinitesimal variation of the fields, the variation of the action vanishes, provided the Euler-Lagrange equations are satisfied. Since ξ represents a specific variation of the fields, we can conclude that

$$\partial_\mu J^\mu = 0$$

if the fields contained in J^μ satisfy their equations of motion. J^μ is called the conserved Noether current associated to the symmetry parametrized by ξ.

(c) Show that $Q = \int \mathrm{d}^3\vec{x}J^0$ is a conserved charge, i.e. $\frac{\partial}{\partial t}Q = 0$, provided the fields evolve according their equations of motion, and the current falls off fast enough.

(d) Show that the Klein–Gordon Lagrangian is invariant under constant translations, $x^\mu \to x^\mu + a^\mu$. Demonstrate that the associated conserved current is given by (use that $\phi(x-\xi) = \phi(x) - \xi^\mu\partial_\mu\phi(x)$, to first order in the variation ξ^μ)

$$T^{\mu\nu} = -\eta^{\mu\nu}\mathcal{L} + \partial^\mu\phi\partial^\nu\phi.$$

The first index can be regarded as labelling the component of the current, whereas a second index arises in this case since there are four independent translations (one in time, three in space). Note, however, that $T^{\mu\nu}$ is symmetric in the two indices. Verify explicitly that $\partial_\mu T^{\mu\nu} = 0$ using the field equation. Calculate the four-vector P^μ of conserved charges. These are the total energy and momentum of the field. For this reason $T^{\mu\nu}$ is called the energy-momentum tensor.

(e) As we saw the Klein–Gordon action is Lorentz invariant. To determine the form of an infinitesimal Lorentz transformation write $L^\mu{}_\nu = \eta^\mu{}_\nu + \xi^\mu{}_\nu$ and use the defining relations of the Lorentz group to determine the form of the infinitesimal parameter $\xi^\mu{}_\nu$. Use this to show that the associated Noether current can be written as

$$J^{\mu\nu\rho} = \frac{1}{2}(x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu}).$$

Write down the conserved charge and interpret its spatial components.