Extra exercise 5

In this exercise, we look at string theory compactified on a circle of self-dual radius $R = \sqrt{\alpha'}$. We will see that at this radius the theory has some symmetries that are not present at arbitrary radius.

We will denote the compactified dimension by X, and all other dimensions by X^μ, so $\mu = 0, \ldots, 24$. For the compact direction, the solutions to the equations of motion are of the form

$$X = x + \frac{\alpha'}{2} p_L(\tau + \sigma) + \frac{\alpha'}{2} p_R(\tau - \sigma) + \sum \text{oscillators}, \tag{1}$$

where the last term stands for the usual oscillator terms involving $\alpha_n, \tilde{\alpha}_n$. To compactify the theory on a circle, we require that the solutions satisfy

$$X(\tau, \sigma + 2\pi) = X(\tau, \sigma) + 2\pi w R, \tag{2}$$

where for now R is an arbitrary radius.

1) How does this requirement restrict the constants in the solution (1)? Quantum mechanically (think of a point particle on a circle), there is a second restriction on these constants; what is it? Write down the expression for p_L, p_R in terms of the two 'quantum numbers' you found. (In the rest of the exercise, we will denote the second quantum number by n)

2) Write the Virasoro constraints $L_0, \tilde{L}_0$ in terms of the quantum numbers n and w, the oscillators α_0^μ and of the oscillator number operator N. (Use the convention $p_L = \sqrt{(2/\alpha')}\alpha_0$.) What are the massless states in the theory for general R?

Now, we will set the compactification radius to $R = \sqrt{\alpha'}$.

3) What are the massless states in the theory at this special radius? If someone told you this theory has an $SU(2) \times SU(2)$ gauge symmetry, would you be surprised? Why not?

To show that this symmetry is indeed present, we define the currents

$$\begin{aligned} j^1(z) &= :\cos[2\alpha'^{-1/2} X_L(z)]: \\ j^2(z) &= :\sin[2\alpha'^{-1/2} X_L(z)]: \\ j^3(z) &= i\alpha^{-1/2}\partial X_L(z) \end{aligned} \tag{3}$$

and similarly for the right-moving part of the theory.

4) Calculate the singular terms in the operator product expansion of $j^2(z)j^3(0)$.

It can be shown that the general expression for the singular terms in the OPE's of these operators is

$$j^i(z)j^j(0) \sim \frac{\delta^{ij}}{2z^2} + i\frac{\epsilon^{ijk}}{z} j^k(0). \tag{4}$$

5) What does this mean for the commutator

$$[Q^i, Q^j] \tag{5}$$

between the corresponding charges? Recall that these charges are given by

$$Q^i = \oint j^i(z) dz, \tag{6}$$

where the integral is over any closed loop around the point $z = 0$. Does this algebra look familiar to you?

It turns out that the enhanced symmetry algebra of the theory at self-dual radius is even much larger: it is the so-called 'level one SU(2) algebra', which is an infinite-dimensional Lie algebra or Kac-Moody algebra. Those who are interested in reading more about this subject are refered to section 8.3 from Polchinski.