Extra Exercise 4

In this exercise we compute the torus partition fucntion for a compactified scalar field.

Let x be a scalar field periodically identified as $x \cong x + 2\pi R$. Consider a two-torus T^2 with modulus $\tau = \tau_1 + i\tau_2$ and complex coordinate z

$$z \cong z + 1, \qquad z \cong z + \tau$$

1. Show that any field configuration is labeled by two integer winding numbers m_1, m_2 defined by

$$x(z+1) = x(z) + 2\pi m_1 R, \qquad x(z+\tau) = x(z) + 2\pi m_2 R.$$

Show that the winding number can be given by the contour integrals

$$\frac{1}{2\pi}\oint_A dx = m_1 R, \qquad \frac{1}{2\pi}\oint_B dx = m_2 R$$

for two curves A, B on T^2 defined as the straight line from 0 to 1 and from 0 to τ respectively.

2. Show that if the field x is split as

$$x = x_{cl} + x_{qu}$$

with x_{cl} satisfying $\partial\overline{\partial} x_{cl} = 0$, then we can choose x_{qu} to be single-valued.

Remark: the splitting above is not unique. In particular, we have to choose where to put the constant term in $x(z, \overline{z})$. For the moment, we will define it to be in the quantum mechanical part x_{qu}.

3. Find the solution x_{cl} with given winding numbers m_1, m_2 by first remarking that dx_{cl} is of the form

$$dx_{cl} = \lambda dz + \overline{\lambda} d\overline{z},$$

where λ and $\overline{\lambda}$ at this stage are independent variables. Then compute

$$\oint_A dz = 1, \qquad \oint_B = dz = \tau$$

and express λ in m_1, m_2 through the relations

$$m_1 = \frac{\lambda + \overline{\lambda}}{2\pi R}, \qquad m_2 = \frac{\tau\lambda + \overline{\tau}\overline{\lambda}}{2\pi R}$$

Combine the ingredients to obtain the result

$$x_{cl} = \Re\left(-2\pi i R z m_2 - m_1 \tau\, \overline{\tau}_2\right)$$

4. Evaluate the action

$$S(x) = \frac{1}{2\pi} \int \partial x \overline{\partial} x$$

for the classical solution to find

$$S_{m_1,m_2} \equiv S(x_{cl}) = \frac{\pi R^2 |m_2 - m_1 \tau|^2}{2\tau_2}$$

5. Show that the partition function can be written as

$$Z = Z_{cl} Z_{qu}$$

with

$$Z_{cl} = \sum_{m_1,m_2} e^{-S_{m_1,m_2}}$$

and Z_{qu} is obtained by performing the path-integral over x_{qu}.

Write Z_{cl} as

$$Z_{cl} = \sum_{m_1,m_2} \exp -\frac{1}{2}\pi R^2 \left(\frac{(m_2 - \tau_1 m_1)^2}{\tau_2} + \tau_2 m_1^2 \right)$$

6. Proof the Poisson resummation formula: if $\hat{f}$ is the Fourier transform of f

$$\hat{f}(y) = \int_{-\infty}^{\infty} dx e^{2\pi i x y} f(x)$$

then

$$\sum_{n \in Z} f(n) = \sum_{m \in Z} \hat{f}(m)$$

Hint: split the integral $\int_{-\infty}^{\infty}$ *into integrals* $\int_{n-1/2}^{n+1/2}$.

Give the formula for the Gaussian

$$f(x) = e^{-\frac{ax^2}{2} + bx}$$

7. We are now going to perform a Poisson resummation on the sum over m_2 in the expression you have found for Z_{cl}. Thereto we have to compute the Fourier transform of the function

$$f(x) = e^{-\frac{\pi R^2 (x - \tau_1 m_1)^2}{2\tau_2}}$$

Show that it is given by

$$\hat{f}(y) = R\sqrt{2\tau_2} \exp(2\pi i \tau_1 m_1 y - \frac{2\pi\tau_2 y^2}{R^2})$$

8. Now show that Z_{cl} can be written as

$$Z_{cl} = R\sqrt{2\tau_2} \sum_{p_L, p_R} e^{i\pi\tau p_L^2 - i\pi\overline{\tau} p_R^2}$$

Remark: we would expect the quantum mechanical partition function - since it has no winding numbers - to be independent of R. In fact it is not, since we included the constant mode in it. Ik we calculate the quantum mechanical wave function, the prefactor of $R\sqrt{\tau_2/2}$ turns out to be cancelled.