Gauge Fixing and the Faddeev-Popov Method

In field theory we often encounter actions which are invariant under certain gauge transformations. If we try and calculate correlation functions, we then usually run into problems. Intuitively, this may be expected, as we should keep in mind that quantum fluctuations along gauge symmetry directions are not physical. Thus, in path integral language, we should not integrate over fluctuations which are related by the gauge symmetry. Actually, one just wants to factor out the contribution to the path integral of the gauge fluctuations. The action and physical observables are independent of gauge fluctuations, and an overall factor in front of the path integral always drops out in amplitudes.

The quantisation of a field theory with gauge symmetry is most intuitively done using the method invented by Faddeev and Popov, which utilises the functional integral formalism. To get some intuition for this procedure, we will compute a very simple toy-partition sum with some symmetry that will be interpreted as gauge symmetry. In parallel with the excercise we will point out the connection of this computation with more physical models.

In quantum field theory the partition sum can be expressed as a functional integral over all physical fields,

$$\int \mathcal{D}[\Phi_i] e^{-S[\Phi_i]}$$

In a gauge theory, some of the field configurations, forming some subset of all configurations, are related to one another by gauge transformations. We should not integrate over different representatives of a single physical field configuration, differing only by a gauge transformation.

Imagine that instead of doing an integral in a space of functions, we are simply performing a finite dimensional integration

$$\int \mathcal{D}[\Phi_i] \to \prod_{k=1}^{n} \int d\phi_k$$

In particular imagine that the different ϕ_k are different components of the gauge field. We now split the collection of ϕ_k's into two groups: one group of components which *do not* transform under gauge transformations (e.g. the transverse components of the electromagnetic vector potential), which we shall call f_i, and the other group, of components which *do* transform under gauge transformations (e.g. the longitudinal and timelike components of the vector potential), these we shall call g_i. Note that in general, a gauge transformation may affect the integral over gauge non-invariant fields through a Jacobian factor. In this excercise, we will restrict ourselves to transformations which leave the measure invariant.

Now instead of integrating over all components of the gauge field

$$(\prod \int df_i)(\prod \int dg_j)$$

we should only integrate over the gauge invariant components. The partition sum of our gauge theory is thus expressed as

$$Z = \prod \int df_i e^{-S} \tag{1}$$

Note that the action is a gauge invariant quantity and it does not depend on the gauge non-invariant fields: $\frac{\partial S}{\partial g_j} = 0$.

Consider the simple 'action'

$$S(x, y) = (x + y)^2$$

which is a function of two 'field components' x and y.

> (i) Write down the expression for the partition sum as a finite dimensional integral over the 'field configurations'. Draw a picture showing the shape of S as a function of x and y. Write down the symmetry transformation on the fields x and y that leaves the action S invariant.

Note that we may just as well write the partition sum of equation (1) as

$$Z = (\prod \int dg_j)(\prod \int df_i) e^{-S} \delta(g_j) \tag{2}$$

where the delta function combined with the additional integrations simply sets the gauge non-invariant fields to zero.

> (ii) We will consider the symmetry found in (i) as a gauge symmetry. Find a new pair of field components $(x, y) \mapsto (f_1, g_1)$, such that f_1 is invariant under the symmetry transformations and only g_1 transforms under the symmetry transformation found in (i). Write down a new partition function, Z_1, analogous to the one in equation (2), where a delta function is used to set the gauge non-invariant field g to zero.

> (iii) The choice of fields (f, g) is not unique! Find a different pair of fields $(x, y) \mapsto (f_2, g_2)$, such that f_2 is invariant under gauge transformations and g_2 is not. Again, write down an expression for the partition sum, now called Z_2, using a delta function to set g_2 to zero. Show that Z_1 and Z_2 are in general *not* identical, but differ by a finite multiplicative factor. Comment on the physical (in-)significance of this ambiguity.

The integration in equation (2) was performed along the surface $g_j = 0$, which is nothing but a spacific gauge choice. We may just as well choose a different gauge, say $g_j = \chi_j(f_i)$ for some functions χ_j. This corresponds to integrating over a different surface. Reintroducing the notation $\prod d\phi_k$ for the combined integration $\prod\int df_j \prod\int dg_j$, we can write the partition function in the new gauge as

$$Z = \prod\int d\phi_k e^{-S}\delta(g_j - \chi(f_i)) \tag{3}$$

Instead of defining a surface over which we integrate by the set of functions $\chi_j(f_i)$, we can define it as the solution to some set of equations $F(\phi_k) = 0$, where F stands for a collection of functions, just enough to fix the gauge dependent fields.

(iv) We want to write the partition sum of our theory, using a general function $F(f, g)$ to fix the gauge. Show that the expression

$$\int df dg e^{-S}(\frac{\partial F}{\partial g})\delta(F(f,g))$$

gives precisely the partition sum Z_1 for any choice of F. Argue that in the case of multiple gauge non-invariant fields, this formula is generalised to

$$\int d\phi_k e^{-S}\det(\frac{\partial F}{\partial g_j})\delta(F(\phi_k))$$

The appropriate way to write the partition sum of a field theory with gauge invariance is thus, by definition,

$$Z = \int \mathcal{D}\Phi e^{-S}(\frac{\partial F}{\partial g})\delta(F(\Phi)) \tag{4}$$

where the F^a is a set of functions of the fields, such that there is precisely one gauge transformation of the fields that makes the equations $F^a(\Phi) = 0$ all true.

Both the contributions of the determinant and of the delta function can be absorbed in extra terms in the Lagrangian. The term $\det(\frac{\partial F}{\partial g})$ can be written as an integral over a complex Grassmann field, using

$$\det(A) = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{-\bar{\psi}A\psi}$$

To write the determinant $\det(\frac{\partial S}{\partial g})$ as coming from a term in the Lagrangian, we need to introduce a field of anti-commuting numbers. These are called Grassmann numbers. They obey the relation $\eta_1\eta_2 = -\eta_2\eta_1$ In particular $\eta\eta = 0$. Furthermore Grassmannian differential obey the anti-commutation relations $\{\eta_1, d\eta_2\} = 0$ and $\{d\eta_1, d\eta_2\} = 0$. Any function of a Grassmann variable has a particularly simple Taylor expansion: it terminates after the linear term $f(\eta) = c_0 + c_1\eta$, where c_0, c_1 are ordinary, commuting numbers.

(v) Write down a general power series expansion of a function of two Grassmann variables, $f(\eta_1, \eta_2)$. *Left* differentiation is defined as

$$\frac{\partial}{\partial\eta_1}(c_0 + c_1\eta_1 + c_2\eta_2 + c_3\eta_1\eta_2) = c_1 + c_3\eta_2$$
$$\frac{\partial}{\partial\eta_2}(c_0 + c_1\eta_1 + c_2\eta_2 + c_3\eta_1\eta_2) = c_2 - c_3\eta_1$$

Multiple integrals are interpreted as iterated integrals, so in particular

$$(\int d\eta)^2 = \int d\eta_1 \int d\eta_2 = -\int d\eta_2 \int d\eta_1 = -(\int d\eta)^2$$

so

$$\int d\eta = 0$$

As there is no other scale to Grassmann variables, we are free to define

$$\int d\eta\eta = 1$$

Show that partial differentiating with respect to η_1 is the same as integration with respect to η_1. Let η and $\bar{\eta}$ be complex Grassmann variables (i.e. $\eta = \frac{\eta_1 + i\eta_2}{\sqrt{2}}$ etc.). Show that

$$\int d\bar{\eta} d\eta e^{-\bar{\eta} c \eta} = c$$

Compare this to the integral over commuting numbers

$$\int d\bar{\theta} d\theta e^{-\bar{\theta} c \theta}$$

(vi) Consider an n-tuple of complex Grassmann variables, η_i and a transformation of variables

$$\eta_i \mapsto \eta_i' = U_{ij}\eta_j$$

where U_{ij} is a unitary matrix. Show that

$$\prod_i \eta_i' = (\det U) \prod_i \eta_i$$

Using this result, prove that for a Hermitean matrix C_{ij}, the Gaussian Grassmann integral satisfies:

$$\prod_i d\bar{\eta}_i d\eta_i e^{-\bar{\eta}_i C_{ij} \eta_j} = \det C$$

Compute the integral

$$\prod_i d\bar{\theta}_i d\theta_i e^{-\bar{\theta}_i C_{ij} \theta_j}$$

where the θ_i and $\bar{\theta}_i$ are n-tuples of commuting c-numbers and C_{ij} is a Hermitean $n \times n$ matrix.

We can now see that we can write the partition sum, equation (4), of our toy theory as

$$Z = \int df\, dg\, d\bar{\eta}\, d\eta\, e^{-S-\bar{\eta}(\frac{\partial F}{\partial g})\eta}\delta(F) \tag{5}$$

The term $\bar{\eta}(\frac{\partial F}{\partial g})\eta$ is called the *Faddeev-Popov* term in the action.

We are free to choose any function F, provided it properly defines a gauge, as we remarked earlier. In practice, we choose such an F, that our calculations simplify. We are however still left with a delta function in our expression, equation (5). In order to get rid of the delta function, at the cost of adding a so-called *gauge fixing term* to the action, we generalise our gauge fixing condition to

$$F(\phi) - \omega = 0$$

That is, in addition to the function F (a function of the gauge fields), we introduce a field ω. Note that the rest of the integral defining Z is completely independent of ω, so instead of choosing one particular ω, we might integrate over all possible ω while adding a weight factor $W(\omega)$. Let us choose the weight factor $W(\omega) = e^{\frac{-1}{2\alpha}\omega^2}$. The expression for our partition sum from equation (3) now is

$$\begin{aligned} Z &= N^{-1}\int (\prod_k d\phi_k)\, d\bar{\eta}\, d\eta\, d\omega\, e^{-S(\phi)-\bar{\eta}(\frac{\partial F}{\partial g})\eta} e^{-\frac{1}{2\alpha}\omega^2}\delta(F-\omega) \\ &= N^{-1}\int (\prod_k d\phi_k)\, d\bar{\eta}\, d\eta\, e^{-S(\phi)-\bar{\eta}(\frac{\partial F}{\partial g})\eta-\frac{1}{2\alpha}F^2} \end{aligned}$$

Here the N^{-1} is some normalisation factor to compensate for the extra weighted integral over ω, which did not appear in equation (3). However, this factor is irrelevant, as is any overall multiplicative factor in such a (path) integral, as in the end the partition sum is defined in such a way, that the expectation value of the unit operator is one: $\langle 1 \rangle = 1$. Besides, note that in excercise (iii) it was already clear that there is a multiplicative ambiguity in the definition of the partition sum for a gauge theory. So we might as well choose to take into account the factor coming from the integration over ω at the stage of defining the partition sum.

(vii) Write down the expression for the partition sum of our original theory, with action $S(x, y) = (x+y)^2$, now including a general gauge fixing term $-\frac{1}{2\alpha}F^2$ and a Faddeev-Popov term in the action. Next, choose $F(x, y) = (x - y)$ as gauge fixing function and evaluate the integral for general value of α multiplying the gauge fixing term. Compute the expectation value of $x_+^2 = (x+y)^2$ and of $x_-^2 = (x-y)^2$. Do these depend on the choice of α? Is this bad and why (not)?