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Introduction to Conformal Field Theory*

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ABSTRACT

An elementary introduction to conformal field theory is given. Topics include free bosons and fermions, orbifolds, affine Lie algebras, coset conformal field theories, superconformal theories, correlation functions on the sphere, partition functions and modular invariance.

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1. Introduction

Conformal field theory has been an important tool in theoretical physics during the last fifteen years. Its origins can be traced back on the one hand to statistical mechanics, and on the other hand to string theory. Historically the most important impetus came from statistical mechanics, where it described and classified critical phenomena. Mainly after 1984 the subject went through a period of rapid development because of its importance for string theory. In addition there has been important input from mathematics, in particular through the work of Kac and collaborators. One can distinguish yet another separate origin of some ideas, namely from work on rigorous approaches to quantum field theory.

At present the subject still continues to develop, though more slowly, and it is still important in all the fields mentioned, plus a few additional branches of mathematics. These lectures are mainly on two-dimensional CFT. Recently conformal field theory appeared in yet another context, namely "AdS/CFT-correspondence", where also higher dimensional (super)conformal invariance is relevant.

I tried to include references to most relevant papers, but the emphasis was on papers I consider to be worth reading even today, and not on papers that are mainly of historical interest. A more detailed account of the history may be found in [1], which was used extensively for the preparation of these notes. In addition to the latter review, other useful general references include the one by J. Cardy from the same proceedings [2]. Other sources I used are [3] and [4]. Some useful results can be found in books on string theory, for example [5] and [6]. Standard reviews on Kac-Moody algebras are [7] and [8]. Finally I mention as a general reference the paper by Belavin, Polyakov and Zamolodchikov [9], which is the starting point of many recent developments.

2. Classical Conformal Invariance

In this section we study classical field theories in an arbitrary number of dimensions. In this space we have a metric $g_{\mu\nu}$. Furthermore we define $g = |\det g_{\mu\nu}|$. We will work in flat space, which means that the coordinates can be chosen in such a way that $g_{\mu\nu} = \eta_{\mu\nu}$, where the latter has the form diag $(-1, \ldots, -1, +1, \ldots + 1)$. The number of eigenvalues -1 or $+1$ is q and p respectively. Our convention is to use -1 in the time direction. Hence in practice q is either 0 (Euclidean space) or 1 (Minkowski space).

2.1. Symmetries

General coordinate invariance

Such theories may have a variety of symmetries. One symmetry that we will assume them to have is general coordinate invariance. Using the action principle this can be used to show that the energy momentum tensor is conserved. In general, this tensor is defined in terms of the variation of the action S under changes of the space-time metric

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu} \ .$$

Then the definition of the energy momentum tensor is

$$\delta S = \tfrac{1}{2} \int d^d x \sqrt{g}\ T^{\mu\nu} \delta g_{\mu\nu} \ . \tag{2.1}$$

If the theory is invariant under general coordinate transformations one can show that

$$(T^{\nu\mu})_{;\nu} = 0 \ .$$

Here (as usual in general relativity) ";ν" denotes a covariant derivative. In flat coordinates the condition reads $\partial_\nu T^{\nu\mu} = 0$.

Weyl invariance

We are not interested in general coordinate invariance, but in a different symmetry which can also be formulated in terms of the metric and the energy momentum tensor. This symmetry is called *Weyl invariance*. The transformation we consider is

$$g_{\mu\nu}(x) \to \Omega(x) g_{\mu\nu}(x) \ , \tag{2.2}$$

or in infinitesimal form

$$g_{\mu\nu} \to g_{\mu\nu}(x) + \omega(x) g_{\mu\nu}(x) \ .$$

The condition for invariance of an action under such a symmetry can also be phrased in terms of the energy momentum tensor. Substituting $\delta g_{\mu\nu} = \omega(x) g_{\mu\nu}(x)$ into (2.1) we find

$$\delta S = \tfrac{1}{2} \int d^d x \sqrt{g}\; T^{\mu}_{\ \mu} \omega(x) \ . \tag{2.3}$$

since this must be true for arbitrary functions ω we conclude that the condition for Weyl invariance is

$$T^{\mu}_{\ \mu} = 0 \ .$$

<u>*Conformal invariance*</u>

A *conformal transformation* can now be defined as a coordinate transformation which acts on the metric as a Weyl transformation. Consider a general coordinate transformation $x \rightarrow x'$, such that $x^{\mu} = f^{\mu}(x'^{\nu})$. This has the following effect on the metric

$$g_{\mu\nu}(x) \rightarrow g'_{\mu\nu}(x') = \frac{\partial f^{\rho}}{\partial x'^{\mu}} \frac{\partial f^{\sigma}}{\partial x'^{\nu}} g_{\rho\sigma}(f(x')) \ . \tag{2.4}$$

We are going to require that the left hand side is proportional to $g_{\mu\nu}$. Rotations and translations do not change the metric at all, and hence preserve all inner products $v \cdot w \equiv v^{\mu} g_{\mu\nu} w^{\nu}$. They are thus part of the group of conformal transformations. A coordinate transformation satisfying (2.2) preserves all angles, $\frac{v \cdot w}{\sqrt{v^2 w^2}}$ (hence the name 'conformal'). Later in this chapter we will determine all such transformations.

If a field theory has a conserved, traceless energy momentum tensor, it is invariant both under general coordinate transformations and Weyl transformations. Suppose the action has the form

$$S = \int d^d x \mathcal{L}(\partial_x, g_{\mu\nu}(x), \phi(x)) \ .$$

Here ϕ denotes generically any field that might appear, except for the metric which we have indicated separately since it plays a special rôle. We have also explicitly indicated space-time derivatives. General coordinate invariance implies that

$$S = S' \equiv \int d^d x' \mathcal{L}(\partial_{x'}, g'_{\mu\nu}(x'), \phi'(x'))$$

Here $g'_{\mu\nu}$ is as defined above, and the transformations of a field ϕ depends on its spin. If it is a tensor of rank n one has

$$\phi'_{\mu_1,\ldots,\mu_n}(x') = \frac{\partial f^{\nu_1}}{\partial x'^{\mu_1}} \cdots \frac{\partial f^{\nu_n}}{\partial x'^{\mu_n}} \phi_{\nu_1,\ldots,\nu_n}(f(x')) \tag{2.5}$$

In particular, for a scalar function $\phi(x)$ we find $\phi'(x') = \phi(f(x')$ and for the derivative of a scalar function we get

$$\frac{\partial}{\partial x^\mu}\phi(x) \to \frac{\partial}{\partial x'^\mu}\phi'(x') = \frac{\partial}{\partial x'^\mu}\phi(f(x')) = \frac{\partial f^\nu}{\partial x'^\mu}\frac{\partial}{\partial_{f^\nu}}\phi(f(x')) \ ,$$

i.e. it transforms like a vector (note, however, that n^{th} order ordinary derivatives do not transform like a tensor of rank n; this is only true if one uses covariant derivatives). If the coordinate transformation $x \to x'$ is of the special type (2.2) we can use Weyl invariance of the action to change the metric back into its original form. Then we have

$$S = S'' \equiv \int d^d x' \mathcal{L}(\partial_{x'}, g_{\mu\nu}(f(x')), \phi'(x')) \tag{2.6}$$

This is the conformal symmetry of the action. Note that the metric now remains unchanged if we start with a flat space metric $g_{\mu\nu} = \eta_{\mu\nu}$. This means that we can define the conformal transformation for theories in flat space that are not coupled to gravity. We may then forget about general coordinate invariance and start with an action in which no "dynamical" metric appears.

The statement of conformal invariance is then that the action of such a theory is unchanged if we integrate the same Lagrangian (or other physical scalar) expressed in terms of the new fields $\phi'(x')$ over the new coordinates x'. If the relation between the old coordinates x and the old ones x' is $x = f(x')$, then the new fields are related to the old fields as in (2.5)

The restriction to flat space is not really a restriction if we are in two dimensions. Then a general metric is given by three functions, $g_{11}(x)$, $g_{22}(x)$ and $g_{12}(x) = g_{21}(x)$. A general coordinate transformation allows us to change this using two functions, $f^1(x)$ and $f^2(x)$, and we can – generically – use this freedom to set $g_{12}(x) = 0$ and $g_{11}(x) = \pm g_{22}(x)$ (depending on the signature of the metric), so that the metric has the form $g(x)\eta_{\mu\nu}$. This is called *conformal gauge*. Then, using a Weyl transformation, we can remove the function $g(x)$ and bring the metric to the form $\eta_{\mu\nu}$. In more than two dimensions we do not have enough freedom to do this, and then the assumption made here is really a restriction to non-gravitational theories in flat space.

On a given two-dimensional manifold the conformal gauge choice can be made locally, but usually not globally. This means that we will be able to use conformal field theory in some coordinate patch, but that additional data may be needed to describe the theory globally.

Fields that transform like (2.5) are called *conformal fields*, or also *primary fields*.

2.2. Conformal transformations in d dimensions

In general the right hand side of (2.4) is of course not proportional to the original metric $g_{\mu\nu}$. To study when it is, consider the infinitesimal transformation $x'^{\mu} = x^{\mu} + \epsilon^{\mu}(x)$ (or rather its inverse, $x^{\mu} = x'^{\mu} - \epsilon^{\mu}(x') + O(\epsilon^2)$. Then

$$\frac{\partial x^{\rho}}{\partial x'^{\mu}} = \delta^{\rho}_{\mu} - \partial_{\mu}\epsilon^{\rho} \ ,$$

and

$$\delta g_{\mu\nu} = -\partial_{\mu}\epsilon_{\nu} - \partial_{\nu}\epsilon_{\mu}$$

This must be equal to $\omega g_{\mu\nu}$. Taking the trace we see then that $\omega = -\frac{2}{d}\partial\cdot\epsilon$ (with $\partial\cdot\epsilon \equiv \partial^{\mu}\epsilon_{\mu}$), so that we get the following equation for ϵ

$$\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = \frac{2}{d}\partial\cdot\epsilon g_{\mu\nu} \tag{2.7}$$

Let us now analyze the solutions to this condition. As a first step, we contract both sides with $\partial^{\mu}\partial^{\nu}$. This yields

$$(1-\frac{1}{d})\Box\partial\cdot\epsilon = 0$$

If $d>1$ this implies that

$$\Box\partial\cdot\epsilon = 0 \tag{2.8}$$

(for $d=1$ (2.7) is satisfied for any ϵ). Next we contract (2.7) with $\partial_{\rho}\partial^{\nu}$. This yields

$$\Box\partial_{\rho}\epsilon_{\mu} + (1-\frac{2}{d})\partial_{\rho}\partial_{\mu}\partial\cdot\epsilon = 0 \ .$$

To this we add the same equation with ρ and μ interchanged, we use (2.7) once more and finally (2.8). The result is

$$(1-\frac{2}{d})\partial_{\rho}\partial_{\mu}\partial\cdot\epsilon = 0 \ . \tag{2.9}$$

We conclude that $\partial_{\rho}\partial_{\mu}\partial\cdot\epsilon = 0$ if $d>2$. The third (and last) step is to take the uncontracted derivative $\partial_{\rho}\partial_{\sigma}$ of (2.7). Define $F_{\rho\sigma\mu\nu} \equiv \partial_{\rho}\partial_{\sigma}\partial_{\mu}\epsilon_{\nu}$. This function is manifestly symmetric in the first three indices. Furthermore, by acting with $\partial_\rho\partial_\sigma$ on (2.7) and using (2.9) we find (for $d > 2$)

$$F_{\rho\sigma\mu\nu} = -F_{\rho\sigma\nu\mu} \ . \tag{2.10}$$

It is now easy to show that a tensor with these symmetries must vanish:

$$F_{\rho\sigma\mu\nu} = F_{\rho\mu\sigma\nu} = -F_{\rho\mu\nu\sigma} = -F_{\rho\nu\mu\sigma} = F_{\rho\nu\sigma\mu} = F_{\rho\sigma\nu\mu} \ ,$$

which contradicts with (2.10) unless $F_{\rho\sigma\mu\nu} = 0$.

Hence we find that for $d > 2$ the full, uncontracted third order derivative of ϵ must vanish, so that it can be of at most second order in x. Therefore we may write

$$\epsilon^\mu(x) = \alpha^\mu + \beta^\mu_\nu x^\nu + \gamma^\mu_{\nu\rho} x^\nu x^\rho \ .$$

Substituting this into (2.7) and collecting the terms of the same order in x we find the conditions

$$\begin{aligned} \beta_{\mu\nu} + \beta_{\nu\mu} &= \frac{2}{d}\beta^\rho{}_\rho g_{\mu\nu} \\ \gamma_{\mu\nu\sigma} + \gamma_{\nu\mu\sigma} &= \frac{2}{d}\gamma^\rho_{\rho\sigma} g_{\mu\nu} \end{aligned}$$

The first one can be solved by splitting $\beta_{\mu\nu}$ into a symmetric and an anti-symmetric part,

$$\beta_{\mu\nu} = \omega_{\mu\nu} + S_{\mu\nu} \ .$$

There is no condition on the anti-symmetric part $\omega_{\mu\nu}$, whereas the symmetric part is found to be proportional to $g_{\mu\nu}$, $S_{\mu\nu} = \sigma g_{\mu\nu}$.

The equation for the quadratic part is somewhat harder to solve. Using the fact that $\gamma_{\mu\nu\sigma}$ is symmetric in the last two indices, we can derive

$$\begin{aligned} \gamma_{\mu\nu\sigma} &= -\gamma_{\nu\mu\sigma} - 2b_\sigma g_{\mu\nu} = -\gamma_{\nu\sigma\mu} - 2b_\sigma g_{\mu\nu} \\ &= \gamma_{\sigma\nu\mu} - 2b_\sigma g_{\mu\nu} + 2b_\mu g_{\nu\sigma} = \gamma_{\sigma\mu\nu} - 2b_\sigma g_{\mu\nu} + 2b_\mu g_{\nu\sigma} \\ &= -\gamma_{\mu\sigma\nu} - 2b_\sigma g_{\mu\nu} + 2b_\mu g_{\nu\sigma} - 2b_\nu g_{\mu\sigma} \\ &= -\gamma_{\mu\nu\sigma} - 2b_\sigma g_{\mu\nu} + 2b_\mu g_{\nu\sigma} - 2b_\nu g_{\mu\sigma} \ , \end{aligned}$$

where $b_\mu = -\frac{1}{d}\gamma^\rho{}_{\rho\mu}$. Therefore

$$\gamma_{\mu\nu\sigma} = -b_\sigma g_{\mu\nu} + b_\mu g_{\nu\sigma} - b_\nu g_{\mu\sigma} \ ,$$

where b_ν is an arbitrary constant vector.

2.3. The conformal group

Most of the transformations we have obtained can easily be identified

- Translations: $x^\mu \to x^\mu + \alpha^\mu$
- (Lorentz) Rotations: $x^\mu \to x^\mu + \omega^\mu{}_\nu x^\nu$
- Scale transformations: $x^\mu \to x^\mu + \sigma x^\mu$

The last transformation is perhaps less familiar and is called a

- Special conformal transformation: $x^\mu \to x^\mu + b^\mu x^2 - 2x^\mu b \cdot x$

Note that $\omega_{\mu\nu}$ is antisymmetric, and this might suggest that it is the parameter of a rotation rather than a Lorentz transformation. However, the correct infinitesimal parameters of the transformation are $\omega^\mu{}_\nu$. Numerically (*i.e.* ignoring its tensor properties) this is equal to the matrix $\Omega = g^{-1}\omega$, which satisfies $\Omega^T = \omega^T g^{T-1} = -\omega g^{-1} = -g\Omega g^{-1}$. Hence $g\Omega + \Omega^T g = 0$, so that to first order in Ω, $(1+\Omega^T)g(1+\Omega) = g$. Hence Ω is indeed an infinitesimal Lorentz transformation.

These are all still in infinitesimal form, but it is fairly straightforward to write their global version. In addition to translations and $SO(p,q)$ Lorentz transformations (or rotations if $q=0$) one has the scale transformation $x \to x' = \lambda x$. The global version of the special conformal transformation has the form

$$x^\mu \to x'^\mu = \frac{x^\mu + b^\mu x^2}{1 + 2b \cdot x + b^2 x^2} \; .$$

The latter transformation can be made a little more intuitive by observing that it can be obtained by the sequence $x \to I(x)$, $x \to x + b$, $x \to I(x)$, where $I(x)$ denotes the space-time inversion $x^\mu \to x^\mu / x^2$. The space-time inversion can be thought of as a global conformal transformation. It preserves angles, but it obviously doesn't have an infinitesimal form, and therefore there is no parameter ϵ_μ corresponding to it, and we didn't find it in the previous analysis.

One can study the action of the infinitesimal conformal transformations on a space of functions of x. For each transformation $x \to x' = x + \epsilon(x)$ one can define a differential operator O_ϵ so that the transformation of a function $f(x)$ is $f(x) \to f(x) + O_\epsilon f(x)$. Clearly these operators are

$$\begin{aligned} P_\mu &= \partial_\mu \\ M_{\mu\nu} &= \tfrac{1}{2}(x_\mu \partial_\nu - x_\nu \partial_\mu) \\ D &= x^\mu \partial_\mu \\ K_\mu &= x^2 \partial_\mu - 2x_\mu x^\nu \partial_\nu \end{aligned} \tag{2.11}$$

These operators are to be contracted respectively with α^μ, $\omega^{\mu\nu}$, σ and b^μ. One can write down the commutators of the operators P, M, D and K, and check that they form a closed algebra which is isomorphic to $SO(p+1, q+1)$.

2.4. The conserved current

Usually symmetries imply the existence of conserved currents. The current of conformal symmetry is

$$J_\mu(\epsilon) = T_{\mu\nu}\epsilon^\nu \tag{2.12}$$

This current is conserved because

$$\partial^\mu J_\mu(\epsilon) = (\partial^\mu T_{\mu\nu})\epsilon^\nu + T_{\mu\nu}((\partial^\mu \epsilon^\nu) ,$$

which vanishes because of (2.7), and because the energy momentum tensor is conserved and traceless.

2.5. The free boson

A standard example is the free boson. The (Euclidean) action is

$$S = \tfrac{1}{2}\int d^d x \sqrt{g} g^{\mu\nu}\partial_\mu \Phi(x)\partial_\nu \Phi(x) , \tag{2.13}$$

where $g \equiv \det g$. To compute the energy momentum tensor we need the variation of $g^{\mu\nu}$ and $\sqrt{g}$, given $\delta g_{\mu\nu}$. To get the former, use $0 = \delta(g^{\mu\nu}g_{\nu\rho})$ to derive $\delta g^{\mu\nu} = -g^{\mu\rho}g^{\nu\sigma}\delta g_{\rho\sigma}$. The second variation is derived as follows

$$\delta\sqrt{g} = \delta e^{\frac{1}{2}\log g} = \tfrac{1}{2}\sqrt{g}\delta\log(g)$$

with

$$\begin{aligned}\delta \log g &= \log\ \det\ (g_{\mu\nu} + \delta g_{\mu\nu}) - \log\ \det\ g_{\mu\nu} \\ &= \log\ \det\ [\delta^\mu_\rho + g^{\mu\nu}\delta g_{\nu\rho}] = \ \mathrm{Tr}\ g^{\mu\nu}\delta g_{\nu\rho} + O(\delta^2) ,\end{aligned}$$

where in the last step the identity $\det\ A = \exp\ \mathrm{Tr}\ \log A$ was used, and the log was expanded in $g^{\mu\nu}\delta g_{\nu\rho}$. In the first and second line the arguments of "det" and "Tr" are matrices with indices (μ,ν) and (μ,ρ) respectively. Putting this all together, and using (2.1) we get

$$T_{\mu\nu} = -\partial_\mu\Phi\partial_\nu\Phi + \tfrac{1}{2}g_{\mu\nu}g^{\rho\sigma}\partial_\rho\Phi\partial_\sigma\Phi \ .$$

It is straightforward to check that $\partial^\mu T_{\mu\nu} = 0$ and that $T^\mu_{\ \mu} \propto (1-\frac{d}{2})$, so that the theory is conformally invariant if (and only if) $d = 2$. Note that to prove $\partial^\mu T_{\mu\nu} = 0$ one has to use equation of motion $\Box\Phi = 0$, whereas tracelessness for $d = 2$ holds also if the equation of motion is not satisfied.

A theory with classical conformal invariance in four dimensions is Yang-Mills theory (both abelian and non-abelian). The verification is left as an exercise.

One may also directly check Weyl invariance of the free bosonic theory. If we transform $g_{\mu\nu}$ to $\Omega(x)g_{\mu\nu}$, the square root of the determinant aquires a factor $\Omega^{d/2}$. This precisely cancels the factor Ω^{-1} from the transformation of $g^{\mu\nu}$ (the inverse of $g_{\mu\nu}$), in two dimensions.

By contrast, a *conformal* transformation does not act on the metric, but changes the integration variables and the derivatives. The simplest non-trivial example is a scale transformation $x = f(x') = \lambda x'$. Then $d^d x' = \lambda^{-d} dx'$ and the new field is $\partial_{x'}\Phi'(x') = (\partial f/\partial_{x'})\partial_f\Phi(f(x')) = \lambda\partial_x\Phi(x)$, where $\Phi'(x') \equiv \Phi(f(x')) = \Phi(\lambda x') = \Phi(x)$. For $d = 2$ the explicit λ-dependence cancels out, and this demonstrates part of the conformal symmetry of the action.

It should be clear that adding a mass term $\int d^2 x m^2\Phi^2$ to the theory breaks conformal invariance. Quantum effects also tend to spoil conformal invariance. Generically they introduce renormalization scale dependence of physical parameters (such as coupling constants) which destroys invariance under scale transformations $q \to \lambda q$ in momentum space. This does indeed happen for Yang-Mills theories, except when the β function vanishes so that the coupling constant is scale independent. The latter occurs for $N = 4$ super-Yang-Mills theory in four dimensions.

2.6. The conformal algebra in two dimensions

In two dimensions the restriction that $\epsilon(x)$ is of at most second order in x does not apply. One can analyze (2.7) directly by writing it out in components. If one does that in Euclidean space, $g_{\mu\nu} = \delta_{\mu\nu}$, one finds

$$\partial_1\epsilon_1 = \partial_2\epsilon_2, \quad \partial_1\epsilon_2 = -\partial_2\epsilon_1 \ .$$

Going to complex variables,

$$\begin{aligned} \epsilon &= \epsilon_1 - i\epsilon_2, \quad \bar\epsilon = \epsilon_1 + i\epsilon_2 \\ z &= x^1 - ix^2, \quad \bar z = x^1 + ix^2 \end{aligned} \tag{2.14}$$

we find

$$\partial_z\bar\epsilon(z,\bar z) = 0; \quad \partial_{\bar z}\epsilon(z,\bar z) = 0 \ ,$$

with $\partial_z \equiv \frac{\partial}{\partial z}$ and analogously for $\bar z$. The general solution to these conditions is that ϵ is an arbitrary function of z (which does not depend on $\bar z$) and $\bar\epsilon$ an arbitrary function of $\bar z$.

The corresponding global transformation is

$$z \to f(z), \qquad \bar{z} \to \bar{f}(\bar{z}) \ ,$$

where $f(z)$ is an arbitrary function of z.

The generators for the infinitesimal transformations can be introduced exactly as before:

$$L_n = -z^{n+1}\partial_z \tag{2.15}$$

generates the transformation

$$z \to z' = z - z^{n+1} \ ,$$

and satisfies the commutation relation

$$[L_n, L_m] = (n-m)L_{m+n} \ .$$

The same holds for the barred quantities, and furthermore one has then

$$[L_n, \bar{L}_m] = 0 \ .$$

The resulting infinitesimal transformations are the most general ones that are analytic near the point $z = 0$. They may introduce poles at $z = 0$, but not branch cuts. We will see later that we will often need contour integrals around $z = 0$, and this is the justification for this restriction.

The generator of an arbitrary conformal transformation is thus

$$\sum_n \left(\epsilon_n L_n + \bar{\epsilon}_n \bar{L}_n\right) \ . \tag{2.16}$$

This operator generates conformal transformations of functions $f(z, \bar{z})$. If we want this transformation to respect complex conjugation of z, we must require that $\bar{\epsilon}_n$ is the complex conjugate of ϵ_n. In that case we can rewrite (2.16) as

$$\sum_n \tfrac{1}{2}\left(\text{ Re } \epsilon_n(L_n + \bar{L}_n) + \text{ Im } \epsilon_n i(L_n - \bar{L}_n)\right)$$

This is in fact the algebra written in terms of the original real coordinates x^1 and x^2 (for example, $L_{-1} + \bar{L}_{-1} = \partial_1$ and $i(L_{-1} - \bar{L}_{-1}) = \partial_2$).

2.7. Complexification and Wick rotation

Usually this reality condition is dropped, and one treats ϵ and $\bar{\epsilon}$ as independent complex parameters. Then the algebra does not map $(x^1, x^2) \in \mathbf{R}^2$ to another point in $\mathbf{R}^2$, but it is a well-defined map on $\mathbf{C}^2$. This is justified if we define our field theory on a complex instead of a real space-time. This allows us to treat the two commuting algebras generated by L_n and $\bar{L}_n$ independently (*i.e.*we may now set $\bar{\epsilon}_n = 0, \epsilon_n \neq 0$ or vice-versa). Even if we are ultimately only interested in the restriction to a real vector space, we can always impose the reality condition at the end.

Note that the distinction between Euclidean space and Minkowski space become irrelevant if we complexify the coordinates. The complex coordinate transformation $x^0 = -ix^2$ changes $\eta_{\mu\nu}$ to $\delta_{\mu\nu}$ [Our convention is to use indices $(0, \ldots d-1)$ in d dimensional Minkowski space, with x^0 as the time coordinate, and $(1, \ldots, d)$ in Euclidean space, with $x^2 = ix^0 (= -ix_0)$. Consequently the indices on δ and η have a different range.] This is known as a *Wick rotation.*

We are usually interested in conformal field theories in Minkowski space, but it is convenient to make use of the powerful theorems that are available for complex functions. For that reason one usually makes a Wick rotation to Euclidean space, which in its turn is mapped to the complex plane. This is not an obviously innocuous transformation though. The Wick rotation changes (in fact, improves) the convergence properties of quantities such as the path integral or the propagator in the quantum theory, which is why it is often used in field theory in four dimensions as well. One has to assume or, if possible, prove that the relevant quantities can indeed by analytically continued to Euclidean space, and if there are singularities one has to find a way to avoid them.

2.8. The global subgroup

An interesting subalgebra of the algebra is the one generated by L_{-1}, L_0, L_1 and their conjugates. This algebra – or rather its restriction to real generators, as discussed above – is isomorphic to $SO(3,1)$, which is precisely the naively expected conformal group $SO(p+1, q+1)$, if one extrapolates from arbitrary d to $d = 2$ (in Euclidean space, with $p = 2$ and $q = 0$).

The precise identification can easily be derived by transforming back to the standard Euclidean coordinates x^1, x^2. The precise relation with the operators defined in (2.11) is

$$\begin{aligned}
P_1 &= \partial_z + \partial_{\bar{z}} = -(L_{-1} + \bar{L}_{-1}) \\
P_2 &= -i(\partial_z - \partial_{\bar{z}}) = i(L_{-1} - \bar{L}_{-1}) \\
M &= -i(z\partial_z - \bar{z}\partial_{\bar{z}}) = i(L_0 - \bar{L}_0) \\
D &= z\partial_z + \bar{z}\partial_{\bar{z}} = -(L_0 + \bar{L}_0) \\
K_1 &= -z^2\partial_z - \bar{z}^2\partial_{\bar{z}} = L_1 + \bar{L}_1 \\
K_2 &= -i(z^2\partial_z - \bar{z}^2\partial_{\bar{z}}) = i(L_1 - \bar{L}_1)
\end{aligned} \tag{2.17}$$

The algebra satisfied by the holomorphic generators is

$$\begin{aligned}[][L_0, L_{-1}] &= L_{-1} \\ [L_0, L_1] &= -L_1 \\ [L_1, L_{-1}] &= 2L_0\end{aligned}$$

This is precisely the $SU(2)$ rotation algebra if we identify L_0 with J_z, iL_1 with $J^- = J_x - iJ_y$ and iL_{-1} with $J^+ = J_x + iJ_y$. The factor i is essential to compensate the sign in $[J^-, J^+] = -2J^0$.

The $SO(3,1)$ generators are the only ones that are globally defined on the complex plane *including* ∞ (this is called the *Riemann sphere*). Clearly the generator $-z^{n+1}\partial_z$ is non-singular at $z = 0$ for $n \geq -1$. To investigate the behavior at infinity it is convenient to make a conformal mapping that interchanges the points $z = 0$ and $z = \infty$. A conformal map that does this is $z = \frac{1}{w}$. Under this transformation the generator L_n transforms to

$$-z^{n+1}\partial_z \rightarrow -w^{-(n+1)}[\frac{dz}{dw}]^{-1}\partial_w = +w^{1-n}\partial_w$$

This operator is non-singular for $n \leq 1$, which combined with the range obtained above leaves $-1 \leq n \leq 1$. For these values of n the generators are defined on the Riemann sphere.

The infinitesimal and global forms of the transformations are as follows

generator	*local transformation*	*global transformation*
$-\epsilon L_{-1}$	$z \rightarrow z + \epsilon$	$z \rightarrow z + \alpha$
$-\epsilon L_0$	$z \rightarrow z + \epsilon z$	$z \rightarrow \lambda z$
$-\epsilon L_1$	$z \rightarrow z + \epsilon z^2$	$z \rightarrow \frac{z}{1-\beta z}$

Combining these transformations we get

$$z \rightarrow \frac{az+b}{cz+d}, \text{ with } ad - bc = 1 \tag{2.18}$$

Note that there are only three independent transformations, and hence there should be only three parameters. This is why we can impose the condition $ad - bc = 1$.* Doing it this way and taking the parameters $\in \mathbf{C}$, we see that the action is that of the group $SL_2(\mathbf{C})/\mathbf{Z}_2$.

* Note that the scale transformation is given by $a = \sqrt{\lambda}, d = 1/\sqrt{\lambda}, b = c = 1$.

The group $SL_2(\mathbf{C})$ is the set of 2×2 complex matrices with determinant 1. The most general such matrix is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \text{with } ad-bc=1 \ .$$

To get the transformation shown above we make it act on complex two-dimensional vectors $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, with vectors related by an overall complex scale identified. In this space only the ratio $z=z_1/z_2$ is a free parameter, and that parameter is easily seen to transform as in (2.18). The transformation (2.18) is clearly unchanged if we multiply the matrix by an overall factor. This freedom is fixed by the determinant condition, except for an overall sign. Therefore the correct group action is $SL_2(\mathbf{C})/\mathbf{Z}_2$ rather than $SL_2(\mathbf{C})$.

In combination with the transformation of the anti-holomorphic sector we get then the group of transformations $SL_2(\mathbf{C})\times SL_2(\mathbf{C})$. This contains as a subgroup $SO(3,1)$, the expected global conformal group, but in terms of real generators $SL_2(\mathbf{C})\times SL_2(\mathbf{C})$ is twice as large as $SO(3,1)$. The reason is of course that we allow the two $SL_2(\mathbf{C})$ transformations to act independently on z and $\bar{z}$. If we impose a reality condition (*i.e.* $\bar{z}$ is the complex conjugate of z) we reduce the number of generators to that of $SO(3,1)$.

2.9. Tensors in complex coordinates

Tensors can be transformed to complex coordinates using the transformation formula (2.5). One should be careful with factors of two in these transformations. A potential source of confusion is the fact that often the same notation is used for coordinates and indices: $z\equiv x^z$; $\bar{z}\equiv x^{\bar{z}}$. The transformation is

$$z\equiv x^z=x^1-ix^2 \quad ;\bar{z}\equiv x^{\bar{z}}=x^1+ix^2$$

The inverse is then

$$z\equiv x^1=\tfrac{1}{2}(x^z+x^{\bar{z}})=\tfrac{1}{2}(z+\bar{z}) \quad ;x^2=\tfrac{1}{2}i(x^z-x^{\bar{z}})=\tfrac{1}{2}i(z-\bar{z})$$

The transformation of a vector to complex coordinates goes as follows

$$V_z=\frac{\partial x^1}{\partial x^z}V_1+\frac{\partial x^2}{\partial x^z}V_2=\tfrac{1}{2}(V_1+iV_2);\quad V_{\bar{z}}=\tfrac{1}{2}(V_1-iV_2) \tag{2.19}$$

The generalization to higher rank tensors is obvious.

The metric $g_{\mu\nu} = \delta_{\mu\nu}$ transforms to $g_{zz} = \frac{1}{4}(g_{11} + ig_{12} + ig_{21} - g_{22}) = 0 = g_{\bar{z}\bar{z}}$, $g_{z\bar{z}} = \frac{1}{4}(g_{11} + ig_{12} - ig_{21} + g_{22}) = \frac{1}{2} = g_{\bar{z}z}$. Hence $g^{z\bar{z}} = g^{\bar{z}z} = 2$. This leads to the akward-looking relation $\partial^{\bar{z}} = 2\partial_z$.†

The same kind of relation holds for the energy-momentum tensor. Since T is traceless we have $T_{11} + T_{22} = 0$, and hence $T_{z\bar{z}} = 0$.

The metric allows us to convert every upper index to a lower one (or vice-versa) at the expense of a simple numerical factor, 2 or $\frac{1}{2}$. One can make use of this freedom to avoid counter-intuitive quantities such as x_z or ∂^z. From now on all tensors and derivatives will be written with lower indices, and all coordinates with upper indices. The latter will be denoted as z or $\bar{z}$.

Conservation of the energy momentum tensor now reads (since $T_{z\bar{z}} = 0$)

$$\partial_{\bar{z}} T_{zz} = \partial_z T_{\bar{z}\bar{z}} = 0 \ ,$$

which implies that T_{zz} is holomorphic and $T_{\bar{z}\bar{z}}$ anti-holomorphic.‡

The infinitesimal parameter for the conformal transformations, ϵ, has been transformed to complex components in (2.14). This definition also requires a bit of care. If we use (2.19) we get

$$\epsilon_z = \tfrac{1}{2}(\epsilon_1 + i\epsilon_2) = \tfrac{1}{2}\bar{\epsilon} \ ,$$

where in the last step the definition (2.14) was used. The "bar" on the right-hand side may look out of place, but the notation was chosen in (2.14) because $\bar{\epsilon}$ is a function only of $\bar{z}$.

The conserved current of conformal symmetry is defined analogously to (2.12)

$$J_\mu(\epsilon) = T_{\mu\nu}\epsilon^\nu \Longrightarrow J_z = 2T_{zz}\epsilon_{\bar{z}} = T_{zz}\epsilon(z); \quad J_{\bar{z}} = T_{\bar{z}\bar{z}}\bar{\epsilon}(\bar{z}) \ . \tag{2.20}$$

Since J_z is holomorphic and $J_{\bar{z}}$ anti-holomorphic, this current is manifestly conserved: $\partial_{\bar{z}} J_z = \partial_z J_{\bar{z}} = 0$

---

† One should keep in mind here that ∂_z can have two meanings, namely the "derivative with respect to the coordinate x^z" ($\partial/\partial x^z$) or the "derivative with respect to the variable z" ($\partial/\partial z$.) Fortunately these two meanings are the same. On the other hand $\partial^{\bar{z}}$ can only mean "derivative with respect to the coordinate $x_{\bar{z}}$", and has nothing to do with a derivative with respect to $\bar{z}$.

‡ The word "holomorphic" has become standard terminology for "depending only on z, not on $\bar{z}$". It does not imply absence of singularities. Mathematicians might prefer the word "meromorphic".

2.10. Conformal Fields in two dimensions

The components of a tensor ϕ of rank n are of the form $\phi_{z\ldots z,\bar{z}\ldots\bar{z}}(z,\bar{z})$. It is easy to see that under conformal transformations this transforms into

$$(\frac{\partial f(z)}{\partial z})^p(\frac{\partial \bar{f}(\bar{z})}{\partial \bar{z}})^q\phi_{z\ldots z,\bar{z}\ldots\bar{z}}(f(z),\bar{f}(z)) ,$$

where p is the number of indices 'z', and $q=n-p$ the number of indices $\bar{z}$. A field that transforms in this way is called a conformal field of weight (p,q).

This rule was derived here for a tensor field, and one may think that p and q should be integers. However, any real value is in fact (*a priori*) allowed. Usually the conformal weight is denoted as $(h,\bar{h})$, where the bar does *not* mean complex conjugation (both numbers are real), but only serves to distinguish the two numbers. Sometimes $h+\bar{h}$ is called the scaling weight, and $h-\bar{h}$ the conformal spin. As we will see later, these are in fact the eigenvalues of (minus) the dilation operator $-D=L_0+\bar{L}_0$ and the $SO(2)$ rotation operator $-iM=L_0-\bar{L}_0$.

2.11. Relation to string theory

Closed bosonic strings are described by means of the bosonic action (in Minkowski space, with $g_{00}=-1$ and $g_{11}=1$)

$$S=-\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-g}g^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X_\mu \ , \tag{2.21}$$

defined on a two-dimensional surface with the topology of a cylinder (for the non-interacting, closed string, at least). Here $X^\mu(\sigma_0,\sigma_1)$ is a map from two dimensional space (called the "world-sheet") to space-time (often called "target space"), and α' is the "Regge-slope parameter". This function defines the embedding of the string in space time, as a function of the proper time σ_0, *i.e.* is specifies where a point σ_1 along the string is located at proper time σ_0. If we take a flat two-dimensional metric $g^{\alpha\beta}$ and a Euclidean flat metric in target space this action is nothing but the action of a free boson in two dimensions. The conformal invariance of that action plays an important rôle in the proper quantization of string theory in Minkowski space (note that X^0 appears then with the "wrong" sign in the two-dimensional action). Furthermore conformal field theory has been used to find alternatives to the free boson action that can be interpreted as consistent string theories.

2.12. Free bosons in complex coordinates

In complex coordinates the free boson action takes the form

$$S = \frac{1}{2\pi\alpha'} \int dz d\bar{z} \partial_z \Phi(z,\bar{z}) \partial_{\bar{z}} \Phi(z,\bar{z}) \tag{2.22}$$

Our convention is that $dzd\bar{z} = 2dx^1dx^2$ (which is indeed what one gets from the Jacobian, but some authors prefer to omit the factor "2"). This complex form of the action is derived from the Euclidean action (2.13) multiplied with an additional factor $1/2\pi\alpha'$, so that one gets the Euclidean action corresponding to (2.21). The factor in front is conventional in string theory.

According to the definition of conformal fields given earlier, $\partial\Phi$ is a conformal field. In complex coordinates its transformation properties are

$$\partial_z \Phi(z,\bar{z}) \rightarrow \frac{\partial f(z)}{\partial z} \partial_f \Phi(f(z), \bar{f}(\bar{z}))\ .$$

For the energy momentum tensor we find, in complex coordinates

$$T_{zz} = -\tfrac{1}{2}(\partial_z \Phi(z))^2$$

and analogously for $T_{\bar{z}\bar{z}}$.

The normalization contains some conventions originating from string theory, namely

— The factor $\frac{1}{2\pi\alpha'}$, as above

— An extra factor 2π multiplying the definition of $T^{\mu\nu}$ (*cf.* footnote on page 24 of part 1 of [5]).

— The convention $\alpha' = 2$.

3. Quantum Conformal Invariance

As discussed in the previous chapter, the theories we consider are defined in Euclidean space, usually obtained after a Wick rotation from Minkowski space. For computational convenience (in particular because some quantities separate into holomorphic and anti-holomorphic parts) we then go to the complex plane. It turns out that to simplify things even more it is convenient to make yet another map, this time a conformal transformation of the complex plane itself.

3.1. Radial quantization

Symmetries in the quantum theory are usually generated by charges, which are space integrals of the zeroth component of a conserved current J^μ, $\partial_\mu J^\mu = 0$. The definition of a charge in a d-dimensional theory is then

$$Q = \int d^{d-1}x J^0(x,t) \ .$$

In the two-dimensional analog of this just one has a one-dimensional integration over x^1.

It is convenient to make the space direction finite, by imposing periodic boundary conditions in the x^1 direction. This is like regulating a quantum system by putting it in a finite box in space. In this case the size of the box will be fixed for convenience to the value 2π, but since the theory is scale invariant that is irrelevant. The Euclidean coordinates $(x^1, x^2) = (x^1, ix^0)$ can then be thought of coordinates on a cylinder. In this situation we get then the following expression for the charge (conveniently normalized to the length of the interval)

$$Q = \frac{1}{2\pi}\int_0^{2\pi} dx^1 J^0 = \frac{1}{2\pi}\int_0^{2\pi} dx^1(-iJ_2) \ .$$

Now we introduce a complex coordinate $z = x^1 - ix^2$ as discussed before. From (2.19) we find that $J_2 = -i(J_z - J_{\bar{z}})$. The charge becomes now

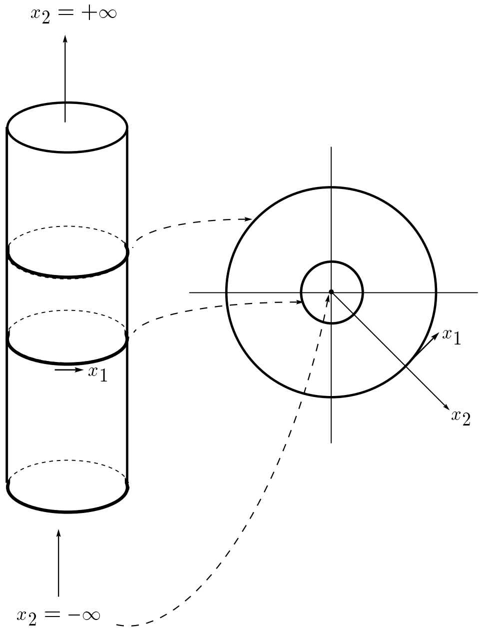
$$Q = -\frac{1}{2\pi}[\oint dz J_z^{\mathrm{cyl}}(z,\bar{z}) - \oint d\bar{z} J_{\bar{z}}^{\mathrm{cyl}}(z,\bar{z})]$$

Here the integration is along a closed contour that encircles the cylinder. For convenience we choose $\bar{z}$ rather than z as the integration variable in the second term. Since we only integrate over Re z this makes no difference. The orientation of these contours is such that $\oint dz = \oint d\bar{z} = 2\pi$. The superscripts "cyl" are added to remind ourselves that the currents are defined on the cylinder.

It turns out to be convenient to perform a conformal transformation

$$w = e^{x^2 + ix^1} = e^{iz} \ . \tag{3.1}$$

Then the surface at the Euclidean time coordinate $x^2 = -\infty$ is mapped to $w = 0$, and the surface at $x^2 = +\infty$ is mapped to the infinite circle at $|w| = \infty$.



To go to the new coordinates we first make a change of integration variables. Then

$$\oint dz = \oint \frac{dw}{iw}; \quad \oint d\bar{z} = \oint \frac{d\bar{w}}{i\bar{w}} \ .$$

It is clear from the picture that in the new coordinates the dz integration becomes a contour integration around the origin. The definition of the integration volume for $d\bar{w}$ implies a choice of orientation for the $\bar{w}$ contour. Rather than remembering the direction of the contour, it is easier to remember the corresponding results for the Cauchy integrals

$$\frac{1}{2\pi i} \oint \frac{dw}{w} = \frac{1}{2\pi i} \oint \frac{d\bar{w}}{\bar{w}} = 1$$

$$Q = -\frac{1}{2\pi} [\oint \frac{dw}{iw} (J_z^{\rm cyl}(z(w), \bar{z}(\bar{w})) - \oint \frac{d\bar{w}}{i\bar{w}} J_{\bar{z}}^{\rm cyl}(z(w), \bar{z}(\bar{w}))]$$

This formula has the disadvantage that Q is still expressed in terms of operators defined on the cylinder. These operators are related to those on the plane by a conformal transformation:

$$\phi^{\rm plane}(w,\bar w) = (\frac{\partial z}{\partial w})^h (\frac{\partial \bar z}{\partial \bar w})^{\bar h} \phi^{\rm cyl}(z(w),\bar z(\bar w))$$

For the transformation considered here this implies

$$\phi^{\rm cyl}(z(w),\bar z(\bar w)) = (iw)^h (-i\bar w)^{\bar h} \phi^{\rm plane}(w,\bar w)$$

The current components considered here, J_z and $J_{\bar z}$ transform as vectors; therefore they have conformal weights $(h,\bar h)=(1,0)$ and $(h,\bar h)=(0,1)$ respectively. For the charge we find then

$$Q = -\frac{1}{2\pi}[\oint dw (iw)^{h-1} J_w^{\rm plane}(w,\bar w) + \oint d\bar w (-i\bar w)^{\bar h-1} J_{\bar w}^{\rm plane}(w\bar w)]$$

Here $h=\bar h=1$, but we left these parameters in the formula for future purposes. We have already seen operators that transform with $h=2$, namely the conformal currents $T_{zz}\epsilon(z)$ (note that only T_{zz} transforms as a conformal field, not $\epsilon(z)$).

Usually the current splits into holomorphic and anti-holomorphic parts, so that we may write $J_z(z,\bar z) \equiv J(z)$ and $J_{\bar z}(z,\bar z) \equiv \bar J(\bar z)$. If a vector current has that property, then it is automatically conserved:

$$\partial^\mu J_\mu = 2(\partial_z J_{\bar z}(\bar z) + \partial_{\bar z} J_z(z)) = 0 \ .$$

The result of the contour integration depends, obviously, on the poles inside the contour. Such poles can arise in the quantum theory when one considers the product of two or more operators.

3.2. Radial ordering

Products of operators only make sense if they are *radially ordered.* This is the analogue of time ordering for field theory on the cylinder. In the classical theory the ordering of fields or charges in a product is of course irrelevant. In the quantum theory they become operators and we have to specify an ordering. The product of two operators $A(x_a,t_a)$ and $B(x_b,t_b)$ can be written, with the help of the Hamiltonian H of the system as

$$A(x_a,t_a)B(x_b,t_b) = e^{iHt_a}A(x_a,0)e^{-iHt_a}e^{iHt_b}A(x_b,0)e^{-iHt_b}$$

The factor $e^{-iH(t_a-t_b)}$ becomes $e^{-H(\tau_a-\tau_b)}$ when we Wick-rotate (here t corresponds to x^0, τ to x^2). Usually the Hamiltonian is bounded from below, but not from above. Then if $\tau_a < \tau_b$ the exponential can take arbitrarily large values, and expectation values of the operator product are then not defined. Hence in operator products one always imposes time ordering, usually denoted as

$$TA(t_a)B(t_b) = \begin{cases} A(t_a)B(t_b) & \text{for } t_a > t_b \\ B(t_b)A(t_a) & \text{for } t_a < t_b \end{cases}$$

After mapping from the cylinder to the plane, the Euclidean time coordinate is mapped to the radial coordinate, and time ordering becomes radial ordering

$$RA(z,\bar z)B(w,\bar w) = \begin{cases} A(z,\bar z)B(w,\bar w) & \text{for } |z| > |w| \\ B(w,\bar w)A(z,\bar z) & \text{for } |z| < |w| \end{cases} .$$

A correlation function in field theory on the cylinder has the form

$$\langle 0|\, T\left(A_1(t_1)\ldots A_n(t_n)\right) |0\rangle$$

where $|0\rangle$ and $\langle 0|$ are "in" and "out" states at $t = -\infty$ and $t = +\infty$ respectively. After the conformal mapping, the correlation functions are

$$\langle 0|\, R\left(A_1(z_1,\bar z_1)\ldots A_n(z_n,\bar z_n)\right) |0\rangle$$

where $|0\rangle$ and $\langle 0|$ are states at $z = 0$ and $z = \infty$ respectively.

3.3. The generator of conformal transformations

Returning now to charge operators, let us consider the generator of the conformal transformations. As we have seen in the previous chapter, the current for an infinitesimal transformation is $T(z)\epsilon(z)$. For the corresponding charge we may then write

$$Q_\epsilon = \frac{1}{2\pi i}\oint dz \epsilon(z)T(z) \tag{3.2}$$

We would expect Q_ϵ to generate the conformal transformation with the global form

$$\phi(w,\bar w) \to \phi'(w,\bar w) = \left(\frac{\partial f(w)}{\partial w}\right)^h \phi(f(w),\bar w) \ ,$$

with $f(w) = w + \epsilon(w)$. Note that any field Φ will in general depend on both w and $\bar w$, but that we are treating w and $\bar{w}$ as independent variables, which can therefore be transformed independently. The infinitesimal form of this transformation is

$$\delta_\epsilon \Phi(w,\bar{w}) = h\partial_w \epsilon(w)\Phi(w,\bar{w}) + \epsilon(w)\partial_w \Phi(w,\bar{w})$$

Consider now the quantum version of this transformation. We may expect the following relation to hold

$$\delta_\epsilon \phi(w,\bar{w}) = [Q_\epsilon, \phi(w,\bar{w})] \tag{3.3}$$

Let us try to evaluate the commutator on the right hand side. Naively we have

$$[Q_\epsilon, \phi(w,\bar{w})] = \frac{1}{2\pi i}\oint dz \epsilon(z)\,[T(z)\phi(w,\bar{w}) - \phi(w,\bar{w})T(z)]\ . \tag{3.4}$$

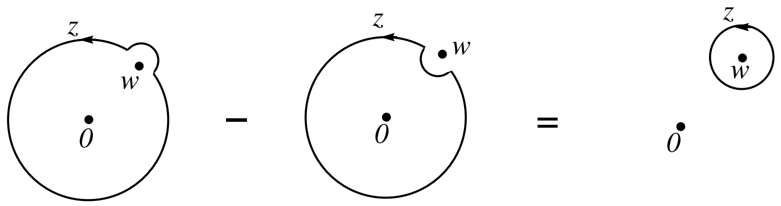
But we have just seen that the first term is defined only if $|z| > |w|$, whereas the second one requires $|z| < |w|$. Note however that z is an integration variable, and that the definition of Q did not include any prescription for the precise contours to be used. Classically Q_ϵ is in fact independent of the contour due to Cauchy's theorem, because the integrand is a holomorphic function. On the cylinder this can be interpreted as charge conservation, *i.e.* evaluating Q_ϵ at two different times gives the same answer. Classically the factor $\phi(w,\bar{w})$ is irrelevant for the evaluation of the integral, and in fact classically the commutator vanishes. In the quantum theory we have to be more careful. As one usually does, we use the freedom we have in the classical theory in order to write the quantity of interest in such a way that it is well-defined after quantization. Nothing forbids us to use different contours in (3.4), so that we get

$$[Q_\epsilon, \phi(w,\bar{w})] = \frac{1}{2\pi i}\oint_{|z|>|w|} dz\epsilon(z)T(z)\phi(w,\bar{w}) - \frac{1}{2\pi i}\oint_{|z|<|w|} dz\epsilon(z)\phi(w,\bar{w})T(z)\ .$$

This can be written as

$$[Q_\epsilon, \phi(w,\bar{w})] = \frac{1}{2\pi i}\left[\oint_{|z|>|w|} - \oint_{|z|<|w|}\right] dz\epsilon(z)R(T(z)\phi(w,\bar{w}))\ .$$

Now we deform the contours as indicated in the following figure



The result is

$$[Q_\epsilon, \phi(w,\bar{w})] = \frac{1}{2\pi i}\oint dz\epsilon(z)R(T(z)\phi(w,\bar{w})) \ , \tag{3.5}$$

where the integration contour encircles the point w. Clearly the integration only makes sense if the radially ordered product is analytic in the neighborhood of the point w. We may thus assume that it can be expanded it in a Laurent series around w

$$R(T(z)\phi(w,\bar{w})) = \sum_n (z-w)^n O_n(w,\bar{w}) \ ,$$

where the coefficients O_n are – in general – operators. It is now easy to verify that the contour integral will produce the desired result if (and only if) the radially ordered product equals

$$R[T(z)\phi(w,\bar{w})] = \frac{h}{(z-w)^2}\phi(w,\bar{w}) + \frac{1}{z-w}\partial_w\phi(w,\bar{w}) + \text{power series in } (z-w) \ . \tag{3.6}$$

The last terms are free of poles at $z = w$, and hence do not contribute to the integral.

The property (3.6) (plus the corresponding one for the anti-holomorphic quantities) defines what we mean by a conformal field.

Often one simplifies the notation by omitting the radial ordering symbol (which however is always implied) and dropping the finite terms.

3.4. Quantization of the free boson

Now we return to the free boson, in order to look at a few concrete examples of the foregoing, rather abstract discussion. On the cylinder, and in Minkowski space-time, the action is

$$S = \frac{1}{8\pi}\int d^2x \sum_i \partial_\alpha \Phi^i \partial^\alpha \Phi^i$$

(the normalization factor is $\frac{1}{4\pi\alpha'}$ with $\alpha' = 2$). The fact that the fields Φ live on a cylinder implies that they satisfy periodic boundary conditions:

$$\Phi^i(x^0, 0) = \Phi^i(x^0, 2\pi)$$

Any field satisfying these boundary condition can be Fourier expanded

$$\Phi^i(x^0, x^1) = \sum_{n=-\infty}^{\infty} e^{inx^1} f_n^i(x^0)$$

The classical equation of motion for Φ is

$$[\partial_0^2 - \partial_1^2]\Phi^i(x^0, x^1) = 0$$

For the Fourier modes of a classical solution this implies

$$\partial_0^2 f_n^i(x^0) = -n^2 f_n^i(x^0) \ .$$

The solution is

$$f_n^i(x^0) = a_n^i e^{inx^0} + b_n^i e^{-inx^0}, n \neq 0$$

and

$$f_0^i(x^0) = p^i x^0 + q^i \ .$$

Putting all this together, and introducing a few convenient factors, we may write the result as

$$\Phi^i(x^0, x^1) = q^i + 2p^i x^0 + i \sum_{n\neq 0} \left\{ \frac{1}{n}(\alpha_n^i e^{-in(x^0+x^1)} + \tilde{\alpha}_n^i e^{-in(x^0-x^1)}) \right\} \ , \tag{3.7}$$

Now we quantize this field using canonical quantization. The canonical momentum is

$$\Pi^i = \frac{1}{4\pi}\partial_0 \Phi^i$$

We impose on it the following commutation relations

$$\begin{aligned} \left[\Phi^i(x^0, x^1), \Pi^j(x^0, y^1)\right] &= i\delta^{ij}\delta(x^1 - y^1) \\ \left[\Phi^i(x^0, x^1), \Phi^j(x^0, y^1)\right] &= 0 \\ \left[\Pi^i(x^0, x^1), \Pi^j(x^0, y^1)\right] &= 0 \ . \end{aligned} \tag{3.8}$$

These relations can be expressed in terms of modes by taking Fourier moments of these conditions with respect to x^1 and x'^1. The result is

$$\begin{aligned} \left[\alpha^i_k, \alpha^j_l\right] = \left[\tilde{\alpha}^i_k, \tilde{\alpha}^j_l\right] &= k\delta^{ij}\delta_{k+l,0} \\ \left[\alpha^i_k, \tilde{\alpha}^j_l\right] &= 0 \end{aligned} \tag{3.9}$$

and

$$\left[q^i, p^j\right] = i\delta^{ij} \tag{3.10}$$

Now transform ϕ to the complex plane. Then we get

$$\Phi^i(z,\bar{z}) = q^i - i(p^i \log(z) + p^i \log(\bar{z})) + i \sum_{n\neq 0} \frac{1}{n}\left[\alpha^i_n z^{-n} + \tilde{\alpha}^i_n \bar{z}^{-n}\right]$$

Note that Φ^i can almost be written as the sum of holomorphic and anti-holomorphic functions. Almost, because $\log(z)$ is not a holomorphic function. However, if we take a derivative of Φ^i we do get purely holomorphic functions, namely $\partial_z\Phi^i(z)$ and $\partial_{\bar{z}}\Phi^i(\bar{z})$.

3.5. The free boson propagator

The first quantity of interest is the product $\Phi^i(z)\Phi^j(w)$. The quantum equivalent of this classical product is $R(\Phi^i(z)\Phi^j(w))$, and we would like to know the behavior of this product as z approaches w. A simple way to probe the short distance behavior is to take expectation values of the operator between two states, for example the vacuum. To compute this we first have to know how the various mode operators act on the vacuum.

Classically the field Φ^i is real. Consequently, in (3.7) q^i and p^i are real, and $\alpha^i_n = (\alpha^i_{-n})^*$. The quantum equivalent of this statement is that q^i and p^i are represented by Hermitean operators, whereas α^i_{-n} is the Hermitean conjugate of α^i_n. Reality conditions are best imposed on the cylinder and in terms of Minkowski coordinates. The complex Wick rotation and subsequent conformal mappings can make reality properties less manifest.

We see now that the commutation relation $[\alpha^i_k, \alpha^j_l] = k\delta^{ij}\delta_{k+l,0}, k > 0$ is the same as that for a set of harmonic oscillators, apart from the factor k, which can be absorbed in the normalization of the operators (note that the commutator for $k < 0$ contains no new information). Indeed, apart from the "zero mode" q^i, p^i the free boson is nothing but an infinite set of harmonic oscillators.

By the usual reasoning for harmonic oscillators, the vacuum satisfies

$$\alpha_k^i \left|0\right\rangle = 0 \quad \text{for} \quad k > 0$$

The algebra of the operators p^i and q^i is also a well-known one, namely the Heisenberg algebra. Hence the vacuum must satisfy

$$p^i \left|0\right\rangle = 0$$

This is all we need to compute the vacuum expectation value. A convenient technique for computing vacuum expectation values is normal ordering. We reorder the oscillators in such a way, using the commutators, that creation operators are always to the left of annihilation operators. Then the vacuum expectation value of normally ordered terms always vanishes for every term that contains at least one harmonic oscillator, and we only have to take into account the contributions picked up from the commutators.

Normally ordered products of oscillators are denoted as

$$: \alpha_{k_1} \ldots \alpha_{k_n} :\equiv \Pi_i \alpha_{k_{\pi(i)}} \ ,$$

where $\pi(i)$ is a permutation of the labels such that $k_{\pi(i)} < k_{\pi(j)}$ if $i < j$ (re-ordering positive and negative labels among each other has no effect, but does not hurt either). Note that oscillators within the normal ordering signs behave as if they are classical. They can be written in any order, since the right hand side is always the same.

The only terms in the product $\Phi^i(z)\Phi^j(w)$ that does not contain oscillators is the zero mode contribution. These terms require some special attention. We define normal ordering of p^i and q^i in such a way that p^i is always to the right of q^i.

Using these rules we get, when $|z| > |w|$,

$$\begin{aligned} R(\Phi^i(z,\bar{z})\Phi^j(w,\bar{w})) =&: \Phi^i(z,\bar{z})\Phi^j(w,\bar{w}) : -i[p^i,q^j](\log z + \log \bar{z}) \\ &+ \left(\left[i \sum_{n>0} \frac{1}{n}\alpha_n^i z^{-n}, i \sum_{m<0} \frac{1}{m}\alpha_m^j w^{-m} \right] + (\text{ anti-holomorphic terms}) \right) \end{aligned}$$

The commutator yields

$$-\sum_{n>0,m<0} \frac{1}{nm} n z^{-n} w^{-m} \delta_{n+m,0} \ ,$$

so that we get

$$: \Phi^i(z,\bar z)\Phi^j(w,\bar w) : + \delta^{ij}(-log(z\bar z) + \sum_{n>0}\frac{1}{n}\Big(\frac{w}{z}\Big)^n + \sum_{n>0}\frac{1}{n}\Big(\frac{\bar w}{\bar z}\Big)^n) \ .$$

The sum converges for $|z| > |w|$. Since the product was radially ordered, this is satisfied. The result is

$$R(\Phi^i(z,\bar z)\Phi^j(w,\bar w)) =: \Phi^i(z,\bar z)\Phi^j(w,\bar w) : -\delta^{ij}\left[\log(z-w) + \log(\bar z - \bar w)\right] \ .$$

This is not quite what one usually gets when evaluating an operator product. Normally the result consists of holomorphic and anti-holomorphic parts, whereas here there is a logarithmic singularity. A more standard result is the operator product $\partial\Phi^i(z,\bar z)\partial\Phi^j(w,\bar w)$, which can be obtained from the above by differentiation (the notation ∂ is short-hand for either ∂_z or ∂_w, depending on what it acts on).

$$R(\partial\Phi^i(z,\bar z)\partial\Phi^j(w,\bar w)) = -\delta^{ij}\frac{1}{(z-w)^2} + : \partial\Phi^i(z,\bar z)\partial\Phi^j(w,\bar w) :$$

since $\partial_z\Phi^i(z,\bar z)$ depends only on z we usually omit the second argument. Furthermore ∂_z is usually written as just ∂, if no confusion is possible. Furthermore the radial ordering is usually not explicitly written, and the finite terms are usually omitted as well. Since the objects within normal ordering signs behave as classical quantities, these are in particular finite as z approaches w. Using all this short-hand notation, the result is then written as

$$\partial\Phi^i(z)\partial\Phi^j(w) = -\frac{\delta^{ij}}{(z-w)^2} \ .$$

3.6. The normally ordered energy momentum tensor

This result shows that we have to be careful with the definition of the quantum energy momentum tensor, which classically is $T(z) = -\frac{1}{2}\sum_i \partial_z\Phi^i(z)\partial_z\Phi^i(z)$, plus the anti-holomorphic term. If we naively quantize Φ^i the product of the two operators is singular. For this reason one defines

$$T(z) \equiv -\tfrac{1}{2} : \sum_i \partial\Phi^i(z)\partial\Phi^i(z) := -\tfrac{1}{2}\sum_i \lim_{z\to w}[\partial\Phi^i(z)\partial\Phi^i(w) + \frac{\delta^{ii}}{(z-w)^2}] \ .$$

This amounts to subtracting an infinite constant from the energy momentum tensor. This sets the energy of the vacuum to zero.

3.7. Operator products for free bosons

We are now ready to compute the operator product of the energy momentum tensor with various operators in the theory. Let us first consider $T(z)\partial\Phi^i(w)$. To compute this operator product we normal order all harmonic oscillators and the zero-mode operators q^i and p^i. The operators within $T(z)$ are already normal ordered, and hence the only ordering to worry about is between $T(z)$ and $\partial\Phi^i(w)$. We may write this as

$$-\tfrac{1}{2}\!:\partial\Phi^i(z)\partial\Phi^i(z)\!:\partial\Phi^j(w) = -\tfrac{1}{2}\!:\partial\Phi^i(z)\partial\Phi^i(z)\partial\Phi^j(w)\!: -\partial\Phi^i(z)\left[-\frac{\delta^{ij}}{(z-w)^2}\right]$$

Note the factor of two in the last term, because there are two factors $\partial\Phi^i(z)$ to order with respect to $\partial\Phi^j(w)$.

To get the operator product in the desired form we wish to express the remaining factor $\partial\Phi^j(z)$ in terms of $\partial\Phi^j(w)$. This is simply a Taylor expansion, $\partial\Phi^j(z) = \partial\Phi^j(w) + (z-w)\partial^2\Phi^j(w) + \frac{1}{2}(z-w)^2\partial^3\Phi^j(w) + \ldots$. The final result may thus be written as

$$T(z)\partial\Phi^j(w) = \frac{1}{(z-w)^2}\partial\Phi^j(w) + \frac{1}{z-w}\partial^2\Phi^j(w)\ ,$$

where as usual we drop all finite terms, and all operators appearing on the right hand side are normally ordered. It follows that $\partial\Phi^j(w)$ is a conformal field with conformal weight 1.

In a similar way one may check that $\partial^2\Phi^i(z)$ is not a conformal field. This is not a surprise, because we have seen before that it is not a conformal field classically.

Now consider the energy momentum tensor itself. It is a simple exercise to compute

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial_w T(w) \tag{3.11}$$

Here c equals the number of bosons Φ^i. If the first term were absent, $T(z)$ would be a conformal field of weight 2, the classical value. In this case quantum effects yield an extra term, an anomaly. This is called the *conformal anomaly.*

3.8. The Virasoro algebra

The operator product (3.11), derived here for free bosons, has a completely general validity. Under quite general assumptions, one may show that the operator product of two energy momentum tensors of a conformal field theory must have the form (3.11).

In (2.20) a current for conformal symmetry was introduced, $J_\epsilon(z) = T(z)\epsilon(z)$. Since $\epsilon(z)$ is an arbitrary holomorphic function, it is natural to expand it in modes. The precise mode expansion one uses depends on the surface one is working on. On the Riemann sphere we require fields and transformations to be continuous on contours around the origin. This was also the surface for which the classical mode expansion (2.15) was written down. We expect thus that $J_\epsilon(z)$ generates the transformation $z \to z' = z - z^{n+1}$ if we choose $\epsilon(z) = z^{n+1}$. We then get an infinite series of currents $J^n(z) = T(z)z^{n+1}$. The correctly normalized operators are in fact

$$L_n = \frac{1}{2\pi i}\oint dz z^{n+1} T(z) \ . \tag{3.12}$$

This relation can be inverted:

$$T(z) = \sum_n z^{-n-2} L_n \ .$$

To check that the normalization and the sign are correct one may compare the quantum algebra with the classical algebra. The commutator of L_n and L_m can be evaluated using contour integrals, as was already done earlier. One finds then the *Virasoro algebra* (the paper by Virasoro [10], to which this algebra owes its name, contains the generators of the algebra as “constraints”, but not the algebra itself)

$$[L_n, L_m] = (n-m)L_{m+n} + \frac{c}{12} n(n^2-1)\delta_{n,-m} \tag{3.13}$$

Not surprisingly, a term proportional to c appears. If that term were absent, the quantum algebra would be identical to the classical one. Strictly speaking, such a constant term is not allowed in an algebra. The commutator of any two elements of the algebra must again be an element of the algebra. We are thus forced to view c not as a number, but as an operator which commutes with any element of the algebra. It follows then that on any representation of the algebra this operator has a constant value, which is also denoted by c, just as the operator itself. Such operators that appear only on the right hand side of commutators are usually called *central charges.*

Note that the $SL(2,\mathbf{C})$ subalgebra generated by L_1, L_0 and L_{-1} is not affected by the extra term. It remains thus meaningful to speak of the conformal weight of $T(z)$.

Because of the central term the classical symmetry is not preserved in quantum mechanics. In particular, the central term prohibits the vacuum to have the full symmetry, because we cannot impose the condition $L_n|0\rangle = 0$ for all n, without getting a contradiction with the algebra. This is analogous to the position and momentum operators in quantum mechanics, which also cannot simultaneously annihilate the vacuum.

Nevertheless we still have all the generators of the Virasoro algebra at our disposal, and they still play a useful rôle. In those cases where conformal invariance is really crucial this is not sufficient though. Presumably this is true in string theory, although there have been attempts to make sense of it without conformal invariance. The simplest string theory, the bosonic string, is constructed out of D free bosons, where D is the number of space-time dimensions. One might think that this is always anomalous, because $c = D$ in this case. However, there is an additional ghost contribution (the ghost is related to gauge fixing for two-dimensional gravity) of -26. This leads to the well-known concept of a critical dimension $D = 26$.

4. Virasoro representation theory

Given any algebra, it is usually important to try and find its representations. The best known example is probably the angular momentum algebra.

In that case all finite dimensional unitary representations are labelled by an integer or half-integer j. The algebra consists of three generators, J^-, J^+ and J_3. All states in a representation are labelled by a J_3 eigenvalue, which is lowered by J^- and increased by J^+. The representation can be built up by starting with the state with maximal J_3 eigenvalue, which is therefore annihilated by J^+. Mathematicians call such a state the *highest weight state*. The other states are obtained by acting on the highest weight state (denoted $|j\rangle$) with J^-. This can only be done a finite number of times if j is integer or half-inter, because one finds that the norm of the state $(J^-)^{2j+1}|j\rangle$ is zero. Such states are called *null states* or *null vectors*. The representation space is defined by setting such states equal to zero.

This is the procedure we wish to mimic for the Virasoro algebra. In general, one starts with determining a (preferably maximal) set of commuting operators (like J^2 and J_3 for angular momentum). A convenient choice is L_0 and the central charge, c.

The Virasoro algebra has many more representations than will be considered here. As is the case for $SU(2)$, the representations of interest are those satisfying a number of physically motived conditions. The ones we will consider here are the so-called unitary highest weight representations.

4.1. Unitarity

A representation of the Virasoro algebra is called unitary if all generators L_n are realized as operators acting on a Hilbert space, with the condition that $L_n^\dagger = L_{-n}$. The latter condition implies in particular that $T(z)$ is a Hermitean operator. This is most easy to see on the cylinder, where we have, classically

$$\tfrac{1}{2}[T_{11}(x_0,x_1) + T_{00}(x_0,x_1)] = \sum_n L_n e^{-in(x_0+x_1)} + \bar{L}_n e^{-in(x_0-x_1)} \ ,$$

and

$$\tfrac{1}{2}[T_{12}(x_0,x_1)+T_{21}(x_0,x_1)] = \sum_n L_n e^{-in(x_0+x_1)} - \bar{L}_n e^{-in(x_0-x_1)} \ ,$$

Reality of $T_{\mu\nu}$ leads to the requirement that $L_n^* = L_{-n}$ (and $\bar{L}_n^* = \bar{L}_{-n}$), which naturally leads to the quantum condition given above. On the complex plane the hermiticity condition looks less natural, because the "in" and "out" states play an asymmetric rôle, and also because we have complexified the coordinates.

In the following we will consider unitary representations. Non-unitary representations have also been studied, in particular in statistical mechanics. Such representations still consist of states in a Hilbert space (in particular having positive norm), but the requirement $L_n^\dagger = L_{-n}$ is dropped.

4.2. HIGHEST WEIGHT REPRESENTATIONS

By definition, a *highest weight representation* is a representation containing a state with a smallest value of L_0. Not all representations have that property, but it is reasonable to expect this in a physical theory, since $L_0 + \bar{L}_0$ is the Hamiltonian, which is usually bounded from below. The term "highest weight" for a state with lowest energy is perhaps somewhat confusing, but has become standard terminology.

It follows from the structure of the algebra that L_n decreases the eigenvalue of L_0 by n,

$$L_0 L_n |\psi\rangle = (L_n L_0 - nL_n)|\psi\rangle = (h-n)L_n|\psi\rangle \ ,$$

if $L_0|\psi\rangle = h|\psi\rangle$.

If $|h\rangle$ is a highest weight state, then obviously $|h\rangle$ is annihilated by all generators L_n with $n > 0$:

$$L_n|h\rangle = 0, \ \text{for } n \geq 1$$

Suppose the operator L_0 acting on the highest weight state $|h\rangle$ creates a state $|h\rangle'$. Then the Virasoro algebra tells us that $L_n|h\rangle' = 0$ for $n \geq 1$, *i.e.* L_0 maps highest weight states to highest weight states. Since L_0 is hermitean we can diagonalize it on the highest weight states, so that we may assume that $L_0|h\rangle = h|h\rangle$ (labelling the state only by its L_0 eigenvalue is inadequate in case of degeneracies, but we will not worry about that now).

The negative modes $L_n, n < 0$ can be used to generate other states in the representation. Usually such states are referred to as *descendants*.

We have in fact already seen an example of a representation that is not a highest weight representations, namely the adjoint representation, defined by the action of the algebra on itself. The commutator

$$[L_0, L_n] = -nL_n$$

tell us that in this representation the eigenvalue of L_0 can take any integer value, whereas

$[c, L_n] = 0$ tells us that the adjoint representation has central charge 0. It is in fact a unitary representation.

4.3. The vacuum

The vacuum of the theory can be defined by the condition that it respects the maximum number of symmetries. This means that it must be annihilated by the maximum number of conserved charges. In the present context this means that we would like it to satisfy $L_n |0\rangle = 0$ for *all* n, but because of the central term that is obviously not possible. For example, if L_2, L_{-2} as well as L_0 annihilate the vacuum, so does the commutator of L_2 and L_{-2}. But this is only consistent with the algebra if $c = 0$. We will soon see that unitary conformal field theories with $c = 0$ are trivial.

The maximal symmetry we can have is

$$L_n |0\rangle = 0 \ , \text{for } n \geq -1 \ .$$

Of course we could also have imposed this for $n \leq 1$, but then $|0\rangle$ is a state with maximal eigenvalue of L_0 rather than one with minimal eigenvalue (a highest weight state).

Because of the commutator $[L_1, L_{-1}] = 2L_0$ any highest weight state which is annihilated by L_0 must be annihilated by L_1 and L_{-1} (and vice-versa). It will always be assumed that there is precisely one state in the theory that has these properties.

We also define its Hermitean conjugate $\langle 0|$. It satisfies $\langle 0| L_n = 0$ for $n \leq 1$.

4.4. Positivity of c and h

The unitary highest weight representations are labelled by two real numbers, h and c. Since all generators commute with c, it has a constant value on a representation. On the other hand L_0 does not have a constant value, but we can define h uniquely as its eigenvalue on the highest weight state. With these two numbers given, we know the Virasoro representation completely, since all states can be created by the action of the Virasoro generators on the ground states, and since the norm of any state can be expressed completely in terms of c and h. Hence any negative or zero norm condition depends only on c and h. It follows that two representations with the same value of c and h are equivalent as Virasoro representations.

In the rest of this chapter we will derive restrictions on c and h by requiring absence of negative norm states. As a modest start we will show that these number are non-negative.

Note first of all the following commutator

$$[L_n, L_{-n}] = \frac{c}{12}(n^3 - n) + 2nL_0$$

Hence we have

$$\| L_{-n} |0\rangle \| = \langle 0| (L_{-n}^{\dagger} L_{-n} |0\rangle = \langle 0| L_n L_{-n} |0\rangle = \langle 0| [L_n, L_{-n}] |0\rangle = \frac{c}{12}(n^3 - n)$$

For $n \geq 2$ this implies that $c \geq 0$ (this follows from the requirement that we work in a Hilbert space, so that all states must have non-negative norm; furthermore zero norm implies that the state vanishes.)

Using the algebra just like we did for the vacuum we find for any other highest weight state

$$\| L_{-n} |h\rangle \| = \langle h| L_n L_{-n} |h\rangle = \langle h| [L_n, L_{-n}] |h\rangle = \left(\frac{c}{12}(n^3 - n) + 2nh\right) \langle h|h\rangle$$

We may assume that $\langle h|h\rangle \neq 0$, since otherwise we would not consider $|h\rangle$ a state in our theory. If the norm of the highest weight state does not vanish this tells us once again that $c \geq 0$ (since the first term dominates the second for large n), while for $n = 1$ we see that either $h > 0$ or $h = 0$ and $\| L_{-1} |h\rangle \| = 0$, *i.e.* $|h\rangle = |0\rangle$.

If $c = 0$ in a unitary theory the vacuum representation contains just one state, $|0\rangle$ itself, since the foregoing argument shows that $L_{-n} |0\rangle = 0$ for $n > 0$. (To rule out non-trivial representations with $c = 0$ and $h > 0$ requires a more sophisticated argument, which we will present later.)

4.5. States and conformal fields

There is a simple connection between highest weight states and conformal fields. Consider a conformal field $\phi(z, \bar{z})$ with weights h and $\bar{h}$. Now define

$$\left|h, \bar{h}\right\rangle = \phi(0, 0) |0\rangle \ ,$$

where of course it is assumed that $\phi(z, \bar{z}) |0\rangle$ is well-behaved at the origin. Now compute $L_n \left|h, \bar{h}\right\rangle$. We find

$$\begin{aligned} [L_n, \phi(w, \bar{w})] &= \frac{1}{2\pi i} \oint z^{n+1} T(z) \phi(w, \bar{w}) \\ &= h(n+1) w^n \phi(w, \bar{w}) + w^{n+1} \partial_w \phi(w, \bar{w}) \ , \end{aligned} \tag{4.1}$$

which vanishes if $w = 0$ and $n > 0$. Hence L_n, $n > 0$ commutes with $\phi(0, 0)$ and it follows that $\left|h, \bar{h}\right\rangle$ is a highest weight state. It is also a highest weight state with respect to the anti-holomorphic sector.

For $n = 0$ we find $[L_0, \phi(0, 0)] = h\phi(0, 0)$, so that the state $\left|h, \bar{h}\right\rangle$ indeed has L_0-eigenvalue h, as the notation suggests.

4.6. Descendant fields

Ground states of Virasoro representations are generated from the vacuum by conformal fields, which are also known as (Virasoro) primary fields. [The addition "Virasoro" is added in more general context, where other algebras are being considered. One can then have a distinction between Virasoro primaries and primaries with respect to other algebras. The name "conformal field" will be used here only in the strict sense of Virasoro primary, which is equivalent to (4.1) being satisfied.]

One can also consider fields that generate descendant states from the vacuum. They are, quite naturally, called *descendant fields*. They can be defined by means of the operator product with the energy momentum tensor

$$T(z)\phi(w,\bar{w}) = \sum_{k\geq 0}(z-w)^{k-2}\phi^{(-k)}(w,\bar{w}) \ .$$

We may project out a term from this sum by

$$\phi^{(-k)}(w,\bar{w}) = \oint \frac{dz}{2\pi i}\frac{1}{(z-w)^{k-1}}T(z)\phi(w,\bar{w})$$

Clearly

$$\phi^{(-k)}(0,0)\left|0\right\rangle = \oint \frac{dz}{2\pi i}\frac{1}{(z)^{k-1}}T(z)\phi(0,0)\left|0\right\rangle = L_{-k}\phi(0,0)\left|0\right\rangle \ , \tag{4.2}$$

so that $\phi^{(-k)}$ does indeed generated the L_{-k} descendant of $\left|h,\bar{h}\right\rangle$. To get descendant states obtained by two Virasoro generators one has to consider operator products of $T(z)$ with $\phi^{(-k)}$, etc.

Descendant fields are of course not conformal fields, but it is interesting to see what the deviation is. It turns out that they behave like conformal fields when commuted with L_0 or L_{-1} (scalings, rotations and translations), but not with L_1 (special conformal transformations). Hence one can assign a conformal weight to them, which is of course equal to the L_0 eigenvalue of the state $\phi^{(-k)}(0,0)\left|0\right\rangle$ they create from the vacuum. This weight is $(h+k,\bar{h})$, where $(h,\bar{h})$ are the conformal weights of ϕ.

For example, the first non-trivial descendant operator of the identity is $T(z)$ itself. It generates a state with $h=2$, $T(0)\left|0\right\rangle = L_{-2}\left|0\right\rangle$.

Note that the first non-trivial descendant of any conformal field ϕ with non-zero conformal weight is $\partial\phi$.

4.7. The Kac determinant

So far we have derived some necessary conditions for the positivity of norms of states. But we have only looked at norms of states $L_{-n}|h\rangle$. At a given excitation level (*i.e.* at a given L_0 eigenvalue $n+h$) there are in general many other descendants, which are linear combinations of states

$$L_{-n_1}\dots L_{-n_k}|h\rangle\ , \qquad \sum_i n_i = n\ . \tag{4.3}$$

Because of the commutation relations of the Virasoro algebra we may in fact assume that the generators are ordered, $n_i \geq n_j$ if $i<j$, since any incorrectly ordered product can be expressed in terms of ordered ones. The collection of states (4.3) for all $n \geq 0$ is called the *Verma module* of $|h\rangle$. Its definition does not make use of any norm on the space of states. If one does have a norm, one can ask whether all states in the Verma module (*i.e.* all linear combinations of the states (4.3)) have positive norm. In general that will not be the case. Note that the set of states in the Verma module is closed with respect to the action of the full set of Virasoro generators, *i.e.* acting with any Virasoro generator on any state in the set produces a linear combination of states in the set. There is no need to include positively moded Virasoro generators, since they can always be commuted to the right where they annihilate $|h\rangle$.

At the first excited level, the only state one can have is $L_{-1}|h\rangle$. We have already seen that this state has positive norm for $h>0$ and norm zero for $h=0$.

At the second level one can have $L_{-2}|h\rangle$ and $(L_{-1})^2|h\rangle$. It is not sufficient to check whether each of these states separately has positive norm, because there could be linear combinations that have zero or negative norm. To deal with this problem in general we consider the matrix

$$K_2 = \begin{pmatrix} \langle h|\,L_{-2}^{\dagger}L_{-2}\,|h\rangle & \langle h|\,L_{-2}^{\dagger}L_{-1}L_{-1}\,|h\rangle \\ \langle h|\,(L_{-1}L_{-1})^{\dagger}L_{-2}\,|h\rangle & \langle h|\,(L_{-1}L_{-1})^{\dagger}L_{-1}L_{-1}\,|h\rangle \end{pmatrix}$$

This matrix is clearly Hermitean. Suppose it has negative or zero determinant. Then there exists an eigenvector $\vec{v}=(\alpha,\beta)$ with zero or negative eigenvalue, *i.e.* $\vec{v}K_2\vec{v}^T \leq 0$. The left hand side is equal to the $\|\alpha L_{-2}|h\rangle + \beta L_{-1}L_{-1}|h\rangle\|$, and we conclude that this quantity is not positive.

At the $n^{\rm th}$ level there is an analogous matrix K_n. The determinant of the matrix K_n is called the Kac determinant. Of course it does not tell us precisely how many positive, zero and negative eigenvalues there are. Even if $\det K > 0$ there could be an even number of negative eigenvalues. Usually one studies the behavior of the Kac determinant as a function of parameters (such as h and c), starting in an asymptotic region where we know that all eigenvalues are positive.

Of special interest are the null vectors, the eigenvectors of zero norm. The vanishing of the norm corresponds to the equality $vK_nv^T = 0$, where $\vec{v}$ is a set of coefficients of the basis states at level n. But if v is a vector in the zero eigenspace of the Hermitean matrix K_n, it is clearly also true that $wK_nv^T = 0$ for *any* vector w, not just $w = v$. It follows that the state defined by the vector v is orthogonal to any state at level n. Furthermore, since the L_0 eigenspaces are all orthogonal, it follows that a null state is orthogonal to any other state in the Verma module. Then in particular, if $|x\rangle$ is a null state $\| L_n |x\rangle \| = 0$ for all n, since this relation can be interpreted as the orthogonality relation between $|x\rangle$ and the Verma module state $L_{-n}L_n |x\rangle$. Thus the Virasoro generators take null states to null states or annihilate them. If we act with positively moded Virasoro generators it must happen that after a finite number of steps we encounter a state $|x_s\rangle$ which is annihilated by all positive L_n's, since $|h\rangle$ has positive norm. The state $|x_s\rangle$ is at the same time a descendant of $|h\rangle$ (as are all states in the Verma module) as a primary, since it is annihilated by all positive L_n's. Such states are called *singular states* or (more frequently) *singular vectors*. (Note that the definition of a singular vector (in contrast to a null vector) does not require a norm.)

Consider now the states obtained by action with all Virasoro generators on the singular state $|x_s\rangle$. Clearly they form a closed subset of the states in the Verma module of $|h\rangle$. This implies that if we remove all these states we will still have a non-trivial representation of the Virasoro algebra. In other words, suppose that $|x_s\rangle$ is generated from the ground state $|h\rangle$ by a combination of Virasoro generators $\mathcal{L}$, $|x_s\rangle = \mathcal{L}\,|h\rangle$. Then we can define a representation of the Virasoro algebra on the subspace of the Verma module obtained by removing $|x_s\rangle$ (and all its descendants) by imposing the condition $\mathcal{L}\,|h\rangle = 0$. This corresponds in $SU(2)$ to the condition $J^-\,|-j\rangle = 0$ in a representation with spin j. Obviously we can only remove a descendant if it has zero norm; otherwise the norms of the left-hand side and the right-hand side of $\mathcal{L}\,|h\rangle = 0$ contradict each other.

We see thus that we can systematically remove all null states from the Verma module by removing all sub-representations whose highest weight states are the singular vectors. On the other hand, negative norm states cannot be removed. If we wish to obtain unitary representations, we are obliged to consider only ground states $|h\rangle$ for which no negative norm states appear at all. This turns out to be very restrictive, at least for $c < 1$.

4.8. The Kac determinant at level 2

The first evidence for that is seen at the second level. The explicit expression for K_2 is

$$K_2 = \begin{pmatrix} 4h + \frac{1}{2}c & 6h \\ 6h & 4h + 8h^2 \end{pmatrix} \langle h|h\rangle \tag{4.4}$$

For large values of c and h the diagonal terms dominate, and the eigenvalues are positive. The determinant is

$$\det\, K_2 = 2\left[16h^3 - 10h^2 + 2h^2c + hc\right] \langle h|h\rangle^2 \; .$$

This can be written as

$$\det\ K_2 = 32(h - h_{11})(h - h_{12})(h - h_{21})\langle h|h\rangle^2 \ ,$$

where we introduce for future purposes

$$h_{pq} = \frac{[(m+1)p - mq]^2 - 1}{4m(m+1)} \quad , \tag{4.5}$$

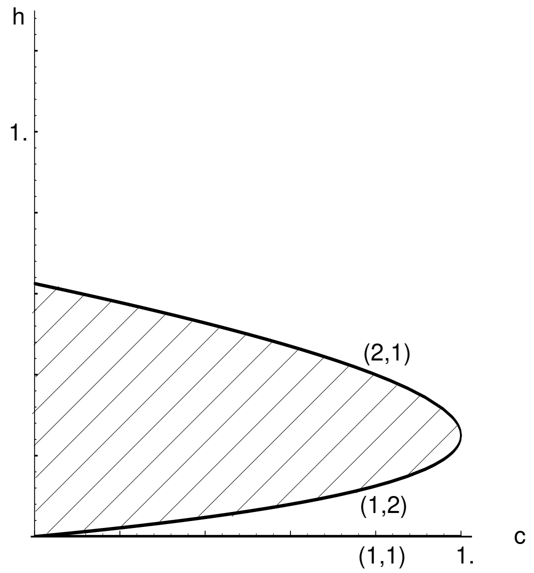
with

$$m = -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{\frac{25-c}{1-c}} \tag{4.6}$$

Note that choosing the $+$ or the $-$ sign has the effect of interchanging m with $-m-1$, which amounts to interchanging p and q.

The determinant is proportional to h (in the second form this is slightly less manifest, but note that $h_{11} = 0$). This is due to that fact that the norm of $L_{-1}|h\rangle$ is proportional to h. Any state built on top of $L_{-1}|h\rangle$ will have a norm proportional to the norm of $L_{-1}|h\rangle$, and hence $\det\ K_2$ is also proportional to h.

The vanishing lines in the (h, c) plane are as follows



Note that the branches $h_{1,2}$ and $h_{2,1}$ join smoothly at $c = 1$. The branch h_{11} coincides with the c-axis, as explained above. Since there are two positive eigenvalues for large c and h, we move into a region with precisely one negative eigenvalue when we cross one of the lines. Apart from the region $h < 0$, which was already ruled out, this eliminates the dashed area to the left of the curve. If h and c are within that area the corresponding representation of the Virasoro algebra has negative norm states. Points on the border of the two regions are acceptable since we can remove the zero norm states present there.

From the second row of K_2 (see (4.4)) we can read off that the null state – if it exists – is of the form

$$\left[L_{-2} - \frac{3}{2(2h+1)} L_{-1}^2\right] |h\rangle \tag{4.7}$$

The first row then gives us an expression for c in terms of h.

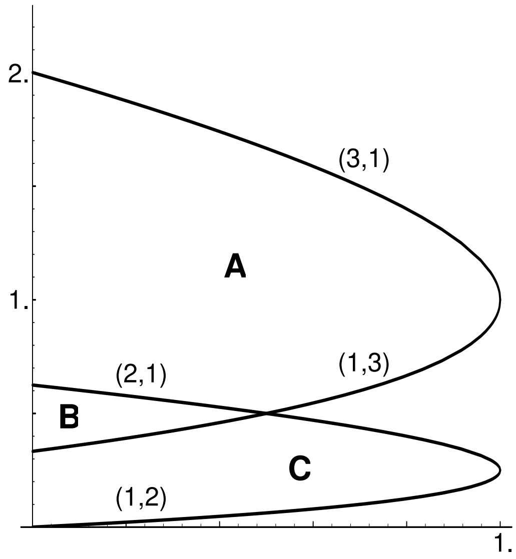
4.9. The Kac determinant at level 3 and 4

At the third level we have to consider the states $L_{-3}\,|h\rangle$, $L_{-2}L_{-1}\,|h\rangle$ and $(L_{-1})^3\,|h\rangle$, etc. A general formula for the Kac determinant can be derived doubref [11] [12], namely

$$\det\, K_n = \alpha_n \Pi_{pq\leq n} (h - h_{pq})^{P(n-pq)} \langle h|h\rangle^n \ ,$$

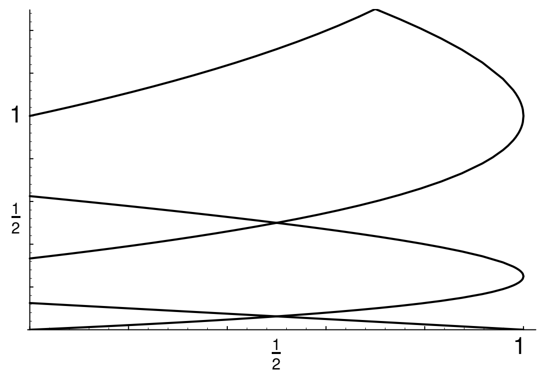
The function $P(N)$ gives the number of partitions of N, *i.e.* the number of ways of writing N as a sum of integers. For example $P(0) = 1, P(1) = 1, P(2) = 2, P(3) = 3$ and $P(4) = 5$, etc. This is equal to the number of states at level N, including null states.

The following picture shows the curves for the third level, together with those for the second one.



At level 2 we had ruled out regions B and C. The Kac determinant at level 3 tell us that regions A and C are ruled out. It says nothing about region B, since we have to pass two vanishing curves to get there, so that the determinant is positive there (but of course there are in fact two negative eigenvalues in this region). But region B was already ruled out. The entire area to the left of the two curves contains negative norm states, and is thus ruled out.

At level 4 we get the following picture



Now an even bigger region gets ruled out, but it should also be clear that the picture at level n always contains all vanishing lines from lower levels.

4.10. The discrete series of minimal unitary Virasoro models

Strictly speaking one can never exclude these lines by looking at the determinant alone. A more detailed argument [13] shows that of the entire region $0 < c \leq 1, h \geq 0$ only a discrete set of points remains. These points are at the following c and h values:

$$c = 1 - \frac{6}{m(m+1)} \,, \qquad m \geq 3$$

with

$$h = \frac{[(m+1)p - mq]^2 - 1}{4m(m+1)} \,, \quad p = 1, \ldots, m-1, 1 \leq q \leq p \,.$$

The last formula looks quite similar to that for the vanishing curves, whereas the inverse of the first formula gives m in terms of c exactly as in (4.6). The main differences are that m is now restricted to integer values and that the range of p and q is limited. This result implies that these values of h and c occur on an infinite number of vanishing lines, *i.e.* they are intersection points of an infinite number of lines. The first such intersections, occurring for $c = \frac{1}{2}$ and $h = \frac{1}{16}$ and $h = \frac{1}{2}$, can be seen at level 3 and 4.

For $c = 0$ most h-values are now eliminated, except for a few discrete points where the vanishing lines reach the h axis. These can be taken care of by considering the set of states $L_{-2n} \left| h \right\rangle$ and $L_{-n}^2 \left| h \right\rangle$ for sufficiently large n [14].

Arguments of this kind can of course only rule out points. To show that conformal field theories with these representations actually exist, the easiest thing to do is to construct examples. We will return to this later.

For $c \geq 1$ unitary representations exist for any positive value of h. For integer values of c, $c = N$, it is quite easy to construct such representations explicitly using free bosons.

5. Correlation functions

The objects we want to calculate in field theory in general, and in conformal field theory in particular are the correlation functions (it is common practice to use statistical mechanics terminology here; in field theory language we would speak of Green's functions). If we know all correlation functions, we can say that we have completely solved the theory; we are then able to compute any scattering amplitude. In four-dimensional field theory this is a very hard problem that we can only address in perturbation theory. In two-dimensional field theory we can go much further.

In path-integral formulation we are interested in expressions of the form

$$\int \mathcal{D}\phi O_1(\phi(z_1)) \ldots O_n(\phi(z_n)) e^{-S(\phi)} , \tag{5.1}$$

where ϕ stands generically for any field in the theory (possibly including ghosts), O_i is some function of the fields, and S is the action, which one continues to Euclidean space to improve convergence.

When computing such an integral one has to specify the two-dimensional surface on which the fields ϕ live. This can be the plane, but it can be any other two-dimensional surface as well. Locally, any such surface looks like a plane, but globally they can have different topologies. In two dimensions, there is a complete classification of the different topologies one can have, the theory of Riemann surfaces. They are classified in terms of a single number, the Euler index, or the genus. The genus g simply counts the number of handles on the surface, with the sphere having $g = 0$, the torus $g = 1$, etc (the Euler index χ is equal to $2(1-g)$). The cylinder is a surface with boundaries, but if we make it infinitely long and at the points at ∞ and $-\infty$ we may think of it topologically as a sphere (with two special points). Similarly the complex plane is topologically a sphere, if we add the point $|z| = \infty$. If one tried to do any of this in four dimensions one would quickly be lost, since there does not exist a corresponding classification of four-manifolds. Nearly all four-dimensional field theory is done on the four-dimensional plane. The possible rôle of other topologies and even how to take them into account properly is still very poorly understood.

In statistical mechanics one usually considers only correlation functions on the plane and the torus. In any case the two-dimensional topology is fixed, and determined by the problem one is studying. If one imposes periodic boundary conditions in space and time directions, one works on the torus. The correlation functions one computes are directly related to quantities one measures in experiments.

In string theory the computation of two-dimensional correlation functions is part of the computation of scattering amplitudes in space-time. The prescription (due to Polyakov

[15]) is to sum over all two-dimensional surfaces that satisfy given boundary conditions. These boundary conditions are a consequence of the external particles for which one wants to compute the scattering amplitude.[⋆] The surfaces of interest have a certain number of handles, with tubes sticking out that correspond to the external particles. If we propagate these particles to infinity, and project on a single particle, we may replace these external lines by single points, just as we did in mapping the cylinder to the sphere. The process of interest is then described by (5.1), where the functions "O_i" operators describe the emission of a certain particle state from the point z_i on the surface. The corresponding operators are known as *vertex operators*.

The topology of the surface corresponds to the order of string perturbation theory. The sphere gives us all tree diagrams, the torus all one-loop diagrams, etc. Note that there is only one diagram for each order of perturbation theory. To get the full space-time scattering amplitude to arbitrary order in perturbation theory, we have to sum first over all topologies, and then integrate over all different surfaces of given topology, as well as over the points z_i. These integration variables are called the *moduli* of the surface.

5.1. Correlation functions on the Riemann sphere

Now we turn to the simplest surface, namely the sphere. As before, we represent it as the complex plane, with infinity added as a single point. This is known as the Riemann sphere.

In this case the path-integral can be expressed as a vacuum-to-vacuum amplitude, or vacuum expectation value,

$$\langle 0|\, O_1(\phi(z_1,\bar z_1))\dots O_n(\phi(z_n,\bar z_n))\,|0\rangle \ , \tag{5.2}$$

where O_i are the quantum mechanical operators representing the functions in (5.1), and radial ordering is implicitly understood. The relation between (5.1) and (5.2) is completely analogous to the more familiar relation in field theory between the path integral and time-ordered perturbation theory.

Conformal invariance puts strong constraints on correlators. Let us first consider correlation functions of primary fields (here we omit for simplicity the dependence on $\bar z_i$)

$$\langle 0|\, \phi(z_1)\dots\phi(z_n)\,|0\rangle \ ,$$

and investigate the consequence of invariance under the $SL_2(\mathbf{C})$ subgroup of the conformal group. We focus on this subgroup first to distinguish the extra information we get in two

⋆ There are also *open* string theories, which have non-trivial boundary conditions in the spatial direction in two dimensions. They are thus defined on two-dimensional strips instead of the cylinder. When they interact they may form, under certain circumstances, non-orientable surfaces like Möbius strips. These theories are not considered here; we restrict ourselves to closed, orientable manifolds.

dimensions from that of conformal invariance in arbitrary dimensions. We have

$$\langle 0|\, L_i = \langle 0|\, L_i^\dagger = \langle 0|\, L_{-i} = 0, \qquad \text{for } i = 0, \pm 1$$

Therefore we can derive (for $i = 0, \pm 1$)

$$\begin{aligned} 0 &= \langle 0|\, L_i \phi(z_1) \ldots \phi(z_n) \,|0\rangle \\ &= \sum_j \langle 0|\, \phi(z_1) \ldots \phi(z_{j-1}) \,[L_i, \phi(z_j)]\, \phi(z_{j+1}) \ldots \phi(z_n) \,|0\rangle \\ &+ \langle 0|\, \phi(z_1) \ldots \phi(z_n) L_i \,|0\rangle \end{aligned}$$

The last term vanishes, and the commutator with L_i generates the infinitesimal conformal transformation δ_i. Hence we get

$$\sum_j \langle 0|\, \phi(z_1) \ldots \phi(z_{j-1}) \delta_i \phi(z_j) \phi(z_{j+1}) \ldots \phi(z_n) \,|0\rangle = 0$$

Now we may use the fact that the fields ϕ_i are conformal fields. Then

$$\delta_\epsilon \phi = \epsilon \partial \phi + h \partial \epsilon \phi$$

If we restrict L_i to $SL_2(\mathbf{C})$, ϵ can be 1, z or z^2.

5.2. Two-point functions

Consider for example the two-point function (propagator)

$$G(z_1, z_2) = \langle \phi_1(z_1) \phi_2(z_2) \rangle$$

We find that this function satisfies the differential equation

$$\left[\epsilon(z_1)\partial_1 + h_1 \partial \epsilon(z_1) + \epsilon(z_2)\partial_2 + h_2 \partial \epsilon(z_2)\right] G(z_1, z_2) = 0 \ ,$$

with ϵ as above. Let us look at each of these choices. The case $i = -1$ ($\epsilon = 1$) yields the equation

$$(\partial_1 + \partial_2) G(z_1, z_2) = 0 \ ,$$

which implies that G depends only on the difference of z_1 and z_2, and not on the sum. Hence $G(z_1, z_2) \equiv \mathcal{G}(x)$, $x = z_1 - z_2$. Then the equation with $i = 0$ can be written as

$$[x\partial_x + h_1 + h_2]\, \mathcal{G}(x) = 0 \ .$$

The solution is (up to a normalization)

$$\mathcal{G}(x) = x^{-h_1 - h_2} \ .$$

Finally we can substitute this solution in the equation for $i = 1$. This yields

$$(h_1 - h_2)(z_1 - z_2) G(z_1, z_2) = 0 \ ,$$

so that h_1 must be equal to h_2, or else the propagator vanishes. The final result is thus that

$$G(z_1, z_2) = C(z_1 - z_2)^{-2h} \ , \qquad h = h_1 = h_2 \ .$$

As usual, we have dropped the anti-holomorphic part, which would have given rise to an additional factor $(\bar{z}_1 - \bar{z}_2)^{-2\bar{h}}$. The coefficient C has no physical relevance, as it can be set to 1 by changing the normalization of the primary fields.

5.3. Three-point functions

To get the three-point function

$$G^{(3)}_{ijk} = \langle 0 |\, \phi_i(z_1) \phi_j(z_2) \phi_k(z_3)\, | 0 \rangle$$

we argue in a similar way. Translation invariance shows that it must be a function of the differences $z_{ij} = z_i - z_j$ (this holds in fact for an arbitrary n-point function. Rotation (L_0) invariance leads to the equation

$$[z_1\partial_1 + z_2\partial_2 + z_3\partial_3 + h_1 + h_2 + h_3]\, G^{(3)}(z_{12}, z_{23}) = 0$$

The correlator is a function of two independent variables instead of three, since $z_{13} = z_{12} - z_{23}$.

If we write the solution as

$$G^{(3)}(z_{12}, z_{23}) = \sum_{ab} D_{ab} z_{12}^a z_{23}^b \ ,$$

we find the condition $a + b = -h_1 - h_2 - h_3$. Finally we use L_1. The solution is

$$G^{(3)}_{ijk}(z_{12}, z_{23}, z_{13}) = C_{ijk} z_{12}^{h_3 - h_1 - h_2} z_{23}^{h_1 - h_2 - h_3} z_{31}^{h_2 - h_3 - h_1} \ ,$$

where we have introduced the redundant variable z_{13} to get the solution in a more symmetric form. The coefficients C_{ijk} depend on the normalization of the two-point function, and we will assume that the latter has been set equal to 1. Again one should multiply this expression with the anti-holomorphic factors.

The foregoing results can be understood as follows. We have three complex transformations at our disposal. Using translations, we can move one of three variables z_1, z_2 and z_3 to any desired point in the complex plane, for example $z_1 = 0$. Then, keeping this point fixed we can use the second symmetry (scalings plus rotations, generated by L_0 and $\bar{L}_0$) to move z_2 to any desired point, and finally we can do the same with z_3 using L_1 and $\bar{L}_1$, the special conformal transformation. Actually one can do this separately for the holomorphic and the anti-holomorphic variables if one allows separate complex transformations for each. Then it is simply a matter of requiring that

$$\frac{az_1 + b}{cz_1 + d} = \alpha_1, \ \frac{az_2 + b}{cz_2 + d} = \alpha_2, \ \frac{az_3 + b}{cz_3 + d} = \alpha_3,$$

where α_1, α_2 and α_3 are three fixed points in the complex plane. Often one chooses $z = 0, z = 1$ and $z = \infty$ for these points. These three equations for the four complex variables a, b, c, d subject to the determinant condition $ad - bc = 1$ have a solution if all z_i are different. Hence the entire answer is determined if we know the three point function in just three points.

5.4. Four-point functions

This tells us immediately that it cannot work for the four-point function. Indeed, the best one can do in that case is

$$G^{(4)}(z_i, \bar{z}_i) = f(x, \bar{x}) \prod_{i<j} z_{ij}^{-h_i - h_j + h/3} \bar{z}_{ij}^{-\bar{h}_i - \bar{h}_j + \bar{h}/3} \ ,$$

where $f(x)$ is a function of

$$x = \frac{z_{12} z_{34}}{z_{13} z_{24}} \ ,$$

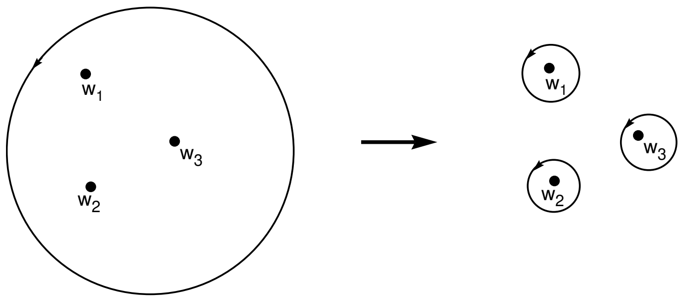
which is the only independent ratio which is invariant under the conformal group.

5.5. Conformal Ward identities

So far we have concentrated on the subgroup $SL_2(\mathbf{C})$. The generalization of the foregoing discussion is obtained by inserting the generator of infinitesimal conformal transformations $\oint dz\epsilon(z)T(z)$ into the correlation function:

$$\langle 0| \oint \frac{dz}{2\pi i}\epsilon(z)T(z)\phi(w_1)\ldots\phi(w_n)|0\rangle \ ,$$

where the contour encircles all the points w_i. By a suitable analytic continuation the contour can be chosen in such a way that the point $z = 0$ is avoided.



We can deform the contour to encircle each w_i separately; then we get

$$\begin{aligned}&\sum_i \langle 0|\phi(w_1)\ldots\oint\frac{dz}{2\pi i}\epsilon(z)T(z)\phi(w_i)\ldots\phi(w_n)|0\rangle\\ =&\sum_i \langle 0|\phi(w_1)\ldots\delta_\epsilon\phi(w_i)\ldots\phi(w_n)|0\rangle \ ,\end{aligned}$$

where the variation of each field $\phi(w_i)$ with respect to ϵ is represented by a contour integral around w_i (*cf.* (3.3), (3.5)).

Since the foregoing holds for any ϵ we may omit the integral; then we get

$$\begin{aligned}&\langle 0|T(z)\phi(w_1)\ldots\phi(w_n)|0\rangle\\ &\quad=\sum_i\left[\frac{h_i}{(z-w_i)^2}+\frac{1}{z-w_i}\frac{\partial}{\partial w_i}\right]\langle 0|\phi(w_1)\ldots\phi(w_n)|0\rangle \ .\end{aligned}$$

5.6. Correlators of descendants

Consider the correlator

$$\langle 0|\,\phi_1(w_1)\dots\phi_{n-1}(w_{n-1})\phi_n^{(-k)}(w_n)\,|0\rangle \ ,$$

where $\phi^{(-k)}$ is the $k^{\rm th}$ descendant of ϕ. Inserting the definition (4.2) and using the conformal Ward identity we may rewrite the result as

$$\oint \frac{dz}{2\pi i}\frac{1}{(z-w_n)^{k-1}}\Bigg\{ \langle 0|\,T(z)\phi_1(w_1)\dots\phi_n(w_n)\,|0\rangle \\ -\sum_{i=1}^{n-1}\left[\frac{h_i}{(z-w_i)^2}+\frac{1}{z-w_i}\frac{\partial}{\partial w_i}\right]\langle 0|\,\phi(w_1)\dots\phi(w_n)\,|0\rangle\Bigg\}$$

In the first term the contour runs outside all points w_i, and hence we can deform it to infinity. Then the integral can be transformed to a contour integral around the point $z=\infty$, To do this explicitly, we can write $T(z)=\sum_n L_n z^{-n-2}$. Then we transform z to $w=\frac{1}{z}$. This yields

$$\oint \frac{dz}{2\pi i}\frac{1}{(z-w_n)^{k-1}}L_n z^{-n-2} = \oint \frac{dw}{2\pi i} -\frac{1}{w^2}\frac{w^{k-1}}{(1-ww_n)^{k-1}}L_n w^{n+2} \\ = -\oint \frac{dw}{2\pi i}\frac{1}{(1-ww_n)^{k-1}}L_n w^{n+k-1}$$

Since $\langle 0|\,L_n=0$ for $n\leq 1$ this integral has no singularity inside its contour, and hence it vanishes. In other words, the contour can be pulled off the back of the Riemann sphere. The other terms can be evaluated as follows. The term

$$\mathcal{L}^i_{-k}\equiv -\oint_w \frac{dz}{2\pi i}\frac{1}{(z-w_n)^{k-1}}\left[\frac{h_i}{(z-w_i)^2}+\frac{1}{z-w_i}\frac{\partial}{\partial w_i}\right]$$

is a contour integral around the point w_i. Using the standard formula

$$\oint_w \frac{dz}{2\pi i}\frac{1}{(z-w)^n}f(z)=\frac{1}{(n-1)!}\partial^{n-1}f(w)$$

we get

$$\mathcal{L}^i_{-k}=-\frac{(1-k)h_i}{(w_i-w_n)^k}+\frac{1}{(w_i-w_n)^{k-1}}\frac{\partial}{\partial w_i}\ .$$

Introducing the operator

$$\mathcal{L}_{-k} \equiv \sum_{i=1}^{n-1} \mathcal{L}^i_{-k}$$

we can write the correlation function as

$$\begin{aligned} &\langle 0|\, \phi_1(w_1) \ldots \phi_{n-1}(w_{n-1}) \phi_n^{(-k)}(w_n)\, |0\rangle \\ &\qquad = \mathcal{L}_{-k} \, \langle 0|\, \phi_1(w_1) \ldots \phi_{n-1}(w_{n-1}) \phi_n(w_n)\, |0\rangle \end{aligned}$$

in other words, the correlator of descendant fields can be expressed entirely in terms of the correlator of primaries. The same is true for more complicated descendants or for correlators involving more than one descendant, but the formulas become more complicated.

This is the extra information we get from the additional conformal symmetry specific to two dimensions. As we have seen, $SL_2(\mathbf{C})$ determines the two-point-functions completely, the three-point functions up to constants C_{ijk}, the four-point functions up to a function, etc. In higher-dimensional conformal field theories one has an analogous symmetry, and analogous restrictions. If we had only $SL_2(\mathbf{C})$ at our disposal, there would still be some relations between correlation functions, but clearly far more limited, since we could only use the Virasoro generators $L_0, L_{\pm 1}$. Sometimes field that transform like conformal fields under $SL(2,C)$ (or the conformal group in higher dimensions) are called *quasi-primary*. Descendant fields can sometimes be quasi-primary. An example is the energy-momentum tensor. Due to the conformal anomaly it is not a primary field, but since the anomaly does not affect $SL_2(\mathbf{C})$ it is a quasi-primary field.

The extra Virasoro generators beyond $SL_2(\mathbf{C})$ allow us to organize the fields in the theory into much larger sets, each consisting out of one primary field and in infinite number of descendants. The task of computing the correlation functions reduces to doing so for correlation functions of primary fields.

5.7. Null state decoupling

There are further constraints on the correlation functions, but they are dependent on the representation we are considering, and do not hold generically.

The foregoing results have important implications in the special case that a descendant of a primary field is a vector of norm zero, a null state. Since this is not a state in the theory, it must decouple from all physical amplitudes. For example, we have found in the previous chapter that certain Virasoro representations – namely those whose c and h values fall on the second level vanishing curve – have a second level null state

$$\left[L_{-2} - \frac{3}{2(2h+1)} L_{-1}^2 \right] |h\rangle = 0 \ .$$

It follows that any amplitudes involving the corresponding descendant field must vanish.

The descendant field is (here ϕ is a conformal field with weight h)

$$\phi^{(-2)} - \frac{3}{2(2h+1)}\partial^2\phi \ ,$$

where we have made use of the fact that the first descendant of a field is simply the derivative. Hence we get

$$\left\{ \sum_{j\neq i} \left[\frac{h_j}{(w_i-w_j)^2} + \frac{1}{w_i-w_j}\partial_j \right] \right. \\ \left. - \frac{3}{2(2h_i+1)}\partial_i^2 \right\} \langle 0|\, \phi_1(w_1)\ldots\phi_n(w_n)\,|0\rangle = 0 \ ,$$

where the first two terms come from $\mathcal{L}_{-2}$, and where we are assuming that ϕ_i has a second level null vector. The third term may be expressed in terms of $(\mathcal{L}_{-1})^2$ using the results of the previous subsection, but that does not simplify the answer.

This is a differential equation that any n-point function involving ϕ_i has to satisfy. Obviously there are higher order differential equations for any other singular state. Thus we see that all correlation functions satisfy a huge number of differential equations.

It is instructive to verify that the two-point and three point functions do indeed satisfy the equation that follows from the second level descendants.

Note that we have these strong constraints only if there are null states in the Virasoro representations *i.e.* only if $0 < c \leq 1$.⋆

5.8. Operator products

Consider the three-point function $G^{(3)}_{ijk}(z_1,z_2,z_3)$ in the limit $z_1 \to z_2$. The leading term is

$$\langle 0|\, \phi_i(z_1)\phi_j(z_2)\phi_k(z_3)\,|0\rangle = G^{(3)}_{ijk}(z_1,z_2,z_3) \approx C_{ijk}(z_1-z_2)^{h_3-h_1-h_2}(z_1-z_3)^{-2h_3}$$

The last term looks like the propagator of the field ϕ_3, and the expression suggests that the two primary fields ϕ_i and ϕ_j contain in their product the field ϕ_3, with strength C_{ijk}. The precise statement of this fact is the operator product expansion, which says that the product of two operators $O_i(x)$ and $O_j(y)$ in field theory can be expanded in a complete set of operators $O_k(y)$ (without loss of generality we can choose our basis of operators at the

⋆ The only exception is the null state $L_{-1}\,|0\rangle$.

point y).

$$O_i(x)O_j(y) = \sum_k \mathcal{C}_{ijk}(x,y)O_k(y)\ .$$

By translation invariance $\mathcal{C}_{ijk}(x,y) = \mathcal{C}_{ijk}(x-y)$.

In conformal field theory we can take as the basis all primaries and a complete set of descendants. Then the operator product expansion has the form

$$\phi_i(z,\bar{z})\phi_j(w,\bar{w}) = \sum_k C_{ijk}(z-w)^{h_k-h_i-h_j}(\bar{z}-\bar{w})^{\bar{h}_k-\bar{h}_i-\bar{h}_j}\phi_k(w,\bar{w})\ . \tag{5.3}$$

The sum on the right-hand side contains both primaries and descendants, and h_k must be interpreted as the ground state conformal weight plus the excitation level of the descendant. If i,j,k are primaries, this expression agrees with the result of the three-point function, and tells us in particular that the coefficients C_{ijk} appearing in the operator product and the three-point function must be the same. In fact, the operator product holds also if all three fields are descendants. The behavior of the coefficients as a function of z and w is completely fixed by translation invariance and the scaling properties ("dimensions") of the fields involved. Both are good symmetries, even for descendants. We have already encountered the operator product of the energy momentum tensor (a descendant) with other fields.

Since correlation functions involving descendants are related to those of primaries, it should not be surprising that one can do the same for operator products. Indeed, the operator product of two primary fields may be written as

$$\phi_i(z,\bar{z})\phi_j(w,\bar{w}) = \sum_{k,p,\bar{p}} C^{p\bar{p}}_{ijk}(z-w)^{h_k+n_p-h_i-h_j}(\bar{z}-\bar{w})^{\bar{h}_k+n_{\bar{p}}-\bar{h}_i-\bar{h}_j}\phi^{p\bar{p}}_k(w,\bar{w})\ , \tag{5.4}$$

where $\phi^{p\bar{p}}_k$ is a descendant of ϕ_k at level $(n_p, n_{\bar{p}})$. One can show that the operator product coefficients $C^{p\bar{p}}_{ijk}$ can be expressed in terms of those of the primary fields,

$$C^{p\bar{p}}_{ijk} = C_{ijk}\beta^{k(p)}_{ij}\bar{\beta}^{k(\bar{p})}_{ij}\ ,$$

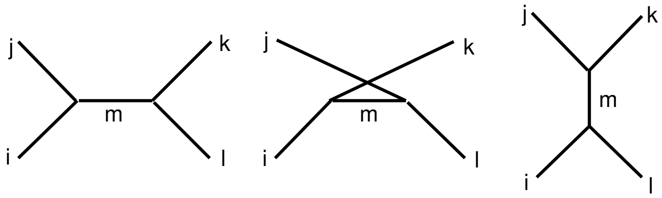
where the coefficients β are determined completely by conformal invariance.

5.9. Duality

The operator product is a useful tool for the computation of four-point functions. However, if we consider the correlator

$$\langle 0|\, \phi_i(z_1,\bar z_1)\phi_j(z_2,\bar z_2)\phi_k(z_3,\bar z_3)\phi_l(z_4,\bar z_4)\, |0\rangle$$

it is not obvious for which pairs of fields we should compute the operator product first. If we combine the fields in pairs there are three alternatives, namely $(i,j)(k,l)$, $(i,k)(j,l)$ or $(i,l)(j,k)$. The result should not depend on how we perform the calculation. This is known as duality (as are many other symmetries). Diagrammatically we may represent this as



where in all three cases a sum over the intermediate states is understood. To write the equations belonging to this picture one defines the *conformal blocks*. First we simplify the four-point function by fixing all but one coordinate using $SL_2(\mathbf{C})$

$$F_{ijkl}(z,\bar z) = \langle 0|\, \phi_i(z,\bar z)\phi_j(0,0)\phi_k(1,1)\phi_l(\infty,\infty)\, |0\rangle \ . \tag{5.5}$$

If we compute this correlator by making the contractions $(i,j)(k,l)$ we can write the result as

$$F_{ijkl}(z,\bar z) = \sum_m C_{ijm}C_{mkl}\mathcal{F}^m_{ijkl}(z)\bar{\mathcal{F}}^m_{ijkl}(\bar z)\ ,$$

where the sum is over all primary fields. All descendant contributions are taken into account by the functions $\mathcal{F}$, called the conformal blocks. Since the Virasoro algebra acts chirally (*i.e.* splits completely in holomorphic and anti-holomorphic generators) we can factorize the contribution of each primary into a holomorphic and an anti-holomorphic contribution, as shown.

Duality imposes the following condition:

$$\begin{aligned}
&\sum_m C_{ijm}C_{mkl}\mathcal{F}^m_{ijkl}(z)\bar{\mathcal{F}}^m_{ijkl}(\bar z)\\
&\qquad\qquad = \sum_m C_{ikm}C_{mjl}\mathcal{F}^m_{ikjl}(1-z)\bar{\mathcal{F}}^m_{ikjl}(1-\bar z)\\
&\qquad\qquad = z^{-2h_i}\bar z^{-2\bar h_i}\sum_m C_{ilm}C_{mkj}\mathcal{F}^m_{ilkj}(\frac{1}{z})\bar{\mathcal{F}}^m_{ilkj}(\frac{1}{z})
\end{aligned}$$

Note that to go from the first to the second (third) line one has to interchange j and k (l). This implies that the choice of the fixed points $0, 1$ and ∞ does not agree anymore with the convention chosen in (5.5), and we have to make a conformal transformation to get back to our conventional coordinates. In going from the first to the second line that transformation is $z \to 1 - z$, which interchanges 0 and 1 while leaving ∞ fixed. In going from the first to the third we use $z \to \frac{1}{z}$, which interchanges 0 and ∞ while leaving 1 fixed. Inevitably these transformations also act on the fourth argument, z, and furthermore they introduce some conformal factors.

Once the conformal blocks are known, these equations impose strong constraints on the operator product coefficients C_{ijk}. The conformal blocks are constraint by decoupling equations for null vectors (assuming there are null vectors). In principle it may be possible to use these constraints to determine the conformal blocks as well as all the operator product coefficients completely. In practice this is quite hard, although the computation can be carried out completely for the simplest conformal field theory, the one at $c = \frac{1}{2}$ (the critical Ising model).

6. Conformal field theory on a torus

Up to now we have seen that a conformal field theory is described algebraically by a set of ground states

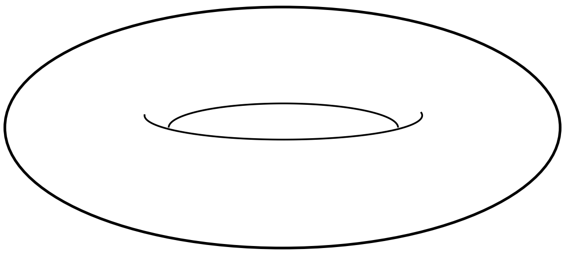
$$\left|h, \bar{h}\right\rangle = \phi_{h,\bar{h}}(0,0)\left|0\right\rangle$$

on which the left- and right Virasoro algebra acts.

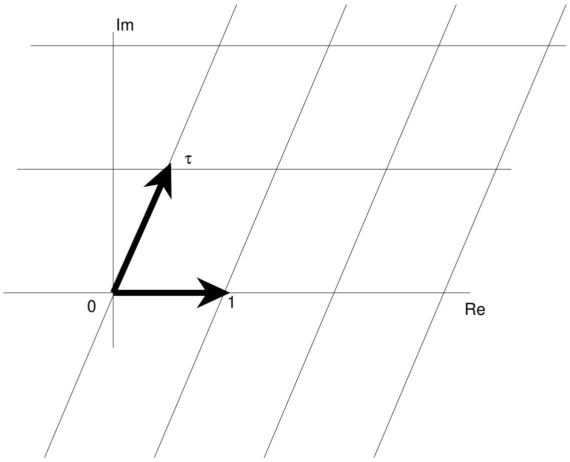
The question which we want to address now is: which combinations of ground states can actually occur in a conformal field theory. Once we know that we have specified the set of physical states in the theory completely: they consist of the ground states plus all the descendants generated by the generators of the Virasoro algebra, minus all null vectors.

6.1. Parametrization of the torus

Intuitively one would expect all the states in a theory to contribute to loop diagrams. Hence loop diagrams should be a useful tool to answer this question. For this reason we are going to study conformal field theory on the simplest loop diagram, the torus.



The torus is a cylinder whose ends have been sewn together, as shown in the "artists impression" above. The most convenient mathematical description of the torus is in terms of the complex plane modulo a lattice, as show below



The meaning of this picture is that all points in the complex plane that differ by a linear combination of the two basic lattice vectors are considered identical. Identification along the real axis has the effect of rolling up the complex plane to a cylinder; then identification along the vector labelled τ rolls up the cylinder to a torus.

A lot of symmetries have been taken into account already in arriving at the picture. First of all two-dimensional general coordinate invariance (also called reparametrization invariance or diffeomorphism invariance) has been used to "straighten" the coordinates so that we get a lattice; rotational invariance has been used to make one direction point along the real axis; translation invariance to put a point of the lattice at the origin and global scale invariance to make the horizontal lattice spacing equal to 1. This means that finally the entire lattice is described by one complex number, τ, which can be chosen in the upper half plane.

6.2. The partition function

We would like to compute the path integral

$$\int \mathcal{D}\phi e^{-S_E(\phi)} \,, \tag{6.1}$$

where S_E is the Euclidean action of a given field configuration on the torus, and the integral is over all field configurations. This notation is only symbolic. If we have a Lagrangian description of a conformal field theory, ϕ stands for all fields in the theory. For example, we could consider free bosonic theories, and in that case the integral is over the bosonic fields Φ^i. However, in many cases such a Lagrangian formulation is either not available, or not practically usable. It is therefore more convenient to express the path-integral in terms of the Hamiltonian of the theory. The Hamiltonian is (related to) $L_0 + \bar{L}_0$, and this is a quantity we always have at our disposal in a conformal field theory.

In ordinary quantum dynamics (in 0+1 dimensions) one can derive the formula

$$\int\limits_{\text{PBC}} \mathcal{D}q e^{-S_E(q)} = \text{ Tr } e^{-\beta H} \ . \tag{6.2}$$

Here S_E is some Euclidean action that satisfies certain conditions; those conditions are satisfied for example for the harmonic oscillator. The integral is over all paths $q(t)$ that start at $t = 0$ and end at $t = \beta$, subject to the periodic boundary condition ("PBC") $q(0) = q(\beta)$. The derivation can be found in many text books on path integrals.

The path integral we are considering is not in 0+1 dimensions, but in 1+1 dimensions. Hence the integration variables ϕ have an extra continuous label x^1. However this is merely a generalization of (6.2) from one degree of freedom (q) to infinitely many ($\phi(x^1)$).

In the lattice description of the torus we regard the real axis as the x^1 direction, and the imaginary axis as the Euclidean time direction. If Re $\tau = 0$ we would find for the path integral (6.1), as a straightforward generalization of (6.2)

$$\int \mathcal{D}\phi e^{-S_E(\phi)} = \text{ Tr } e^{-2\pi \text{ Im } \tau H} \ .$$

The only small subtlety here is the factor 2π. It appears because the torus as depicted earlier has a periodicity 1 rather than 2π along the x^1 direction. To make contact with earlier conventions we had to scale up the entire lattice by a factor of 2π, so that we get $2\pi\tau$ instead of τ.

What happens if Re $\tau \neq 0$? In that case we have to twist the torus before gluing it together again, and the periodic boundary conditions in the Euclidean time direction are defined including such a shift. The operator performing such a shift in the x^1 direction is the momentum operator P. A shift by an amount 2π Re τ is achieved by the operator

$$e^{iP(2\pi \text{ Re } \tau)}$$

The correct result is obtained by inserting this shift operator in the trace, so that we get

$$\int \mathcal{D}\phi e^{-S_E(\phi)} = \text{ Tr } e^{-2\pi \text{ Im } \tau H} e^{iP(2\pi \text{ Re } \tau)} \ . \tag{6.3}$$

The operators H and P are the time and space translation operators on the cylinder, and they can be derived from the energy-momentum tensor.

6.3. The cylinder versus the Riemann sphere

Up to now we have defined quantum conformal field theory, and in particular the energy-momentum tensor, on the complex plane (Riemann sphere), related to the cylinder via a conformal mapping. We now want to examine carefully how the energy-momentum tensor defined on the complex plane is related to that on the cylinder.

To do so we need the transformation of the energy momentum tensor itself under conformal transformations. Since it is not a conformal field, we have to work this out explicitly. Infinitesimal conformal transformations are generated by

$$Q_\epsilon = \frac{1}{2\pi i}\oint dz \epsilon(z) T(z) ,$$

as we have seen before. The infinitesimal conformal transformation of the energy momentum tensor is thus

$$\delta_\epsilon T(w) = [Q_\epsilon, T(w)] = \frac{1}{2\pi i}\oint dz \epsilon(z) T(z) T(w) .$$

Into this expression we insert the operator product of $T(z)$ and $T(w)$. The result is

$$\delta_\epsilon T(w) = \epsilon(w)\partial T(w) + 2\partial\epsilon(w) T(w) + \frac{c}{12}\partial^3\epsilon(w)$$

The global form of this transformation is

$$T(w) \to (\partial f)^2 T(f(w)) + \frac{c}{12} S(f, w) \tag{6.4}$$

The factor multiplying $\frac{c}{12}$ is known as the Schwartzian derivative,

$$S(f, z) = \frac{\partial f \partial^3 f - \frac{3}{2}(\partial^2 f)^2}{(\partial f)^2} .$$

It is easy to see that its infinitesimal form is indeed $\partial^3\epsilon$ (just substitute $f(w) = w + \epsilon(w)$), but the exact expression is a bit harder to understand. Note however the following. If we apply a second conformal transformation, $w \to g(w)$, we get in the first step (6.4), and after the second step

$$(\partial_w g(w))^2 \left[(\partial_g f(g))^2 T(f(g(w))) + \frac{c}{12} S(f(g), g) \right] + \frac{c}{12} S(g(w), w) \Big]$$

On the other hand, defining the function $h = f \circ g$ (which means $h(w) \equiv f(g(w))$) we would expect to get exactly (6.4) with f replaced by h. For the terms involving T this is manifestly true, since $\partial_w g(w)\partial_g f(g) = \partial_w f(g(w) = \partial_w h(w)$, but for the constant terms we get the non-trivial condition

$$S(f\circ g,w) = (\partial_w g(w))^2 S(f(g),g) + S(g(w),w)$$

One may check that the Schwartzian derivative does indeed satisfy this condition.

We are now ready to apply this to the map from the plane to the cylinder. The map we will use is $w = e^{iz}$, as in Eqn. (3.1). Hence w is the plane coordinate and z the cylinder coordinate. We get

$$T_{\rm cyl}(z) = [\partial_z w(z)]^2 T(w(z) + \frac{c}{12}S(w,z) = -[w^2 T(w) - \frac{c}{24}] \tag{6.5}$$

Now we can substitute the mode expansion for $T(w)$ on the plane

$$T(w) = \sum_n w^{-n-2} L_n \ .$$

Then we find

$$T_{\rm cyl}(z) = -[\sum_n e^{-nz}(L_n) - \frac{c}{24}] \ ,$$

and analogously for the anti-holomorphic component.

Our next task is to find the precise definition of H and P. They are derived from the energy-momentum tensor on the cylinder in the following way*

$$H = \frac{1}{2\pi}\int dx^1 T^M_{00} = \frac{1}{2\pi}\int dx^1 T^E_{22} = -\frac{1}{2\pi}\int d{\rm Re}\, z\ [T^{\rm cyl}(z) + T^{\rm cyl}(\bar z)]$$

and

$$P = \frac{1}{2\pi}\int dx^1 T^M_{01} = \frac{1}{2\pi}\int dx^1(-iT^E_{21}) = -\frac{1}{2\pi}\int d{\rm Re}\, z\ [T^{\rm cyl}(z) - T^{\rm cyl}(\bar z)]$$

* For a generic rank 2 tensor $X_{00} = -X_{22}$ and $X_{01} = iX_{21}$. Here there is an additional $-$ sign because the Euclidean action is defined with an extra $-$ sign w.r.t. the Minkowski action. The Euclidean and Minkowski energy momentum tensors T^E and T^M are both derived from these respective actions using formula (2.1).

Hence we find

$$H = L_0 - \frac{c}{24} + \bar{L}_0 - \frac{\bar{c}}{24} \ ,$$

where we have allowed for the possibility that the holomorphic and anti-holomorphic components have different central charges. For the momentum we find

$$P = (L_0 - \frac{c}{24}) - (\bar{L}_0 - \frac{\bar{c}}{24}) \ .$$

Substituting this into (6.3) we get

$$\int \mathcal{D}\phi e^{-S_E(\phi)} = \ \mathrm{Tr}\ e^{2\pi i\tau(L_0 - \frac{c}{24})} e^{-2\pi i\bar{\tau}(\bar{L}_0 - \frac{\bar{c}}{24})} \equiv P(\tau, \bar{\tau}) \ .$$

The right hand side of this expression will serve as the definition of the partition function for general conformal field theories. As promised, this expression does not require a Lagrangian formulation, and only uses the Virasoro generators themselves. The trace is over all states in the Hilbert space (*i.e.* not including zero norm states).

6.4. Virasoro characters

The partition function can be expressed in terms of primary fields and descendants, in the following way

$$P(\tau, \bar{\tau}) = \sum_{i,j} M_{ij} \mathcal{X}_i(\tau) \mathcal{X}_j(\bar{\tau}) \ . \tag{6.6}$$

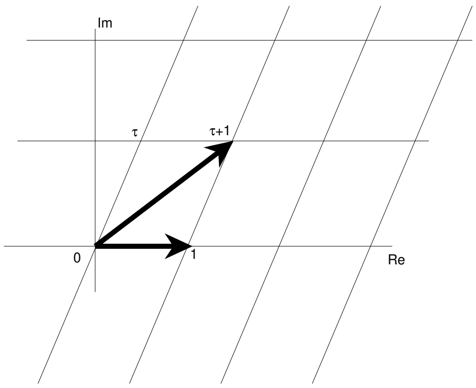
Here i and j label a certain highest weight states $|i, j\rangle$. The label i is used for representations of the holomorphic algebra, and j for the anti-holomorphic algebra. The multiplicity of such a state is M_{ij}, a non-negative integer. The functions $\mathcal{X}$ are the (Virasoro) characters of the representation. They are defined as

$$\mathcal{X}_i(\tau) = \ \mathrm{Tr}_{\ \text{desc. of } i} e^{2\pi i\tau(L_0 - \frac{c}{24})}$$

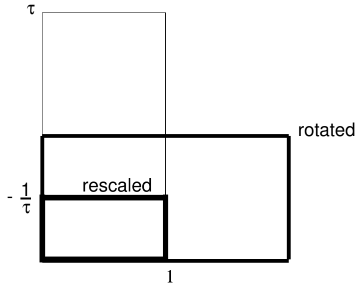
Thus the trace is over all (positive norm) states in the highest weight representation labelled i. If we know the content of the representation, the missing information is thus contained in the integers M_{ij}.

6.5. Modular invariance

Up to now we had defined the torus in terms of a lattice. This lattice was defined by two basis vectors, corresponding to the points "1" and "τ" in the complex plane. However, the same lattice – and hence the same torus – can be described just as well by choosing different basis vectors. For example the choice "1", "$\tau+1$" clearly describes the same lattice



This can be generalized further. One should keep in mind that the torus was defined by rotating one basis vector along the real axis in the complex plane, and scaling it to 1. The choice of this basis vector is free; we can also choose the direction "τ". This has the effect of interchanging the two basis vectors. This rotation, combined with a rescaling the new basis vector along the real axis, has the effect of replacing τ by $-\frac{1}{\tau}$. This is most easily illustrated by taking τ purely imaginary:



The set of such transformations of the torus forms a group, called the *modular group*. We have identified two elements of that group, namely

$$
\begin{aligned}
T: &\qquad \tau \to \tau + 1 \\
S: &\qquad \tau \to -\frac{1}{\tau}
\end{aligned}
$$

It turns out that these two transformations generate the entire group. The most general modular transformation has the form

$$\tau \to \frac{a\tau + b}{c\tau + d}, \qquad a, b, c, d \in \mathbf{Z}; \quad ad - bc = 1 \ .$$

This group is isomorphic to $SL_2(\mathbf{Z})/\mathbf{Z}_2$. The group SL_2 can be defined by the set of 2×2 matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with determinant 1 (we have already encountered the group $SL_2(\mathbf{C})$, where the matrix elements are complex numbers). The group SL_2 contains the element $-\mathbf{1}$. In the modular transformation this is indistinguishable from the identity, and for this reason the modular group is actually isomorphic to $SL_2(\mathbf{Z})/\mathbf{Z}_2$ rather than $SL_2(\mathbf{Z})$. One may check that the modular transformations satisfy

$$(ST)^3 = S^2 = 1 \ .$$

It is now natural to ask how the partition function behaves under transformations in the modular group. If we start with a well-defined two-dimensional theory on the torus, in which all fields are periodic along all cycles around the torus, the result of the path-integral should not depend on how that torus was parametrized. Hence the partition function should be invariant under modular transformations. For example, if we compute the path integral for free bosons on the torus, we will automatically get a modular invariant partition function.

6.6. Modular transformations of the characters

On the other hand, if we just choose some multiplicities in (6.6) we will in general not get a modular invariant partition function. To verify if a partition function written in terms of characters is modular invariant we need to know how the characters transform. This is easy for the transformation T:

$$\mathcal{X}_i(\tau+1) = e^{2\pi i(h_i - \frac{c}{24})}\mathcal{X}_i(\tau)$$

This is often written in matrix form,

$$\mathcal{X}_i(\tau+1) = \sum_j T_{ij}\mathcal{X}_j(\tau) \ ,$$

where T is a diagonal matrix of phases. The transformation S is much harder to compute, and it is not even obvious that one can express the transformed characters in the original ones. One can, however, and the result is

$$\mathcal{X}_i(-\frac{1}{\tau}) = \sum_j S_{ij}\mathcal{X}_j(\tau) \ ,$$

where S is a unitary and symmetric matrix.

6.7. Conditions for modular invariance

The conditions for modular invariance of the partition function can now be phrased in the following simple matrix form

$$[M,T]=[M,S]=0\ ,$$

with

$$M_{ij}\in\mathbf{Z},\qquad M_{ij}\geq 0\ .$$

Furthermore this is usually supplemented with the additional physical requirement that the vacuum is present in the theory, and is unique. If we label the vacuum by "0", we get thus the condition

$$M_{00}=1\ .$$

6.8. The diagonal invariant

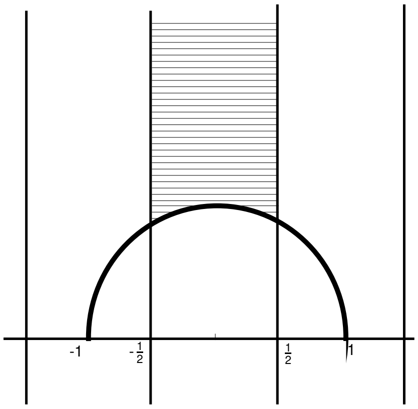
These conditions have a trivial solution

$$M_{ij}=\delta_{ij}\ .$$

This is called the diagonal invariant. Consider for example a conformal field theory with central charge $\frac{1}{2}$. We have seen that at this value for c there are just three Virasoro representations, with $h=0,\frac{1}{2}$ and $\frac{1}{16}$. A modular invariant partition function of this system is thus obtained by choosing the three ground states $\left|0,0\right\rangle,\left|\frac{1}{2},\frac{1}{2}\right\rangle$ and $\left|\frac{1}{16},\frac{1}{16}\right\rangle$. Corresponding to these ground states there are three primary fields that create the ground states from the vacuum, often denoted as $\mathbf{1}$, ψ and σ respectively. We have now specified the theory completely. It is known as the Ising model.

6.9. Integration over moduli

As remarked above, it is not hard to construct partition functions that are not modular invariant. Usually these are rejected. We have already argued that they can not correspond to well-defined two-dimensional theory on the torus. In string theory there is another reason, namely that one has to integrate over the parameter τ. String perturbation theory is a summation over all two-dimensional surfaces. This sum splits in a sum over all different topologies, and integrals over all different shapes of surfaces with a given topology, the *moduli.* In two dimensions the topology can be described by a single parameter, the number of handles, or the genus. The genus is 0 for the sphere, 1 for the torus, etc. At genus 1 there is one complex modulus, the parameter τ. The integral over τ is not over the full positive upper half plane, but should be restricted to a region that covers the set of distinct tori just once. An example of such a region is shown below



The entire upper half plane is covered with an infinite number of such regions of different shapes and sizes. For example, the lower part of the strip $-\frac{1}{2} \leq$ Re $\tau < \frac{1}{2}$ contains an infinite number of such regions. The integral over τ should not depend on the choice of the region, or otherwise the theory is not well-defined. If the theory is modular invariant, this problem does not arise.

6.10. Operator products and locality

An important consequence of modular invariance is that operator products are local, *i.e.* that it has no branch cuts as a function of $z-w$. This is true because one always finds combinations like $(z-w)^h(\bar{z}-\bar{w})^{\bar{h}} = |z-w|^{2\bar{h}}(z-w)^{h-\bar{h}}$, with $h-\bar{h}$ an integer.

This implies that integrals over the positions z and w around the cylinder (or along cycles of the torus or higher Riemann surfaces) are well-defined. Such integral occur always in the application to string theory, where one has to integrate not only over the moduli of the Riemann surface, but also over the positions of the vertex operators. If the operator product were not local, there would inevitably exist correlation functions which have branch cuts in some of their variables, and one would encounter integrals like $\oint dz\sqrt{z}$ around $z=0$. Since the integrand is not periodic around the origin, the integral depends on the choice of the beginning and the end of the interval, and is thus not defined.

In applications to statistical mechanics the existence of such branch cuts is less obviously fatal, and indeed even in the application to string theory it can be useful to drop the requirement of locality (and modular invariance) in intermediate results.

6.11. Fusion rules

The coefficients C_{ijk} in three-point functions or in the operator product satisfy certain selection rules. Note that i, j and k label fields $\phi_i(z,\bar{z})$ *etc.*. The label "i" stands thus for some combination of holomorphic and anti-holomorphic representations of the Virasoro algebra. The selection rules depend on those representations. To discuss them it is thus better to label the fields as $\phi_{i,\bar{\imath}}$.

The selection rules imposed by the Virasoro algebra are called *fusion rules* [9]. They are written as follows

$$[i] \times [j] = \sum_k N_{ij}^k [k] \ .$$

Here N_{ijk} is a set of non-negative integers, and $[i] \ldots$ label representations of the Virasoro algebra. The notation with raised indices is introduced for future purposes (when we consider extended algebras), and has no relevance here.

If the fusion coefficient N_{ij}^k vanishes, this means that the OPE-coefficients $C_{i\bar{\imath},j\bar{\jmath},k\bar{k}}$ vanish. If the coefficient does not vanish, the corresponding OPE-coefficient is allowed, and usually it is then indeed non-zero.

From this discussion one might think that N_{ij}^k must be either zero or 1, but actually it can be any non-negative integers. Values higher than 1 indicate that there exists more than one way of coupling the fields. This should be compared to the tensor product rules in group theory; for example in $SU(3)$ there are two distinct ways of coupling two 8's to a third 8. The interpretation in terms of three point functions tells us that N_{ijk} must be symmetric in all three indices.

Although the fusion rules are similar to rules for tensor products in several respects, it would not be correct to refer to them as the tensor product rules of the Virasoro algebra. If one tensors two Virasoro representations, one would have to add up the central charge and the conformal weight, which is not the case for the fusion rules.

6.12. The Verlinde formula

The reason fusion rules are discussed in this chapter is that there turns out to be a relation between the fusion rule coefficients and the matrix S, discovered by E. Verlinde [16]. His formula is⋆

$$N_{ijk} = \sum_n \frac{S_{in} S_{jn} S_{kn}}{S_{0n}} \ . \tag{6.7}$$

It is remarkable that such a bizarre-looking expression involving matrix elements of a unitary matrix actually produces non-negative integers, but it does!

Another useful way of looking at this formula is to regard N_{ijk} as a collection of matrices $(N_i)_{jk} = N_{ijk}$. Then (6.7) can be rewritten as

$$(S^\dagger N_i S)_{pq} = (\frac{S_{iq}}{S_{0q}}) \delta_{pq} \ .$$

In words, the matrix S simultaneously diagonalizes the fusion rules for all fields i. In deriving this relation we have used the fact that the matrix S on Virasoro representations is real.

⋆ We are restricting ourselves here to theories with trivial charge conjugation, which is equivalent to S being real. This includes the minimal series of Virasoro theories. The general case will be discussed later.

The ratios

$$\lambda_i^{(n)} \equiv \frac{S_{in}}{S_{0n}}$$

are sometimes called the (generalized) *quantum dimensions* of the field i (there is a more restricted definition where one only calls $\lambda_i^{(0)}$ the quantum dimension). Yet another way of writing the Verlinde formula is as

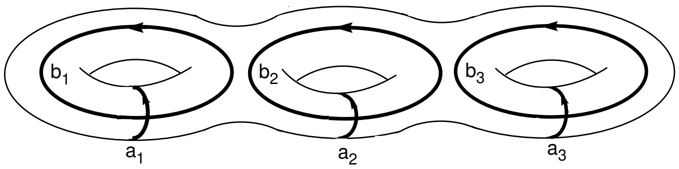
$$\lambda_i^{(n)}\lambda_j^{(n)} = \sum_k N_{ijk}\lambda_k^{(n)} \ ,$$

with no sum over n. This equation states that the quantum dimensions for each value of n form a one-dimensional representation of the fusion rules.

Formula (6.7) was conjectured by Verlinde, and proved by Moore and Seiberg [17].

6.13. Higher genus partition functions

The basic ideas discussed here have generalizations to higher genus. On a surface with n handles one can define a basis of homology cycles a_i, b_i, $i = 1, \ldots n$



as shown in the figure. One can choose a set of n holomorphic 1-forms on the surface, and normalize them so that

$$\int_{a_i} \omega_j = \delta_{ij}$$

Then the integral along the b cycles defines the *period matrix* Ω_{ij}

$$\int_{b_i} \omega_j = \Omega_{ij}$$

This is the higher genus generalization of τ. In the lattice picture of the torus, the a-cycle is the line from 0 to 1, and the b cycle the line from 0 to τ. The holomorphic 1-form is dz. Since it is constant it is periodic, hence well-defined on the torus. Clearly $\int_a dz = \int_0^1 dz = 1$ so that it is properly normalized. Then $\Omega_{11} = \int_b dz = \int_0^\tau dz = \tau$. The modular transformation S corresponds to the mapping $a \to -b$ and $b \to a$, while the transformation T corresponds to replacing b by $a + b$, without changing a.

The higher genus generalization of the modular transformation is

$$\Omega \to (A\Omega + D)(C\Omega + D)^{-1} \ ,$$

where the $n \times n$ matrices A, B, C, D must be such that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2n}(\mathbf{Z})$$

For $n = 1$ one has the isomorphism $Sp_{2n} \sim SL_2$.

The condition for higher genus modular invariance is thus simply invariance of the partition function $P(\Omega, \bar{\Omega})$ under this group. The computation of the partition function is much harder, and in practice has only been done for theories of free bosons and fermions, plus a few isolated other cases.

The most important constraint come from the partition function at genus 1. The genus-2 partition function adds some further restrictions, in particular to quantities which vanish at genus 1. There are believed to be no further constraints from genus three and higher.

The definition of the modular domain which contains all inequivalent surfaces exactly once becomes much harder than it is for the torus.

7. Extensions of the Virasoro algebra

In most applications one is dealing with theories that have more symmetries than just the Virasoro algebra. Such symmetries are called extensions of the Virasoro algebra. These theories have a larger algebra that contains the Virasoro algebra as a subalgebra. These generalized algebras are often referred to as *chiral algebras*, since they are generated by currents that are holomorphic or anti-holomorphic, as we will see. When going back to the cylinder and to Minkowski space, holomorphic dependence on z translates to dependence on $x + t$, so that the corresponding modes are purely left-moving; hence the word chiral. The chiral algebra may in fact be different in the holomorphic and the anti-holomorphic sector; then one speaks sometimes of a *heterotic* theory.

Since the algebra contains the Virasoro algebra, all representations can be decomposed into Virasoro representations. In most cases the number of Virasoro representations contained in an extended algebra representation is infinite.

7.1. Rational conformal field theories

We have seen that the Virasoro algebra with $0 < c < 1$ has a finite number of unitary representations. In general, a conformal field theory with an extended algebra with a finite number of unitary representations is called a *rational* conformal field theory. One can show that the values of c and h in such a theory must be rational numbers [18,19].

For $c \geq 1$ the number of Virasoro representations is infinite. This implies that a diagonal modular invariant partition function necessarily contains an infinite number of terms.

Since extended algebras can group an infinite number of Virasoro characters into extended algebra representations, it may happen that a conformal field theory with $c > 1$, which has an infinite number of Virasoro representations, becomes a rational conformal field theory with respect to an extended algebra.

This clearly has positive consequences for the solvability of the theory. First of all one would expect that there are fewer distinct correlation functions to be calculated. Just as for the Virasoro algebra one would expect that it should be sufficient to know the correlation functions for just one (highest weight) state in each representation. In a rational conformal field theory the number of distinct n-point functions is then finite. This implies also that the number of operator product coefficients is finite, so that one has a much better chance of determining them all from duality arguments. But perhaps the most important consequence of additional symmetry is the appearance of additional null vectors. This implies more constraints on correlation functions than the Virasoro algebra gives by itself, so that correlators have to satisfy additional differential equations.

7.2. Currents

The Virasoro algebra is generated by operators L_n, which are modes of a current $T(z)$, which has conformal spin 2. Extended algebras are generated by modes of other currents. These currents may have integer spin (bosons), half-integer spin (fermions) or fractional (rational, but not half-integer) spin (para-fermions [20]). An important difference between bosonic currents and (para)-fermionic ones is that the former can satisfy the condition $h - \bar{h} = 0 \bmod 1$. Hence the currents $J(z)$ can appear as conformal fields in a modular invariant theory. The converse is also true. Suppose a conformal field theory contains a conformal field $J(z, \bar{z})$ with $h \in \mathbf{Z}$ and $\bar{h} = 0$. Then we know that the state $J(0,0)\,|0\rangle$ satisfies $\bar{L}_{-1} J(0,0)\,|0\rangle = 0$, since this would-be descendant has zero norm. Hence the corresponding descendant field must vanish. This field is $\partial_{\bar{z}} J(z, \bar{z})$. Since it must vanish, $J(z, \bar{z})$ must be holomorphic. .

7.3. Fermionic currents

Fermionic currents are often discussed in a similar way, as if they were holomorphic operators. However, they can never appear like that in a modular invariant theory. One way of thinking about them is in terms of a theory that is conformally invariant only under a maximal subgroup of the modular group that does not contain the element T but does contain T^2. Then fermionic operators are allowed, but correlation functions on the cylinder may have branch cuts. We will soon see the consequences of this fact. Similar remarks apply to para-fermionic theories, but will not be discussed here.

7.4. Mode expansions

Modes of the currents are defined as follows

$$J_r = \oint \frac{dz}{2\pi i} z^{r+h-1} J(z)$$

The inverse relation is

$$J(z) = \sum_r z^{-r-h} J_r \tag{7.1}$$

Since the currents are conformal fields, it is straightforward to compute their commutator with the Virasoro generators,

$$[L_n, J_r] = (n(h-1) - r) J_{r+n} \ , \tag{7.2}$$

where h is the conformal weight of $J(z)$. It follows that acting with J_r decreases the conformal weight of a state by r. The commutators of the current modes themselves define the extended algebra. To compute them requires more detailed knowledge, namely the operator product of two currents with each other.

7.5. Integer and half-integer modes

The parameter r that defines the modes of a current is not necessarily an integer. Suppose on some primary field $\phi(w, \bar{w})$

$$J(z)\phi(w,\bar{w}) = (z-w)^{\alpha} \phi'(w,\bar{w}) + \dots$$

Then we would like to define the $J(z)$-charges of the state created by ϕ from the vacuum by means of the contour integral

$$\oint dz z^{r+h-1} J(z)\phi(0,0)\,|0\rangle = \oint dz z^{r+h-1} z^{\alpha}\phi'(0,0)\,|0\rangle \ .$$

writing $z = e^{i\theta}, dz = e^{i\theta} i d\theta$ we get a phase integral

$$\int_0^{2\pi} i d\theta e^{i(r+h+\alpha)\theta}$$

This integral is well-defined (*i.e.* independent of the choice of the θ interval) only if $r+h+\alpha \in \mathbf{Z}$. In modular invariant theories α is always an integer, and it follows then that $h+r$ must be an integer as well. In theories with fermionic currents, α can have both integer and half-integer values, and hence we must choose integer or half-integer modes for the currents, depending on which representation they act. Note that the right-hand side of an operator product does not contain just one term, but in general an infinite number of terms. A mode expansion can only be defined if the fractional parts of the exponents α are the same for all terms. One calls such a universal phase $\exp 2\pi i\alpha$ the *monodromy* of J around ϕ.

Note that the periodicity changes if we go from the complex plane to the cylinder. Because of the factor $(\frac{\partial f}{\partial z})^h$ in the conformal transformation currents on the plane and the cylinder are related as

$$J^{\rm cyl}(w) = z^h J(z) \ ,$$

where $z = e^w$. If h is half-integer, the periodicity changes.

For historical reasons [21], representations on which J is half-integer moded are called Neveu-Schwarz representations. On the plane, the current acting on such ground states is periodic around the origin (*cf.* (7.1)), but it is anti-periodic around the cylinder. Representations which allow integer modes are called Ramond representations. The current is anti-periodic on the plane, but periodic on the cylinder. This is summarized in the following table

	$r \in$	Plane	Cylinder
Neveu-Schwarz	$\mathbf{Z}+\frac{1}{2}$	periodic	anti-periodic
Ramond	$\mathbf{Z}$	anti-periodic	periodic

7.6. Types of chiral algebra extensions

A rough classification of the possible extensions of the Virasoro algebra is in terms of the conformal spin of the currents. The following possibilities are of interest

- $h = \frac{1}{2}$ Free fermions
- $h = 1$ Affine Lie algebras (also called Kac-Moody algebras in the physics literature)
- $h = \frac{3}{2}$ Superconformal algebras
- $h = 2$ Virasoro tensor products
- $h > 2$ W-algebras.

7.7. Properties of extended Virasoro representations

In the foregoing five lectures we have extensively studied Virasoro representations. Fortunately most of what we learned remains valid for the extended theories.

Extended symmetries affect the representation theory in two ways. First the number of ground states is typically reduced, because one imposes extra conditions on them. These conditions usually take the form $J_n\,|\phi\rangle = 0$ for positive modes of the extra currents $J(z)$. The rationale behind this should be clear: just as we did for the Virasoro algebra we would like to build highest weight representations. Since J_n, $n > 0$ decreases the conformal weight, any state not annihilated by J_n is obviously not a highest weight state. Having determined the highest weight states, one uses the negative modes of the extra currents in addition to those of the Virasoro algebra to build representations. Since extra currents are used, the representations can only get larger, even though additional null-states appear.

There is thus a new notion of primary field required. Primary fields not only have operator products of a prescribed form with $T(z)$ (namely (3.6)), but there are additional operator products with the currents $J(z)$ that must have a certain form. Furthermore descendant fields are now not only created by $T(z)$ but also by all currents $J(z)$ in the chiral algebra. These operator products are equivalent to the aforementioned requirements on highest weight states, since that latter are still created from the vacuum by $\phi(0)\,|0\rangle$, if $\phi(z,\bar{z})$ is an (extended algebra) primary field.

Explicitly these primary field conditions take the following form for integer-moded currents of integer spin h

$$J(z)\phi(w,\bar{w}) \propto (z-w)^{-h}\phi'(w,\bar{w}) + \text{higher order in } z-w\ , \tag{7.3}$$

This means that the operator product cannot be more singular than indicated. If it is more singular $\phi(w,\bar{w})$ is a descendant field. However, the operator product can be *less* than indicated in (7.3). For example, if the zero-mode J_0 annihilates the state, the leading power is $-h+1$ or less; if J_{-1} also annihilates the states it is $-h+2$ or less, etc. Usually there is only one state that is annihilated by J_0, namely the vacuum, created by the operator $\varphi(z,\bar{z}) = \mathbf{1}$. The leading power is then in fact 0, and for example if we consider the Virasoro algebra this implies that both L_0 and L_{-1} must annihilate the vacuum. Fields which have powers of $(z-w)^{-1}$ larger than h in their operator products are descendants; the corresponding states are not annihilated by $J_n, n>0$.

For half-integer spin algebras we have to distinguish half-integer moded (Neveu-Schwarz) and integer moded (Ramond) operators. In the former case the operator product of a primary field $\phi(w)$ is

$$J(z)\phi(w,\bar{w}) = (z-w)^{-h+\frac{1}{2}}\phi'(w,\bar{w}) + \text{higher order in } z-w \ ,$$

where ϕ' is a descendant that has a conformal weight that is $\frac{1}{2}$ larger than that of ϕ. It may happen that there is no such field. Then $J_{-\frac{1}{2}}$ annihilates the ground state $\phi(0)\,|0\rangle$, and the leading power in the operator product is lower. But in any case fields with a power higher than h are descendants.

In the Ramond sector one has

$$J(z)\phi(w,\bar{w}) \propto (z-w)^{-h}\phi'(w,\bar{w}) + \text{higher order in } z-w \ .$$

Only states that are annihilated by the zero mode generators do not satisfy this formula. They have a leading power $(z-w)^{-h+1}$ (or less). There can be arbitrarily many such states.

Note that the zero-modes in of integer-moded operators either annihilate a state, or they transform ground states of a given value of h into each other. The ground states form in this way a representation of an algebra generated by the zero-mode generators.

All these notions have been developed explicitly for Kac-Moody algebras, free fermions and superconformal algebras. In the application to W algebras there are several footnotes to be added to this general picture.

7.8. Charge conjugation

Everything discussed in the section 4 and 5 goes through for extended algebras, apart from one important difference. We have seen that two-point functions of Virasoro primaries are diagonal in the sense that

$$\langle 0|\,\phi_h(z)\phi_{h'}(w)\,|0\rangle = \delta_{h,h'}(z-w)^{-2h} \tag{7.4}$$

for the holomorphic part. Since Virasoro representations are uniquely determined by c and h, the Kronecker δ implies that only identical representations have a non-trivial propagator connecting them.

In the extended case (7.4) still holds, but now there can be representations with identical values of h that are different with respect to other generators of the algebra. In particular it may happen that the propagator does not act diagonally within each set of h values. One can always choose a basis of fields so that they come in pairs connected by the propagator. The two members of such a pair are called each others charge conjugates.

Charge conjugation thus defines a matrix C which is symmetric, whose entries are 0 or 1, and which satisfies $C^2 = 1$. It either takes a field into itself (such a field is called self-conjugate), or to its charge conjugate. The vacuum is necessarily self-conjugate, since it is non-degenerate.

If charge conjugation is non-trivial, the duality diagrams of the previous chapter must be modified by assigning arrows to each line.

7.9. Characters and modular transformations

Virasoro characters of extended algebras are defined exactly as for Virasoro representations. One can generalize the notion of the character by inserting exponentials of zero-modes of other currents into the trace, but we will not consider that here.

The matrix T requires no further discussion. There also is a matrix S with the property

$$\chi_i(-\frac{1}{\tau}) = \sum_j S_{ij}\chi_j(\tau) \ .$$

Note however that this transformation as it stands does not always determine S completely, because it is now possible that several representations j have the same character. This was excluded for Virasoro representations because all representations have different conformal weights. One can nevertheless define S completely by taking into account extra variables in the characters (as mentioned above) and by requiring it to be a unitary and symmetric matrix.

The relation among the generators in the general case is

$$(ST)^3 = S^2 = C \quad \text{with} \ \ C^2 = 1$$

Note that the modular transformation S acting on the variable τ ($\tau \to -\frac{1}{\tau}$) squares to 1. However the transformation on the a and b cycles is

$$a \to -b; \quad b \to a$$

and squares to -1. It is thus a double cover of the transformation on the positive upper half plane in which τ is defined. The transformation S^2, *i.e.* $a \to -a$, $b \to -b$ is non-trivial, but it acts trivially on the "period matrix" τ. Intuitively S^2 flips the time (and the space) direction on the torus, and this is why a field goes into its charge conjugate rather than itself.

Because S is still unitary and symmetric, we have

$$S = CS^\dagger = S^\dagger C = S^* C = CS^*$$

so that in particular reality of S is equivalent to C being equal to the identity.

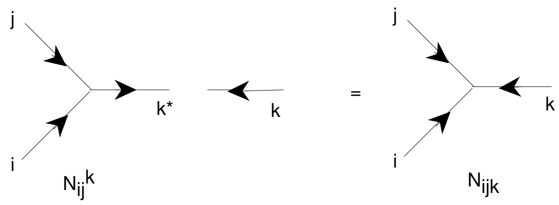
The Verlinde formula in its general form reads

$$N_{ij}{}^k = \sum_n \frac{S_{in} S_{jn} S^\dagger_{nk}}{S_{0n}} \ .$$

Here the raised index indicates charge conjugation. We may also define

$$N_{ijk} = \sum_l N_{ij}{}^l C_{lk} = \sum_n \frac{S_{in} S_{jn} S_{nl}}{S_{0n}}$$

Because S is symmetric, N_{ijk} is symmetric in all its indices. This is the quantity that counts the number of couplings in the three point vertex. In other words, if $N_{ij}{}^k$ does not vanish, $[i]\times[j]$ contains the representation $[k]$. Hence they can be coupled to the representation $[k^*]$ to form a non-vanishing three point coupling, by insertion of the $k-k^*$ propagator. This is illustrated below.



7.10. Virasoro tensor products

A simple example of an extended chiral algebra is obtained by taking the tensor product of two Virasoro representations, with central charges c_1 and c_2. The resulting theory has a Virasoro algebra generated by $L_n^{(1)} + L_n^{(2)}$ with central charge $c_1 + c_2$. The representations are simply all pairs of representations of the two algebras, and have conformal weights $h_i^{(1)} + h_i^{(2)}$.

It is easy to check that in such a theory there is a conformal field with weights $(2,0)$, namely

$$J(z) = c_2 T^{(1)}(z) - c_1 T^{(2)}(z) \ .$$

This is the current of the extended symmetry.

The simplest modular invariant partition function of such a system is the diagonal one, which is the product of the diagonal invariant of the two systems. However, in principle there can be many additional modular invariants, and in general there are.

Of course one can also consider tensor products of representations of other extended algebras.

7.11. Extensions and off-diagonal partition functions

Sometimes possible extensions of the chiral algebra can be read off directly from the existence of modular invariant partition functions. A typical such partition function has the form of a sum of squares,

$$P(\tau,\bar\tau) = \sum_{l=1}^{M} |\sum_{a=1}^{N} \chi_{l,a}|^2 \tag{7.5}$$

Such an expression can often be interpreted in terms of an extension of the original chiral algebra (which itself may be an extension of the Virasoro algebra), in such a way that the characters of the new algebra are equal to sums of characters of the original algebra

$$\chi_l^{\rm new} = \sum_a \chi_{l,a} \ . \tag{7.6}$$

The new theory as M characters, whereas the original one had at least NM characters. In fact it always has more, because a general feature of a partition function of the form (7.5) is that certain representations of the original algebra are "projected out", *i.e.* they do not appear at all in the off-diagonal partition function.

Of special interest is the identity character. If it is a sum of several characters of the old theory, then the extra terms imply the existence of matrix elements $M_{i0} \neq 0$, where M is the multiplicity matrix in the modular invariant. The corresponding primary fields have $h = h_i \neq 0, \bar h = 0$. They can thus be interpreted as currents, and they are in fact precisely the currents that extend the chiral algebra.

Although in practice one only deals with explicit modular invariant partition functions for extensions from one rational conformal field theory to another one, conceptually the chiral algebra extension that make a non-rational conformal field theory rational work in the same way. In that case M is finite and N is infinite.

7.12. The new S and T matrices

Once we have a new, smaller set of characters, one expects to have a new set of modular transformation matrices S and T. The new T matrix is trivial to get, since by modular invariance all terms in (7.6) have the same T-eigenvalue. The new matrix $S^{\rm new}$ can be obtained easily from the original one, $S^{\rm old}$, in the simplest case, where all linear combinations (7.6) have the same number of terms (each with coefficient 1). It is then not hard to show that the matrix

$$S^{\rm new}_{l,m} = \frac{1}{N}\sum_{a,b} S^{\rm old}_{(l,a)(m,b)}$$

transforms the new characters if $S^{\rm old}$ is the transformation matrix for the original one.

Many off-diagonal invariant invariants have a more complicated from. For example, it may happen (although it rarely does) that the linear combinations in (7.6) have coefficients larger than 1. A more serious complication occurs when the linear combinations have different lengths. The typical form of such a partition function – in this example with linear combinations of either N terms or 1 – is something like

$$P(\tau,\bar\tau) = \sum_{l=1}^{M} |\sum_{a=1}^{N} \chi_{l,a}|^2 + \sum_{f=1}^{N_f} N|\chi_f|^2$$

This partition function can – usually – be interpreted in terms of a new, extended algebra with $M + N_f \times N$ representations. Note that the last $N_f \times N$ representations have characters that are identical in groups of N. This means that it is not obvious which matrix $S^{\rm new}$ to use for the transformations among these characters. Indeed, since they are identical in groups of N the transformation $\tau \to -\frac{1}{\tau}$ does not determine $S^{\rm new}$ completely. This problem can be solved by imposing unitarity as well as the modular group property $(ST)^3 = S^2$ on $S^{\rm new}$, but it turns out that in this case the matrix elements of $S^{\rm new}$ are not simply linear combinations of those of $S^{\rm old}$.

7.13. Extensions and automorphisms

The matrices M that define a modular invariant partition function can be divided into two main groups: those with $M_{0i} = M_{i0} = 0$ for $i \neq 0$, and all others. It can be shown that in the former case there is no extension of the chiral algebra, and that all characters must appear in the partition function. However, they may appear non-diagonally, as

$$\sum_l \chi_l \Pi_{lm} \bar\chi_m$$

where Π is a permutation of the labels. It is not hard to see that Π is then an automorphism of the fusion rules, *i.e.* the fusion coefficients N_{ijk} are invariant when Π acts simultaneously on all labels.

If some matrix elements M_{0i} or M_{i0} are non-zero, the modular invariant can always be interpreted as an extension of the chiral algebra. If one re-writes it in terms of characters of the new algebra one either gets the diagonal invariant of the new algebra, or a fusion rule automorphism of the new algebra.

7.14. Simple currents

Many conformal field theories have representations $[J]$ with the property that

$$[J] \times [i] = [i']$$

for all other representations $[i]$. The special property is thus that there is just one term on the right hand side. Then $[J]$ is referred to as a *simple current* [22].

The word "current" anticipates the fact that it may be used to extend the chiral algebra, or at least plays the rôle of a (para)fermionic current.

Simple currents organize the fields in a conformal field theory in an obvious way into orbits, and one can in an equally obvious way assign an order N to them. Among themselves they generate an abelian group called the *center* of the conformal field theory.

Simple currents can always be used to extend the chiral algebra. In the simplest cases – N prime – it is furthermore true that currents of fractional spin $\frac{\ell}{N}$ generate fusion rule automorphisms. In more complicated cases one gets combinations of automorphisms and extensions.

The number of simple current invariants of a given conformal field theory grows very rapidly with the number of abelian factors of the center, but all solutions have now been classified.

It seems that most modular invariant partition functions can be described in terms of simple currents, but there are exceptions. These are called, quite naturally, exceptional invariants.

8. Free fermions

Free fermions are described by the two-dimensional action

$$S = \frac{1}{8\pi} \int d^2 z \left(\psi \partial_{\bar{z}} \psi + \bar{\psi} \partial_z \bar{\psi} \right) \ ,$$

where we have already switched to Euclidean space and to complex coordinates. We will focus on the fields ψ from here on; all equations that follow are also valid with bars on all relevant quantities.

8.1. The propagator

The equations of motion for $\psi(z,\bar{z})$ are $\partial_{\bar{z}}\psi(z,\bar{z}) = 0$, so that we may write $\psi(z)$ instead of $\psi(z,\bar{z})$. The operator product of two fermions is

$$\psi(z)\psi(w) = \frac{1}{z-w}$$

8.2. Energy-Momentum tensor and central charge

The energy-momentum tensor is

$$T(z) = -\tfrac{1}{2} : \psi(z)\partial_z\psi(z) : , \tag{8.1}$$

where as usual normal ordering implies that the vacuum expectation value of $T(z)$ is zero. This requires the subtraction of the singular terms in the operator product.

It is a simple exercise to verify that the central charge is equal to $\frac{1}{2}$, and that $\psi(z)$ has conformal weight $(\frac{1}{2}, 0)$. This is an interesting result in view of the classification of Virasoro representations. We have seen that for $c = \frac{1}{2}$ three representations exist: with $h = 0$, $h = \frac{1}{2}$ and $h = \frac{1}{16}$. The conformal field $\psi(z)$ clearly creates an $h = \frac{1}{2}$ state from the vacuum:

$$\left|\tfrac{1}{2}\right\rangle = \psi(z)\left|0\right\rangle$$

8.3. Mode expansion

The free fermion can be expanded in modes. On the complex plane the mode expansion is

$$\psi(z) = \sum_n b_n z^{-n-\frac{1}{2}} , \tag{8.2}$$

which can be inverted in the usual way. When going to the cylinder the free fermion picks up a conformal factor $\left(\frac{\partial z}{\partial w}\right)^{\frac{1}{2}}$. Hence we get

$$\psi^{\mathrm{cyl}}(w) = z^{\frac{1}{2}} \sum_n b_n z^{-n-\frac{1}{2}} = \sum_n b_n e^{-nw} .$$

Here we see explicitly the aforementioned periodicity flip.

8.4. The spin field

The field $\psi(z)$ has local operator products with all primary fields we have seen so far (namely $\psi(z)$ itself and the identity). We expect there to exist also fields with which it has a square root branch cut, so that $\psi(z)$ is realized *à la* Ramond.

Furthermore we expect fields with conformal weight $\frac{1}{16}$, since that is another allowed Virasoro representation at $c = \frac{1}{2}$. Indeed, we will see that modular invariance forces such fields to exist.

Let us therefore introduce a field $\sigma(z, \bar{z})$ with $h = \bar{h} = \frac{1}{16}$. Its operator product with $\psi(z)$ has the form

$$\psi(z)\sigma(w, \bar{w}) = (z - w)^{h_\mu - \frac{1}{2} - \frac{1}{16}}\mu(w, \bar{w}) \ ,$$

where μ is some other field in the theory. Since we know all Virasoro representations its conformal weight h_μ can only be $0, \frac{1}{2}$ or $\frac{1}{16}$, perhaps up to integers if we allow μ to be a descendant. Clearly only the choice $\frac{1}{16}$ leads to an acceptable branch cut, since for fermions only square root branch cuts (or no cuts at all) are allowed. We find thus that the field σ does indeed introduce the expected branch cut. This field is often referred to as a *spin field.*

Acting on the vacuum the field σ produces a state $\left|\frac{1}{16}, \frac{1}{16}\right\rangle = \sigma(0, 0)\,|0\rangle$. The field $\psi(z)$, acting on such a state is anti-periodic on the plane, and hence integer-moded.

It is tempting to argue that μ and σ are in fact one and the same field. In the present context that is in fact not quite correct, since we are not dealing with a modular invariant partition function. When we make the theory modular invariant, $\psi(z)$ and either σ or μ are removed from the spectrum (*i.e.* all the states they create are removed). The primary fields in the modular invariant theory are 1, $\psi(z)\bar{\psi}(\bar{z})$ and $\sigma(z, \bar{z})$. Each creates one state from the vacuum, namely the vacuum itself, the state $\left|\frac{1}{2}, \frac{1}{2}\right\rangle$ and $\left|\frac{1}{16}, \frac{1}{16}\right\rangle$. On these states one builds Virasoro representations. The ground states are non-degenerate, *i.e.* there is just one state with the corresponding values of $h, \bar{h}$.

The partially modular invariant theory has in addition the primary fields $\psi(z)$, $\bar{\psi}(\bar{z})$ and $\mu(z, \bar{z})$. The operator products are now non-local. In addition to the ones already mentioned one has

$$\begin{aligned} \sigma(z, \bar{z})\sigma(w, \bar{w}) &\to 1, \psi(w)\bar{\psi}(\bar{w}) \\ \mu(z, \bar{z})\mu(w, \bar{w}) &\to 1, \psi(w)\bar{\psi}(\bar{w}) \\ \sigma(z, \bar{z})\mu(w, \bar{w}) &\to \psi(w), \bar{\psi}(\bar{w}) \end{aligned} \tag{8.3}$$

If one substitutes the conformal weight factors $(z-w)^{h_k - h_i - h_j}$ one finds that the last operator product is non-local, indicating that one cannot have both σ and μ in the same modular invariant theory. Removing the free fermions and either μ or σ solves the non-locality problem in a consistent way, *i.e.* the operator product closes after this truncation. This will be made explicit later in this chapter.

8.5. Free fermion characters

One advantage of the free fermion formulation of the $c = \frac{1}{2}$ theory is that it is straightforward to compute the characters. The reason why this is not straightforward for Virasoro representations is the existence of null vectors. Let us compare the lowest lying states in the three representations.

<u>*Neveu-Schwarz states*</u>

Consider first the Neveu-Schwarz sector. Using the operator product of two free fermions and the mode-expansion (8.2), one can easily derive that the modes satisfy the following anti-commutator

$$\{b_r, b_s\} = \delta_{r+s,0} \ ,$$

where r and s are half-integers. Clearly we cannot impose $b_r |0\rangle = 0$ for all r, and hence we only do so for $r > 0$. This is also the natural definition for highest weight modules, since the positively moded b_r's decrease the L_0 eigenvalue.

The Virasoro generator L_0 can be expressed in terms of the fermionic oscillators. Classically, the result is

$$L_0 = \tfrac{1}{2} \sum_r r b_{-r} b_r \ .$$

Quantum mechanically we have to be more careful, since the operators b_r and b_{-r} do not commute. Changing their order only affects L_0 by a constant, so that we get

$$L_0 = \sum_{r>0} r b_{-r} b_r + \text{constant} \ .$$

Here we have normal ordered the fermionic oscillators. Since we have already defined normal ordering in (8.1), the constant is not a free parameter. It must be chosen in such a way that $\langle 0| L_0 |0\rangle = 0$, *i.e.* the constant must be zero. It follows immediately that $L_0 |h\rangle = 0$ if $|h\rangle$ is a highest weight state of the fermionic algebra, *i.e.* if $b_r |h\rangle = 0$ for positive r. Hence the fermionic algebra can have just one representation in the Neveu-Schwarz sector, namely the one built on the vacuum.

At the first few levels, this representation contains the following states:

$h = 0$	$\|0\rangle$
$h = \frac{1}{2}$	$b_{-\frac{1}{2}} \|0\rangle$
$h = 1$	none
$h = \frac{3}{2}$	$b_{-\frac{3}{2}} \|0\rangle$
$h = 2$	$b_{-\frac{3}{2}} b_{-\frac{1}{2}} \|0\rangle$
$h = \frac{5}{2}$	$b_{-\frac{5}{2}} \|0\rangle$
$h = 3$	$b_{-\frac{5}{2}} b_{-\frac{1}{2}} \|0\rangle$
$h = \frac{7}{2}$	$b_{-\frac{7}{2}} \|0\rangle$
$h = 4$	$b_{-\frac{7}{2}} b_{-\frac{1}{2}} \|0\rangle$; $b_{-\frac{5}{2}} b_{-\frac{3}{2}} \|0\rangle$

Note that fermionic oscillators must satisfy the Pauli exclusion principle, so that for example $b_{1/2}b_{1/2}$ is zero. For this reason there is no state at level $h = 1$, and we have to go to $h = 4$ to find more than one state.

An important question is whether all these states have positive norm. Due to the simplicity of the free fermion algebra it is not hard to show that indeed the norm of every state is exactly 1, and that all distinct states are orthogonal. The fact that $\psi(z)$ as a field on the cylinder is real implies that $b_r^\dagger = b_{-r}$. It is then trivial to prove that the states are indeed orthonormal.

Hence we may expect them to fit exactly into one or more Virasoro representations. The relevant Virasoro representations are, for the ground state representation

$h = 0$	$\|0\rangle$
$h = 1$	$L_{-1} \|0\rangle$
$h = 2$	$L_{-2} \|0\rangle$, $(L_{-1})^2 \|0\rangle$
$h = 3$	$L_{-3} \|0\rangle , L_{-1}L_{-2}, \|0\rangle \, (L_{-1})^3 \|0\rangle$
$h = 4$	$L_{-4} \|0\rangle , L_{-3}L_{-1} \|0\rangle , L_{-2}L_{-2}, \|0\rangle \, (L_{-1})^4 \|0\rangle \, L_{-2}(L_{-1})^2 \|0\rangle$

For the representation with ground state weight $h = \frac{1}{2}$ we find exactly the same result, with $|0\rangle$ replaced by $\left|\frac{1}{2}\right\rangle$, and all conformal weights shifted up by half a unit. However, we have already seen that not all these states have positive norm. The ground state representation has a null state at its first excited level (which propagates trough to all higher levels), while the $h = \frac{1}{2}$ representation has a null state at its second level. This agrees precisely with the assumption that the fermionic representation is the sum of the two Virasoro representations, and also gives us a quick way of counting the number of Virasoro null states at higher levels.

<u>*Neveu-Schwarz characters*</u>

It is straightforward to compute the character for the fermionic representation, since there are no null states to be taken into account. Each oscillator b_{-r} can act once or zero times on the ground state. If there were just one oscillator b_{-r} there would just be two states, $|0\rangle$ and $b_{-r}|0\rangle$ with $h=0$ and $h=r$. The character is thus $\mathrm{Tr}\ q^{L_0}=1+q^r$. All oscillators with different modes acts independently, and it is easy to see that each contributes via additional factors of this form. Furthermore we have to take into account the subtraction $-c/24$. The result is thus

$$\chi_0+\chi_{\frac{1}{2}}=\ \mathrm{Tr}\ q^{L_0-\frac{c}{24}}=q^{-\frac{1}{48}}\prod_{r=\frac{1}{2}}^{\infty}(1+q^r)\ . \tag{8.4}$$

This gives us the sum of the characters of two Virasoro representations. Their difference is also easy to compute. Just observe that states created by an odd number of fermions contribute to the spin-$\frac{1}{2}$ representation, and the remaining ones to the vacuum representation. Hence we can get the difference by changing the sign of the contribution of each single fermion to the trace,

$$\chi_0-\chi_{\frac{1}{2}}=q^{-\frac{1}{48}}\prod_{r=\frac{1}{2}}^{\infty}(1-q^r)\ .$$

This expression can also be written as a trace over the fermion representation, namely as

$$\chi_0-\chi_{\frac{1}{2}}=\ \mathrm{Tr}\ (-1)^F q^{L_0-\frac{c}{24}}\ . \tag{8.5}$$

Here F is the fermion number operator. We have now succeeded in computing both the $h=0$ and $h=\frac{1}{2}$ character at $c=\frac{1}{2}$.

<u>*Ramond states*</u>

In the Ramond sector the fermionic oscillators are integer moded, which has the interesting consequence that there exists a zero mode oscillator b_0, which satisfies $\{b_0,b_0\}=1$. The expression for L_0 is

$$L_0=\sum_{n>0}nb_{-n}b_n+\text{constant}\ .$$

Obviously highest weight states $|h\rangle$ must satisfy $b_n|h\rangle=0$ for $n>0$, and this implies that all highest weight representations in the Ramond sector must have the same highest weight, namely "constant". Since we know that the Ramond sector is realized on states created by the field σ with $h=\frac{1}{16}$, it follows that "constant" must be equal to $\frac{1}{16}$ in this case.

<u>*The Ramond ground state*</u>

It is fairly obvious how to build up the representation, the only slight problem being the action of the operator b_0. This operator changes the fermion number of the state it acts on. To realize this we need thus two states, one with $(-1)^F = +$ and one with $(-1)^F = -$. Denoting these states as $|+\rangle$ and $|-\rangle$ we have thus

$$b_0\,|+\rangle = \frac{1}{\sqrt{2}}\,|-\rangle\,, \qquad b_0\,|-\rangle = \frac{1}{\sqrt{2}}\,|+\rangle\ ,$$

so that $b_0^2 = \frac{1}{2}$. Of course we can realize this operator algebra on even more states, but two is the minimum required.

<u>Ramond characters</u>

Having done this, we get for the character

$$\chi_{\frac{1}{16}} \propto \ \mathrm{Tr}\ q^{L_0-\frac{c}{24}} = 2q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1+q^n)\ , \tag{8.6}$$

by exactly the same arguments as used above. The correct normalization will be discussed in a moment. In principle the ground state of a Virasoro (or extended Virasoro) algebra can be degenerate, so in principle it could be possible that the factor 2 should be absorbed into the character itself. It is also possible to define this trace with a factor $(-1)^F$, but it is clear that the result is then zero: the operator b_0 maps any state into a degenerate state, while flipping the fermion number.

8.6. The partition function

Let us now assemble the partition function, by combining it with the anti-holomorphic fields. This is trivial in the Neveu-Schwarz sector. The oscillators $\bar{b}_r$ contribute addition factors $(1+\bar{q}^r)$ so that we get

$$P_{\mathrm{NS}} = (q\bar{q})^{-\frac{1}{48}}\prod_{r,s=\frac{1}{2}}^{\infty}(1+q^r)(1+\bar{q}^s)\ .$$

Now consider the Ramond sector. Here some further thought is needed. Do we again double the ground state to deal with the action of $\bar{b}_0$ (in other words, do we take the absolute value squared of (8.6))? Clearly this is not needed, because we already have two ground states, and that is sufficient to realize simultaneously the b_0 and the $\bar{b}_0$ algebra. If we define the fermion number operator F to count the *total* fermion number (for $\psi(z)$ as well as $\bar{\psi}(\bar{z})$, and we choose two ground states $|+\rangle$ and $|-\rangle$ with opposite total fermion numbers, everything will work automatically. Hence we define

$$P_{\mathrm{R}} = 2(q\bar{q})^{\frac{1}{24}}\prod_{n,m=1}^{\infty}(1+q^n)(1+\bar{q}^m)\ .$$

8.7. Theta-functions

Altogether we have now defined four kinds of partition functions on the torus: with R or NS boundary conditions along the space direction, and with or without $(-1)^F$ operator inserted. This latter operator can be interpreted in terms of periodicity along the Euclidean time direction of the torus. The normal trace corresponds to a fermion path integral with anti-periodic boundary conditions (this boundary condition has the same origin as the usual $-$ sign in fermion loops; it can be computed by repeating the calculation that yields (6.2) for fermions). The insertion of $(-1)^F$ gives an extra $-$ sign for every fermion in the loop, so it flips the boundary condition to periodic. Hence we have

$$\begin{array}{lll} \mathrm{AA} & \mathrm{Tr}_{\,\mathrm{NS}} q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}} & \left|\frac{\theta_3}{\eta}\right| \\ \mathrm{AP} & \mathrm{Tr}_{\,\mathrm{NS}}(-1)^F q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}} & \left|\frac{\theta_4}{\eta}\right| \\ \mathrm{PA} & \mathrm{Tr}_{\,\mathrm{R}} q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}} & \left|\frac{\theta_2}{\eta}\right| \\ \mathrm{PP} & \mathrm{Tr}_{\,\mathrm{R}}(-1)^F q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}} & \left|\frac{\theta_1}{\eta}\right| \end{array}$$

Here the letters "AP" indicate anti-periodicity along the "space" direction and periodicity along the "time" direction on the torus, etc. It turns out that these four partition functions can be expressed in terms of standard mathematical functions, namely the Jacobi θ-functions and the Dedekind η function. These functions are defined as follows

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(z|\tau) = \sum_n e^{i\pi[(n+a)^2\tau+2(n+a)(z+b)]}$$

with the additional definitions

$$\theta_1 = \theta\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}; \quad \theta_2 = \theta\begin{bmatrix} 1/2 \\ 0 \end{bmatrix}; \quad \theta_3 = \theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}; \quad \theta_4 = \theta\begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$$

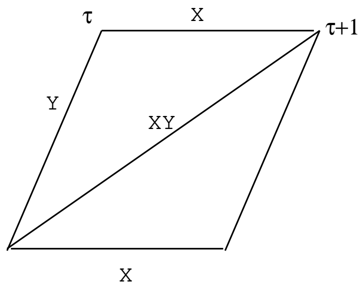
and

$$\eta(q) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)$$

with $q = e^{2\pi i\tau}$. The last column above indicates the identification of each partition function with ratios of θ and η functions. The Jacobi θ-functions have two arguments, but we are only using them at $z = 0$ here. The function $\theta_1(z|\tau)$ vanishes for $z = 0$, as does the partition function in the PP sector, but it can be made plausible that the identification given here is the correct one. The fact that these functions are identical is far from obvious, but is one of many remarkable identities that modular functions enjoy.

8.8. Modular transformations

Finally we discuss modular invariance. Clearly modular transformations change the fermion boundary conditions. For example, the transformation S interchanges the two cycles ("space" and "time") on the torus, and hence it interchanges AP and PA. The transformation T maps XY to X(XY) as shown in the figure, where X and Y stand for A or P, and the multiplication rule is AA=P, AP=A and PP=P. In other words, it interchanges AA and AP. Since S and T generate the modular group we generate all permutations of AA, AP and PA, whereas PP transforms into itself.



These transformations are clearly sensitive to the correct normalization of the partition functions. They can be computed explicitly for the θ and η functions, and one find

$$\theta_1(-\frac{1}{\tau}) = -i\sqrt{-i\tau}\theta_1(\tau); \quad \theta_2(-\frac{1}{\tau}) = \sqrt{-i\tau}\theta_4(\tau);$$
$$\theta_3(-\frac{1}{\tau}) = \sqrt{-i\tau}\theta_3(\tau); \quad \theta_4(-\frac{1}{\tau}) = \sqrt{-i\tau}\theta_2(\tau)$$

$$\theta_1(\tau+1) = e^{i\pi/4}\theta_1(\tau); \quad \theta_2(\tau+1) = e^{i\pi/4}\theta_2(\tau);$$
$$\theta_3(\tau+1) = \theta_4(\tau); \quad \theta_4(\tau+1) = \theta_3(\tau)$$

$$\eta(-\frac{1}{\tau}) = \sqrt{-i\tau}\eta(\tau); \quad \eta(\tau+1) = e^{i\pi/12}\eta(\tau)$$

It follows that the partition function $P_{\rm R} + P_{\rm NS} =\mid \frac{\theta_3}{\eta} \mid + \mid \frac{\theta_2}{\eta} \mid$ is not modular invariant, as expected. It is in fact invariant under a subgroup of the modular group generated by TST and T^2. This is clearly a subgroup of order 2, since by adding the element T we get the full modular group. This shows in particular that we have chosen the correct normalization for the ground state in the Ramond sector.

8.9. The modular invariant partition function

It is also clear that the following partition function *is* fully modular invariant

$$\tfrac{1}{2}\left\{ | \frac{\theta_3}{\eta} | + | \frac{\theta_4}{\eta} | + | \frac{\theta_2}{\eta} | \pm | \frac{\theta_1}{\eta} | \right\}$$

The factor $\frac{1}{2}$ was added to make sure that the vacuum appears with the correct multiplicity, namely 1. The last term can be added with any factor, since it is (a) modular invariant by itself and (b) zero. However, consistency of higher loop diagrams as well as one-loop diagrams with external legs force this term to appear exactly as it does. The two signs have a simple interpretation: the Ramond ground state appears in the partition function with a factor $\frac{1}{2}(1 \pm (-1)^F)$ so that depending on the sign either the ground state with positive or the one with negative fermion number survives. Note that the modular invariant partition function has just one Ramond ground state. This is no problem, since the operators b_0 and $\bar{b}_0$ (zero modes of $\psi(z)$ and $\bar{\psi}(\bar{z})$ are not in the theory anymore. The first two terms only have contributions from the state $\psi(0)\bar{\psi}(\bar{0})\,|0\rangle$ and its descendants, and this operator does not change fermion number by an odd amount. The partially modular invariant partition function has two Ramond ground states, corresponding to the fields σ and μ. Depending on the sign choice, either one of these is projected out. This sort of operation (for going from a partially modular invariant partition function to a modular invariant one) is sometimes called a *GSO-projection* (GSO stands for Gliozzi, Scherk and Olive, whose paper [23] was the starting point of superstring theory).

8.10. Ising characters

We can write the modular invariant partition function as

$$|\chi_0|^2 + |\chi_{\frac{1}{2}}|^2 + |\chi_{\frac{1}{16}}|^2$$

by making the identifications

$$\begin{aligned}
\chi_0 &= \tfrac{1}{2}\Big(\sqrt{\frac{\theta_3}{\eta}} + \sqrt{\frac{\theta_4}{\eta}}\Big) \\
\chi_{\frac{1}{2}} &= \tfrac{1}{2}\Big(\sqrt{\frac{\theta_3}{\eta}} - \sqrt{\frac{\theta_4}{\eta}}\Big) \\
\chi_{\frac{1}{16}} &= \frac{1}{\sqrt{2}}\sqrt{\frac{\theta_2}{\eta}}
\end{aligned}$$

The first two equations follow already from (8.4) and (8.5).

8.11. The matrix S and the fusion rules

Using the transformation properties of the θ functions and the η function it is now easy to get the matrix S for the $c = \frac{1}{2}$ system. On the basis $(1, \psi, \sigma)$ the result is

$$S = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\sqrt{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} & 0 \end{pmatrix}$$

Using this matrix and the Verlinde formula we can compute the fusion rules:

$$\begin{aligned}
[1] \times [1] &= [1] \\
[1] \times [\psi] &= [\psi] \\
[1] \times [\sigma] &= [\sigma] \\
[\psi] \times [\psi] &= [1] \\
[\psi] \times [\sigma] &= [\sigma] \\
[\sigma] \times [\sigma] &= [1] + [\psi]
\end{aligned}$$

This result should be compared with (8.3).

8.12. Multi-fermion systems

It can be shown that if there is more than one fermionic current with spin $\frac{1}{2}$, then a corresponding part of the theory can be described as a free fermionic theory with $c = N/2$, where N is the number of fermions. Of course this $c = N/2$ theory can appear as part of a tensor product with other (extended) Virasoro representations, but at least the free fermion part is easy to describe, and exactly solvable.

In such a free fermion theory each fermion can have its own boundary conditions on the torus and higher Riemann surfaces, but there are constraints from modular invariance. These constraints have been solved in general when the number of fermions is even, but there is still some controversy regarding the odd fermion number case.

The number of modular invariant partition functions one can write down grows extremely rapidly with N. Systematic studies of conformal field theories built out of free fermions (in the context of heterotic string construction) were presented in [24,25].

9. Free Boson partition functions

The free boson provides another simple example of a theory with an extended algebra. In this case the current has spin 1,

$$J(z) = \partial\Phi(z) \ .$$

The mode expansion has already been discussed before.

As we will see later, such a current can be interpreted as a generator of a $U(1)$ symmetry, with the momenta as charges.

9.1. The spectrum

The discussion of the spectrum is quite similar to that of the free fermion. The ground states are defined by the condition

$$\alpha_n \left|x\right\rangle = 0, \quad n > 0$$

The representations are built up by acting with the negatively moded oscillators. It is not hard to see that any state gotten this way has positive norm.

The Virasoro generators are dependent on the bosonic oscillators,

$$L_n = \tfrac{1}{2}\{\sum_m \alpha_{n-m}\alpha_m\} \ ,$$

where $\alpha_0 = p$, and the sum is over all integers. For the Virasoro zero mode we get thus

$$L_0 = \tfrac{1}{2}p^2 + \sum_{m>0} \alpha_{-m}\alpha_m \ .$$

In principle we would have to worry about normal ordering, but since we know that $L_0 \left|0\right\rangle = 0$ we see immediately that there is no additional constant.

The ground state $\left|x\right\rangle$ is completely determined by the action of the zero-mode generator p. Once this is fixed, we know the entire representation, and the action of the Virasoro generators. Hence we define

$$\left|p\right\rangle : \quad p_{\rm op} \left|p\right\rangle = p \left|p\right\rangle \ ,$$

where p on the left-hand side is the operator, and on the right hand side the eigenvalue. Note that there is no separate holomorphic and anti-holomorphic zero-mode algebra: $\alpha_0 = \bar\alpha_0 = p$.

9.2. The characters and the diagonal invariant

It is straightforward to derive the character formula, since just as for the free fermion all oscillators act independently and without generating null vectors. The result is

$$\chi_p(q) = q^{\frac{1}{2}p^2-\frac{1}{24}} \prod_{n=1}^{\infty} \frac{1}{(1-q^n)} = \frac{q^{\frac{1}{2}p^2}}{\eta(q)} .$$

Note that the expansion of $(1-q^n)^{-1}$ yields exactly one contribution at any level that is a multiple n. Thus each such factor describes the contribution of one bosonic oscillator α_{-n} acting any number of times on $|p\rangle$.

Since any real value of p is allowed, there exists an infinite number of characters. The diagonal partition function is therefore not a sum, but an integral

$$P(\tau,\bar\tau) = \int_0^\infty dp\, \frac{e^{i\pi\tau p^2}}{\eta(\tau)} \frac{e^{-i\pi\bar\tau p^2}}{\eta(\bar\tau)} \propto \frac{\sqrt{\text{Im }\tau}}{\eta(\tau)\eta(\bar\tau)} .$$

The proper derivation requires of course a discussion of the measure and the normalization, but the result is correct. This factor appears in the partition function of the bosonic string, which is described by a tensor product of 26 free bosonic theories (plus ghosts).

Note that in this partition function we are exactly using all the ground states we have at our disposal. Although the algebra is extended by $\partial\Phi$ we do not get a finite number of primary fields, *i.e.* a rational conformal field theory. In many of the representations the extension does not even make any difference. If there are no null vectors in a Virasoro representation, the Virasoro algebra acts just like a free bosonic oscillator, and one gets a partition function

$$q^{h-\frac{c}{24}} \prod_{n=1}^{\infty} \frac{1}{(1-q^n)} ,$$

where now every factor represents a single Virasoro generator L_{-n} instead of a free bosons α_{-n}. Hence if on $|p\rangle$ the Virasoro algebra has no null vectors, the Virasoro representation is equal to the "Virasoro+$\partial\Phi$"- representation.

The existence of Virasoro null vectors follows from the same curves we used for $c < 1$. These curves hit the line $c = 1$ at several values of h, and only for those values the Virasoro representation has a null state. From (4.5) and (4.6) we see that this happens for $m \to \infty$,

$$h = \frac{1}{4}(p-q)^2$$

For example for $h = 0$, $L_{-1}|0\rangle$ is a null state. The state $\alpha_{-1}|0\rangle$ is of course not null, so that the identity representation is indeed non-trivially extended by $\partial\Phi$.

9.3. Chiral bosons

The free boson mode expansion can be generalized by adding separate momenta for the holomorphic and anti-holomorphic terms:

$$\Phi(z,\bar{z}) = q - i(p_{\mathrm{L}}\log(z) + p_{\mathrm{R}}\log(\bar{z})) + i\sum_{n\neq 0}\frac{1}{n}\left[\alpha_n z^{-n} + \tilde{\alpha}_n \bar{z}^{-n}\right]$$

We have denoted these momenta as "L" (left) and "R" (right) because z and $\bar{z}$ originate from left- and right-moving modes on the cylinder. We may straightforwardly split also q in left- and rightmoving operators by writing $q = q_L + q_R$. Furthermore we define the canonical commutators $[q_L, p_L] = [q_R, p_R] = i$, while left and right operators commute. If we identify $p_L = p_R = p$ this leads again to old commutator $[q,p] = i$. Having done this we can now split the boson completely in left and right components

$$\Phi(z,\bar{z}) = \Phi_L(z) + \Phi_R(\bar{z})$$

with

$$\Phi_L(z) = q_L - ip_L\log(z) + i\sum_{n\neq 0}\frac{1}{n}\alpha_n z^{-n} \ ,$$

and analogously for Φ_R.

These manipulations do not influence any previous results that depend only on $\partial\Phi$ or $\bar{\partial}\Phi$, but we can now give meaning to chiral (holomorphic) objects like

$$e^{i\lambda\Phi_R(z)}$$

It may be checked that this is a conformal field of weight $\frac{1}{2}\vec{\lambda}^2$.

To see what the meaning is of the separate left and right momenta we can express the field back into cylinder coordinates. Then we get

$$\Phi(x_0, x_1) = q + 2px_0 + Lx_1 + \text{oscillators} \ ,$$

where

$$p_L = p + \tfrac{1}{2}L \ , \qquad p_R = p - \tfrac{1}{2}L \tag{9.1}$$

Previously we did not have the extra x_1 term because we required Φ to be periodic,

$\Phi(x_0, x_1) = \Phi(x_0, x_1 + 2\pi)$. The extra term destroys the periodicity unless we impose it as a symmetry on the field Φ: $\Phi = \Phi + 2\pi L$. This must hold for any eigenvalue that the operator L can have, and obviously also for all integer linear combinations of those eigenvalues. If we want Φ to have a non-trivial dependence on x_1, the only possibility is then that the L eigenvalues are quantized on a lattice of dimension equal to the number of free bosons.

This has a natural interpretation in closed string theory, where Φ is viewed as the coordinate of a space in which the string is embedded (this space is called *target space*). The existence of a lattice means that the space is compactified on a torus (a D-dimensional torus can be defined as D-dimensional Euclidean space in which points differing by vectors on a lattice are identified). If L is a non-trivial lattice vector this means that the string is not closed in the Euclidean space, but it is closed on the torus, *i.e.* the string winds around a couple of times around the torus and ends in a point identified with its beginning.

9.4. Further extensions of the chiral algebra

We arrive at the same lattice description naturally by extending the chiral algebra further. In addition to $\partial\Phi$ we add integer spin currents

$$V_\lambda = e^{i\lambda\Phi(z)}, \quad \lambda^2 \in 2\mathbf{Z} \tag{9.2}$$

to both the left and the right algebra. Note that such a current corresponds to momenta $(\lambda, 0)$, so that it is only after introducing separate p_L and p_R that we have this possibility.

It is easy to check that V_λ satisfies the operator product

$$V_\lambda(z)V_{\lambda'}(w) = (z-w)^{\lambda\lambda'}V_{\lambda+\lambda'}(w) + \ldots \tag{9.3}$$

Therefore, closure of the operator product requires $V_{2\lambda}(z)$ to be an operator in the theory if $V_\lambda(z)$ is. More generally we see that the set of λ's such that V_λ is in the chiral algebra must close under addition. It forms thus a one-dimensional even lattice, which we will call Λ. Note that the operator product (9.3) is automatically local.

9.5. Representations of the extended algebra

Since the chiral algebra contains in any case the Virasoro algebra and the operator ∂X, any other states in the theory are built on ground states $|p_L, p_R\rangle$. We have to restrict this set by imposing on it highest weight conditions with respect to the extended algebra. The field creating these states from the vacuum are

$$V_{p_L p_R}(z, \bar{z}) = e^{ip_L\Phi_L(z)}e^{ip_L\Phi_R(\bar{z})} \ ,$$

because

$$e^{ip_L\Phi_L(0)}e^{ip_L\Phi_R(\bar{0})}\left|0\right\rangle = e^{ip_Lq_L+ip_Rq_R}\left|0\right\rangle = \left|p_L,p_R\right\rangle \ .$$

Locality with respect to the left and right chiral algebra requires that $\lambda p_L \in \mathbf{Z}$ and $\lambda p_R \in \mathbf{Z}$. This immediately restricts the set of left and right momenta that we can ever encounter to the set

$$p_L \in \Lambda^* \ ,$$

where Λ^* is the *dual* (or reciprocal) lattice of λ,

$$\Lambda^* = \{\mu \in \mathbf{R}|\mu\lambda \in \mathbf{Z}, \quad \forall \lambda \in \Lambda\}$$

The lattice Λ is necessarily of the form nR, $n \in \mathbf{Z}$ and R^2 even. The lattice Λ^* has the form m/R, $m \in \mathbf{Z}$. For example, if Λ is the set of even integers, Λ^* is the set of integer and half-integers. In this description R denotes the smallest positive value of λ on the lattice.

Now let us try to find which fields are primary with respect to the full extended algebra. As we have seen in (7.3), any field with a singularity stronger than $(z-w)^{-h}$ in its operator product with a current of spin h is a descendant. A field $V_{p_Lp_R}(z,\bar{z})$ has singularity $(z-w)^{p_L\lambda}$ with V_λ. Hence we find the condition

$$p_L\lambda \leq \tfrac{1}{2}\lambda^2, \qquad \forall\ \lambda \in \Lambda \tag{9.4}$$

and the same for p_R. The vectors on Λ^* satisfying the highest weight condition (9.4) are thus those with

$$-\tfrac{1}{2}R^2 \leq m \leq \tfrac{1}{2}R^2$$

We see thus that there is – in both the left as the right chiral algebra – only a finite number of highest weight representations. Hence the theory we are constructing is a rational conformal field theory.

Note that each highest weight completely fixes the corresponding representation, since it determines completely how all the oscillators and the operators p and q act on a state. Now we can build these representations by acting with all negative modes of $\partial\Phi$ and $V_{\lambda\Phi}$. Doing this in an unrestricted way would certainly lead to null states, since there is an infinite number of chiral algebra generators. However, writing everything in terms of oscillators and momenta, one sees that the only states one can ever get starting from $\left|p_L,p_R\right\rangle$ are of the form

$$(\text{oscillators})\left|p_L+\lambda,p_R+\lambda'\right\rangle \ , \lambda,\lambda' \in \Lambda \ .$$

Furthermore any state of this form is indeed generated by the chiral algebra.

Note that the highest weight condition (7.3) is saturated only for $m = \pm\frac{1}{2}R^2$, and furthermore this only happens for $V_{\mp R}$, not for any other operators in the chiral algebra. These are thus the only highest weights which are not annihilated by the zero mode of $V_{\mp R}$. It is easy to verify that

$$[V_{\mp R}]_0 \left|\pm\tfrac{1}{2}R\right\rangle = \int dz z^{\frac{1}{2}R^2} V_{\mp R}(z) \left|\pm\tfrac{1}{2}R\right\rangle = \left|\pm\tfrac{1}{2}R\right\rangle$$

so that these two highest weight states are actually in the same representation of the horizontal algebra.

This brings us then finally to the following characterization of the representations of the chiral algebra. If the algebra is specified by a lattice Λ with spacing R, satisfying $R^2 = 2N$, then the representations are labelled by the integers $m, -N < m \leq N$, and have characters

$$\chi_m(q) = \frac{1}{\eta(q)} \sum_n q^{\frac{1}{2}(\frac{m}{R}+nR)^2} \tag{9.5}$$

Note that we may define m modulo $2N$, since a shift $m \to m+2N = m+R^2$ can be cancelled by a shift in the summation index n. It is sometimes convenient to choose m in the range $0 \leq m < 2N$. There are in total $2N$ representations. The ground state multiplicity for each of them except one is 1, the ground states being $|p\rangle = |m/R\rangle$ with $-\frac{1}{2}R^2 < m < \frac{1}{2}R^2$. The exception is the representation labelled by $m = N(\sim -N)$. Here the ground state multiplicity is two, because the states $\left|\pm\frac{1}{2}R\right\rangle$ are degenerate.

9.6. The matrix S

This is the condition for T invariance. To examine S invariance we have to determine first how the characters transform. We know this already for the η-function. To deal with the infinite sum one can use a trick called *Poisson resummation.* Taking into account the η function we get finally

$$\chi_i(-\frac{1}{\tau}) = \sum_j S_{ij}\chi_i(\tau)$$

with

$$S_{ij} = \frac{1}{R}e^{-2\pi i\mu_i\mu_j} = \frac{1}{\sqrt{2N}}e^{-2\pi i\frac{ij}{2N}}$$

This is a unitary, symmetric $2N \times 2N$ matrix. It is not real, a reflection of the fact that the theory does not have charge conjugation symmetry. Indeed, only the representations $i = 0$ and $i = N$ are self-conjugate.

Since the characters do indeed transform into each other, the diagonal partition function is indeed a modular invariant. Another modular invariant is defined by the charge conjugation matrix C, which always commutes with S and T.

For a given R there are usually many more modular invariant partition functions.

9.7. Relation with circle compactification

The modular invariant partition functions we have found (without claiming uniqueness) can be described most conveniently by introducing a new lattice Γ with momenta (p_L, p_R). This lattice contains all combinations of p_L and p_R that occur, and once we know it, we know the full partition function:

$$P(\tau,\bar\tau) = \frac{1}{\eta(\tau)\eta(\bar\tau)} \sum_{p_L,p_R\in\Gamma} e^{2\pi i\tau p_L^2} e^{-2\pi i\bar\tau p_R^2} \ .$$

where the sum is over all vectors in the two-dimensional lattice. It is easy to show (again using Poisson resummation) that this partition function is modular invariant if and only if Γ is an even self-dual lattice with respect to the Lorentzian metric $(-,+)$. Here "even" means of course that for all lattice vectors $p_L^2 - p_R^2$ must be an even integer, and self-dual means that $\Gamma = \Gamma^*$ (but with duality defined using the Lorentzian metric). One of the conditions for modular invariance is locality. It is easy to verify that

$$V_{p_Lp_R}(z,\bar z)V_{p'_Lp'_R}(w,\bar w) = (z-w)^{p_Lp'_L}(\bar z-\bar w)^{p_Rp'_R}V_{p_L+p'_L,p_R+p'_R} + \ldots,$$

so that locality clearly requires that $p_Lp'_L - p_Rp'_R \in \mathbf{Z}$. This follows indeed from the condition that the lattice is Lorentzian even, by considering the vector $(p-p')$.

The momenta occurring in our partition functions are

$$(\frac{i}{R} + nR, \frac{i}{R} + mR); \qquad (\frac{i}{R} + nR, -\frac{i}{R} + mR)$$

for the diagonal and charge conjugation invariant respectively. Here i lies in the range $0, 2N-1$ and n, m are arbitrary integers. It may be verified that this defines an even self-dual Lorentzian lattice. One can also characterize these partition functions by two unrestricted integers as

$$(\frac{n}{R}, \frac{n}{R} + mR); \qquad (\frac{n}{R}, -\frac{n}{R} + mR)$$

To make the result look more symmetric one can subtract $mR/2$ from both p_L and p_R (*i.e.* one writes $n = n' - \frac{1}{2}R^2m$) to get

$$(\frac{n}{R} - \tfrac{1}{2}mR, \frac{n}{R} + \tfrac{1}{2}mR); \qquad (\frac{n}{R} + \tfrac{1}{2}mR, -\frac{n}{R} + \tfrac{1}{2}mR)\ ,$$

where in the second term the lattice vector $(mR, 0)$ was added. If we compare this to (9.1) we see that the first partition function can be identified with it if L takes the values mR.

Because of the interpretation of L this implies that we are in a compact space with radius R, defined by the lattice Λ. The momenta p in such a space must be such that $\exp(ipx)$ respects the periodicity $x \to x + nR$ of that space, and this implies that p must lie on the dual lattice.

9.8. $R \to 2/R$ DUALITY

In interesting feature of these partition functions is duality. If one replaces R by $\frac{2}{R}$ and interchanges the variables n and m (which are summed over in the partition function), the two partition function (diagonal and charge conjugation) are switched. However these two partition functions are indistinguishable, since charge conjugation does not change the conformal weight, it only flips the $U(1)$ charge. But our choice for the left and right $U(1)$ generator is just a convention. One arrives thus at the surprising conclusion that two theories that are *priori* distinct are in fact indistinguishable.

9.9. RATIONALITY

Note that earlier in this chapter we had found that R^2 should be an even integer. However, from the point of view of circle compactification it does not make any difference what R is. There is an interesting subset of values of R for which the conformal field theory is rational. This happens if the lattice contains vectors $(p_L, 0)$ or $(0, p_R)$, which correspond to operators in the chiral algebra. The condition for rationality is thus

$$\frac{n}{R} + \tfrac{1}{2} m R = 0 \ ,$$

for at least one non-trivial set of integers. The most general solution is $R^2 = \frac{2p}{q}$, where $\frac{p}{q}$ can be any rational number.

Although we only constructed the special cases $q = 1$ and $p = 1$ (the latter is obtained from duality) explicitly, all other cases can be obtained by constructing all other modular invariant partition functions out of the characters.

Note that there is an infinite number of irrational values. Nevertheless, as far as exact solvability is concerned these values are not worse than the rational ones.

The rational theories can all be obtained as modular invariant partition functions of theories with the extended algebras of the form (9.2). The generators of this algebra are thus

$$\partial X; \quad e^{inR\Phi} \ n \in \mathbf{Z}, n \neq 0 \tag{9.6}$$

This are clearly the only operators we have at our disposal. These extended algebras are characterized by a number R with R^2 an even integer. It follows that if we allow rational values of R it cannot be true that one should substitute those values in (9.6). This would lead to non-integer conformal weights for the extended algebra generators. Instead, the theory for other rational R values is realized as a non-diagonal modular invariant of a theory satisfying $R^2 \in 2\mathbf{Z}$.

9.10. Theories with more than one free boson

All the foregoing results have a simple generalization to theories with more than one free boson. The most general modular invariant partition function is described by a Lorentzian even self-dual lattice $\Gamma_{N,N}$ with metric $((-)^N, (+)^N)$ (this is sometimes called a Narain lattice) [26].

To get the most general theory of this kind from a compactification on on N-dimensional torus requires the addition of a term to the Lagrangian, namely

$$\sim \int d^2x B_{ij} \epsilon^{\alpha\beta} \partial_\alpha \Phi^i \partial_\beta \Phi^j$$

where B_{ij} is an arbitrary set of constants.

It is not hard to write down partition functions for these theories at arbitrary genus, and check modular invariance.

10. Orbifolds

There are still more conformal field theories one can construct with one boson. From the point of view of the target space interpretation the additional freedom consists of another choice for the "manifold". Most manifolds are unsuitable since the resulting theory would not be conformally invariant. The torus is always a solution to these conditions, since it is flat and affects the theory only via boundary conditions. In one dimension there is not much choice, and the only proper manifold one can use is the circle. However it turns out that one can still get sensible conformal field theories (and string theories) using spaces that are not proper manifolds, but manifolds with singularities called *orbifolds*. This notion was first used in heterotic string compactification [27], but rapidly acquired a much more general significance.

10.1. Orbifolds as singular manifolds

The definition of an orbifold is as follows. Consider a manifold which has a discrete symmetry. Such a symmetry is said to act freely if it moves every point to a different point. Now we define a new "manifold" by regarding points related to each other by the symmetry as identical. If one uses a symmetry that does not act freely then the fixed points of that symmetry introduce conical singularities. This object is not a manifold, but is called an "orbifold".

10.2. Orbifolds in conformal field theory

In conformal field theory the name "orbifold construction" is often used in a more general sense for a method that allows one to modify conformal field theories by adding new fields, while removing some others. In some cases this procedure has an interpretation in terms of manifolds. There is no need to distinguish freely acting or non-freely acting symmetries, although the latter are usually more difficult to deal with.

Intuitively the orbifold procedure implies the following changes to the theory

— Some states do not respect the discrete symmetry. They have to be removed from the theory (they are "projected out").

— Since some points are identified one can relax the boundary conditions of the boson. Rather than $\Phi^i(x^1 + 2\pi) = \Phi^i(x^1)$ (for the uncompactified boson), or $\Phi^i(x^1 + 2\pi) = \Phi^i(x^1) + 2\pi L^i$ (for the boson on the torus), one must now also allow $\Phi^i(x^1 + 2\pi) = G_{ij}\Phi^j(x^1) + 2\pi L^i$, where G_{ij} is a matrix representing the symmetry. This implies that new states are added to the theory. This new set of states is called the *twisted sector*.

The two items mentioned above are closely related. Roughly speaking, a modular invariant theory contains the maximal set of mutually local fields. They must be mutually local to have T-invariance, and maximal for S-invariance. This same structure is seen in the requirements "even" and "self-dual" that a modular invariant torus compactification must satisfy. Thus if we remove some fields from a modular invariant theory, we can only maintain modular invariance by adding some other fields. Such fields are called *twist fields* [28,29].

10.3. Orbifolds of the circle

In one dimension we have to consider the discrete symmetries of the circle. There are two obvious $\mathbf{Z}_2$ symmetries, namely the "anti-podal map" and the reflection with respect to some axis. The anti-podal map is a special case of an infinite series of Z_N symmetries, which can be realized by shifts $\Phi \to \Phi + 2\pi\ell\frac{R}{N}$. These maps do not have fixed points. One can use them in an orbifold construction, but one finds that they simply lead to a theory on a circle with a different radius, and not to anything new. The reflection corresponds to the symmetry $\Phi \to -\Phi$. This map has two fixed points, $\Phi = 0$ and $\Phi = \pi R$ (note that $-\pi R = \pi R$ because of the lattice identification), and does lead to a new series of theories.

We will discuss these theories here starting from the diagonal partition function of a circle theory. Hence we will assume that R^2 is an even integer. Other radii can then be obtained by means of non-diagonal modular invariants.

<u>*The twist fields*</u>

It follows from the general reasoning that the twist fields must be non-local with respect to the fields that are projected out.

The discrete symmetry acts by taking the conformal field $\partial\Phi$ to $-\partial\Phi$. Thus this operator must be removed. This is done by introducing a twist field with respect to which $\partial\Phi$ is non-local:

$$\partial\Phi(z)\sigma(w,\bar{w}) = (z-w)^{\Delta h}\sigma'(w,\bar{w}) + \dots\ , \tag{10.1}$$

where Δh is non-integer. The branch cut must be such that it still respects the periodicities of the new manifold. This means that when $\partial\Phi(z)$ is moved once around w it can only change sign. Note that moving $\partial\Phi(z)$ around the origin on the complex plane is related by a conformal mapping to moving $\partial\Phi$ once around the cylinder. In the latter case $\partial\Phi$ can return to itself with or without a sign change. Hence we require that $\partial\Phi(z)$ in the complex plane also changes by at most a sign when carried along a circle around the origin. Whether or not there is a sign change depends on the state inserted at the origin (the initial state in the cylinder picture). Since we want $\partial\Phi(z)$ to be non-local w.r.t. σ we require that Δh must be half-integer.

Just as we did for the free fermion, we can describe the construction in terms of an intermediate partially modular invariant invariant theory, in which $\partial\Phi$ and the twist field can co-exist. In this theory, if we require that σ is primary with respect to $\partial\Phi$ we find that Δh must in fact be $-\frac{1}{2}$ (ff it were not primary, we expect another operator to exist which *is* primary, and which we would use instead.) One can then show (see [28]) that $h_\sigma = \frac{1}{16}$. Consequently $h_{\sigma'} = \frac{9}{16}$.

We will assume that there exists a modular invariant diagonal theory in which all ground states have equal holomorphic and anti-holomorphic conformal weights. That theory will ultimately be obtained by making a projection on our T^2-invariant, but not T-invariant theory. Such a theory must contain an operator σ whose anti-holomorphic conformal weight is also $\frac{1}{16}$.

Note however that this implies that $\bar{h}_{\sigma'} = \frac{1}{16}$, since $\partial\Phi(z)$ has $\bar{h}$=0, and it cannot introduce an anti-holomorphic branch cut. Hence the operator σ' has conformal weights $(\frac{9}{16},\frac{1}{16})$ and must disappear in the modular invariant theory. This is consistent with the operator product (10.1) since also $\partial\phi(z)$ will not be an operator in the final theory, because it is odd under the orbifold symmetry.

Now we also need a field $\tilde{\sigma}$ with the operator product

$$\bar{\partial}\Phi(\bar{z})\tilde{\sigma}(w,\bar{w}) = (\bar{z}-\bar{w})^{-\frac{1}{2}}\tilde{\sigma}'(w,\bar{w}) + \dots\ .$$

The conformal weights of these operators must, by arguments similar to the foregoing ones, be $(\frac{1}{16},\frac{1}{16})$ for σ' and $(\frac{1}{16},\frac{9}{16})$ for $\tilde{\sigma}'$. Clearly $\sigma' \neq \tilde{\sigma}'$, so that we clearly need at least one new twist field. In fact it turns out we need two: both $\tilde{\sigma}$ and $\tilde{\sigma}'$ must be new fields.

Projecting on the invariant states

The partially modular invariant theory is now obtained by acting on these twisted ground states with all combinations of oscillators (note that the oscillators are half-integer moded in the twisted sector), and including all states in the untwisted sector.

Now we remove all states that are not invariant under the symmetry $\Phi \to -\Phi$, and the corresponding operators. In particular this removes the operator $\partial\Phi$ and hence the branch cut (10.1) causes no problems anymore. Note that this symmetry changes the sign of all the oscillators as well as the momentum operator.

We do not only wish to remove all states that are odd under the discrete symmetry, but also organize the remaining ones into representations of the chiral algebra of the orbifold theory. In particular this means that we write the new partition function in the standard diagonal form. The chiral algebra of the orbifold theory does not contain the current $\partial\Phi$, but it does contain some other operators, namely the symmetric combinations

$$e^{inR\Phi} + e^{-inR\Phi} \; , n > 0 \; .$$

The operator of lowest conformal weight in this set has conformal weight $\frac{1}{2}R^2$. For $R = \sqrt{2}$ this current has spin 1.

The ground states in the untwisted sector transform as follows. We start from the diagonal partition function of the circle theory, which is created by oscillators acting on the states $|m, m\rangle$ with $-N < m < N$, as well as the states $|\pm N, \pm N\rangle$. The latter four come from the terms $|\chi_N|^2$ in the diagonal partition function. Here the notation is as in (9.5), *i.e.* m denotes a representation with ground state momentum m/R.

A state $|m, m\rangle\, (m \neq 0, m \neq \pm N)$ is mapped to $|-m, -m\rangle$, so that only the linear combination $|m, m\rangle + |-m, -m\rangle$ is left in the orbifold theory. At the first excited level there were four states, $\alpha_{-1}\,|m, m\rangle$ $\bar{\alpha}_{-1}\,|m, m\rangle$, $\alpha_{-1}\,|-m, -m\rangle$ and $\bar{\alpha}_{-1}\,|-m, -m\rangle$. In this case the linear combinations $\alpha_{-1}\,|m, m\rangle - \alpha_{-1}\,|-m, -m\rangle$ and $\bar{\alpha}_{-1}\,|m, m\rangle - \bar{\alpha}_{-1}\,|-m, -m\rangle$ survive the projection. These two states are created from the ground state by the mode L_{-1} of the energy-momentum tensor $-\frac{1}{2}(\partial\Phi)^2$ and its anti-holomorphic partner.⋆ Thus we see that the structure of the lowest lying states is consistent with a contribution to the partition function of the form $|q^{h-c/24}(1+q+\ldots)|^2$, the square of a single character. With some more work one can show that this structure persists to higher excitation levels. Thus for each value of m in the range $0 < m < N$ we find precisely one representation of the orbifold chiral algebra.

The states with charges $\pm N$ are slightly more subtle. Of the four states $|\pm N, \pm N\rangle$ two linear combinations survive, namely $|+\rangle = |N, N\rangle + |-N, -N\rangle$ and $|-\rangle = |N, -N\rangle + |-N, N\rangle$. These two states are mapped into each other by the operator $J = e^{iR\Phi} + e^{-iR\Phi}$, but the linear combinations $|+\rangle \pm |-\rangle$ are eigenstates of J. They form two separate ground states, each of one representation. This is as it should be: in the diagonal partition function ground states are represented by the square of a character, and hence the multiplicity of any ground state $|h, h\rangle$ must be a square. If one finds a ground state with multiplicity 2, it must be obtained as $1 + 1$, since 2 is not a square.

The vacuum sector also requires more attention. Here we have to distinguish two cases. For $R^2 > 2$ the first excited states are are $\alpha_{-1}\,|0, 0\rangle$ and $\bar{\alpha}_{-1}\,|0, 0\rangle$. They are both odd under the

⋆ Note that L_{-1} contains a term $\propto \alpha_0\alpha_{-1}$, and that $\alpha_0\,|m, m'\rangle = m\,|m, m'\rangle$.

symmetry $\Phi \to -\Phi$ and disappear. However, the symmetric excitation $\alpha_{-1}\bar{\alpha}_{-1}\,|0,0\rangle$ does survive. This contribution to the diagonal partition function starts thus as $[q\bar{q}]^{-c/24}(1+q\bar{q})$, and does not factorize (the circle partition function has as its leading terms $|q^{-c/24}(1+q)|^2$.) Hence we are forced to introduce a new ground state, denoted as $\left|\partial\Phi,\bar{\partial}\bar{\Phi}\right\rangle$, that corresponds to the circle state $\alpha_{-1}\bar{\alpha}_{-1}\,|0,0\rangle$. It may then be shown that all further excitations factorize in a sum of two terms, one corresponding the vacuum representation and one to the representation built on the ground state $\left|\partial\Phi,\bar{\partial}\bar{\Phi}\right\rangle$.

If $R=\sqrt{2}$ the circle chiral algebra contains 3 spin 1 currents, $\partial\Phi$ and $\exp(\pm iR\Phi)$, and hence the leading terms in the circle partition function are $[q\bar{q}]^{-c/24}(1+3q+3\bar{q}+9q\bar{q})$. Only one of the three currents survive the projection, and of the nine current-current states five survive. Hence the orbifold partition function starts with

$$[q\bar{q}]^{-c/24}(1+q+\bar{q}+5\bar{q}) = [q\bar{q}]^{-c/24}|1+q+\dots|^2 + [q\bar{q}]^{1-c/24}|2+\dots|^2$$

In this case the ground state of the representation denoted "$\left|\partial\Phi,\bar{\partial}\bar{\Phi}\right\rangle$" has thus multiplicity 2, and contributes to the full partition function with multiplicity 4.

In the twisted sector we have to define the action of the symmetry on the ground states; then the rest is fixed. The unprimed and primed twist fields must transform with an opposite sign, as is clear from (10.1). Since $\sigma(0)\,|0,0\rangle = \left|\frac{1}{16},\frac{1}{16}\right\rangle$ is a desirable state and $\sigma'(0)\,|0,0\rangle = \left|\frac{9}{16},\frac{1}{16}\right\rangle$ is not, we choose σ (as well as $\tilde{\sigma}$) to transform with a $+$ sign. Then the state $\sigma'(0)\,|0,0\rangle$ transforms with a $-$ sign and is removed, while for example $\bar{\alpha}_{-1/2}\sigma'(0)\,|0,0\rangle$ with conformal weight $h=\bar{h}=\frac{9}{16}$ remains. Since we have removed σ' as well as $\partial\phi$ we need a new operator that creates the state $\left|\frac{9}{16},\frac{9}{16}\right\rangle$ from the vacuum. We will call this operator once again σ'.

10.4. The partition function

To summarize, we find thus the following partition function

$$P_{\text{orb}} = \sum_r |\chi_r|^2$$

where the label r stands for the following representations (the notation is inspired by the foregoing discussion in an obvious way, but note that here we are only considering one chiral sector)

r	h	r	h
0	0	$\partial\Phi$	1
σ	$\frac{1}{16}$	$\tilde{\sigma}$	$\frac{1}{16}$
σ'	$\frac{9}{16}$	$\tilde{\sigma}'$	$\frac{9}{16}$
$N_{(1)}$	$\frac{N}{4}$	$N_{(2)}$	$\frac{N}{4}$
$m \ \ (0 < m < N)$	$\frac{m^2}{4N}$		

There are in total $1 + 1 + 4 + 2 + N - 1 = N + 7$ representations. Each has ground state multiplicity 1, except $r = \partial\Phi$ for $R^2 = 2$, as noted above.

10.5. The geometric description

Although the presentation given above was a bit intuitive and not completely rigorous, it is not hard to show that it actually leads to a modular invariant partition function. The partition function we were constructing can be summarized as follows

$$P_{\rm orb} = \tfrac{1}{2}(P_B^{\rm PP} + P_B^{\rm PA} + P_B^{\rm AA} + P_B^{\rm AP})$$

Here P_B represents the free boson path integral on the (world-sheet) torus, with boundary conditions as indicated. The sum over boundary conditions is as for the free fermion, and is modular invariant for the same reason.

The term $P_B^{\rm PP}$ is the circle partition function. The second term is anti-periodic in the time direction, which means that odd numbers of bosons contribute with a $-$ sign. The third and fourth term are anti-periodic in the space direction, and represent the twisted sector. The combination of terms projects out the unwanted states in that sector. The first term and the sum over the last three terms are separately modular invariant. The precise combination of these two modular invariant sets is dictated by the requirement of having a unique vacuum and positive integral multiplicities for all other states. In particular a relative $-$ sign between these modular invariant sets (which was allowed for the free fermion) is not allowed here because it would project out the vacuum.

The partition function may also be written as

$$P_{\rm orb} = \ {\rm Tr}\ _P \tfrac{1}{2}(1+g) q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - \bar{c}/24} + \ {\rm Tr}\ _A \tfrac{1}{2}(1+g) q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - \bar{c}/24} \ ,$$

where g represents the non-trivial $\mathbf{Z}_2$ element that sends Φ to $-\Phi$. This formula has an immediate generalization to arbitrary discrete abelian groups, often written suggestively as

$$P_{\rm orb} = \frac{1}{|G|} \sum_{g,h\in G} g\underset{h}{\square}\ ,$$

where $|G|$ is the number of elements in the group G. The sum over h is over all possible twisted sectors, whereas the sum over g performs the projections. Modular invariance of this expression is intuitively clear.

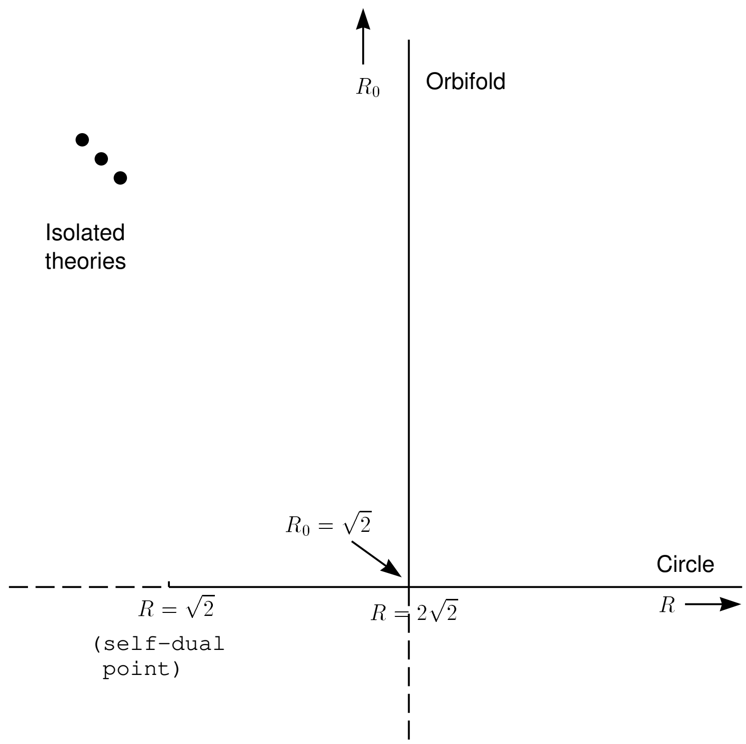
The advantage of this formulation applied to the $c = 1$ orbifolds is that it works immediately for arbitrary (even non-rational) R. The disadvantages is that it does not give direct information on the chiral algebra and the representations.

10.6. The $c = 1$ conformal field theories

We have now identified two sets of $c = 1$ conformal field theories, each parametrized by a real number R. Furthermore there is a duality in both spectra, since R and $\frac{2}{R}$ are giving rise to the same spectrum. The self-dual point occurs at $R = \sqrt{2}$.

One may think that the orbifold and the circle theories are all different, but in fact they are not. It can be shown that the orbifold of the $R = \sqrt{2}$ theory and the circle with $R = 2\sqrt{2}$ describe one and the same theory. This cannot happen at any other point, since only for $R = \sqrt{2}$ the orbifold theory has a spin-1 current. It is easy to verify that the spectra of the orbifold and circle theories are indeed the same, and not much harder to show that they are in fact the same theory.

Hence the two lines are not separate, but connected, as shown in the following picture [30], [31]. Note that only the topology of the picture matters, not the geometry. The dashed lines indicate values of R that have already been taken into account because of $R \to 2/R$ duality. The orbifold radius is denoted by R_o.



Apart from this continuum there also exist three isolated theories. They can be obtained by an orbifold procedure applied to the $R = \sqrt{2}$ circle theory [32]. With these points included the picture is conjectured to be complete.

10.7. Moduli and marginal deformations

This picture provides the simplest example of moduli in conformal field theory. Moduli are free parameters which can be varied continuously without affecting conformal invariance. Apart from the three isolated points, every point on the diagram corresponds to a conformal field theory with moduli. The point where the circle and the orbifold meet is characterized by the existence of an additional modulus.

One can detect the existence of such conformal invariant deformations within a given theory by looking for conformal fields of dimension $(1, 1)$, called *marginal operators*. Such operators have precisely the correct weights to yield a conformal invariant result when integrated over dz and $d\bar{z}$.* This implies that they can be added as a perturbation to the action,

$$\delta S \propto \int dzzd\bar{z}V_{1,1}(z, \bar{z}) ,$$

where $V_{1,1}$ is a marginal operator.

⋆ Marginal operators must satisfy additional constraints not explained here.

In the circle and orbifold theories, this operator is $\partial\Phi(z)\partial\Phi(\bar{z})$. The additional marginal operator in the meeting point of the lines is due to combinations with the additional spin-1 field $e^{i\sqrt{2}\Phi}$.

11. Kac-Moody algebras

In this chapter we consider extensions of the chiral algebra by a set of (anti)-holomorphic spin-1 currents.

11.1. Spin one operator products

These currents are conformal fields with respect to the Virasoro algebra. The operator product of two such currents in a modular invariant conformal field theory must be local. Since the currents are holomorphic their operator product is holomorphic as well. Hence it is an expansion of integer powers of $(z-w)$ multiplied by integer spin operators. Since the lowest spin an operator in a unitary conformal field theory can have is zero (the identity), the leading term is a constant times $(z-w)^{-2}$. The next one is $(z-w)^{-1}$ times a holomorphic spin-1 operator, which must therefore be one of the currents. Hence we get

$$J^a(z)J^b(w) = \frac{d^{ab}}{(z-w)^2} + \frac{if^{abc}J^c}{z-w} + \ldots \tag{11.1}$$

Note that the next term has spin 2, and hence is a candidate for a Virasoro generator; we will return to it later.

Since integer spin currents are bosons the left-hand side is symmetric under interchange $((z,a) \leftrightarrow (w,b))$. It follows that d^{ab} must be symmetric and f^{abc} anti-symmetric in a and b. Since f^{abc} appears in the three-point function it must then be anti-symmetric in all three indices [This is true provided a Hermitean basis is chosen]. Using duality relations one can then show that the coefficients f^{abc} must satisfy Jacobi identities. It follows then that they are structure constants of a Lie algebra. This Lie algebra must be a direct product of some simple Lie algebras and some $U(1)$ factors (here "some" includes the possibility that there are no such factors).

The argument given here is due to A. Zamolodchikov [33].

11.2. Intermezzo: some Lie algebra facts

We will fix some standard normalizations for simple Lie algebras. The algebra is

$$\left[T^a, T^b\right] = if^{abc}T^c$$

It is satisfied in particular by the matrices

$$(T^a_{\rm adj})_{bc} = -if^{abc} \ ,$$

which are the generators of the adjoint representation. Their commutator is in fact nothing but the Jacobi identity. The generators in the adjoint representation act on the algebra via the commutator.

The root system is defined by selecting out of the generators T^a the maximally commuting set H^i, the Cartan subalgebra. The number of such generators is called the rank of the algebra. In the adjoint representation we may simultaneously diagonalize the Cartan subalgebra acting on the remaining ones, so that

$$\left[H^i, E_{\vec\alpha}\right] = \alpha^i E_{\vec\alpha} \tag{11.2}$$

The set of vectors $\vec\alpha$ is called the root system of the algebra.

In a compact Lie algebra a basis can be chosen so that

$$\mathrm{Tr}\ T^a_{\rm adj} T^b_{\rm adj} = N\delta^{ab} \ , \tag{11.3}$$

where N is a normalization, to be fixed in a moment. The left-hand side is called the Killing metric of the Lie algebra. It will be assumed that the Cartan subalgebra generators are elements of the basis. Then (11.3) induces a natural inner product on the root space, namely $\vec\alpha\cdot\vec\beta = \sum_i \alpha_i\beta_i$.

Given a root system we can choose a plane which divides the roots into positive and negative ones (this plane must thus be chosen in such a way that none of the roots lies in the plane). Then one defines a set of *simple roots* which form a basis of the root system with the property that all positive roots are linear combinations with positive coefficients of the simple roots. One also defines a highest root ψ as the unique positive root from which all other roots can be obtained by subtracting simple roots. Now we define the dual Coxeter number g:

$$g = \frac{(\vec\psi + 2\vec\rho)\cdot\vec\psi}{\vec\psi\cdot\vec\psi} \ ,$$

where $\vec\rho$ is half the sum of the positive roots. Note that this definition is independent of the normalization of the inner product. The values of the dual Coxeter number for all simple Lie algebras are listed in the following table

$\text{Type}_{\text{rank}}$	Algebra	Value of g	Adjoint dimension
A_{N-1}	$SU(N)$	N	N^2-1
$B_{\frac{N-1}{2}}$	$SO(N),\ N>4, \text{odd}$	$N-2$	$\frac{1}{2}N(N-1)$
C_N	$Sp(2N)$	$N+1$	$N(2N+1)$
$D_{\frac{N}{2}}$	$SO(N),\ N>3, \text{even}$	$N-2$	$\frac{1}{2}N(N-1)$
G_2		4	14
F_4		9	52
E_6		12	78
E_7		18	133
E_8		30	248

The first column gives the Dynkin classification, while the second one gives the identification with the perhaps more familiar classical Lie algebras.

We now fix the normalization of the generators by requiring that $N = 2g$ in (11.3). This normalization implies that the highest root has norm 2. To conclude this section we write down the remaining commutators among the generators in this basis. For the commutator between the root generators one has

$$\left[E^{\vec{\alpha}}, E^{\vec{\beta}}\right] = \epsilon(\vec{\alpha},\vec{\beta})E^{\vec{\alpha}+\vec{\beta}} \ ,$$

if $\vec{\alpha}+\vec{\beta}$ is a root, and

$$\left[E^{\vec{\alpha}}, E^{-\vec{\alpha}}\right] = \vec{\alpha}\cdot\vec{H} \ ,$$

and zero in all other cases. The coefficients $\epsilon(\vec{\alpha},\vec{\beta})$ are non-zero real numbers.

11.3. The central term

The first tensor in (11.1) must be symmetric in a and b, and furthermore the Lie algebra structure we have just identified requires it to be an invariant tensor of the Lie algebra. Hence it must be proportional to the Killing form, which in our conventions means it is proportional to δ^{ab}. Since we have already fixed the normalization of the structure constants, the normalization of the first term is fixed. Note that the first term determines the current-current propagator, and that this has a positive residue only if the Lie-algebra is compact (if it is not compact the Killing form has negative eigenvalues). If the propagator had a wrong-sign residue this would violate unitarity. Thus in unitary conformal field theories the Lie algebra must be compact.

If the Lie algebra is semi-simple the term d^{ab} takes the form $k^a\delta^{ab}$, where k^a is constant on each simple factor. From now on we will focus on simple Lie algebras; the index a on k^a can then be dropped.

11.4. Modes

The mode expansion of the currents is as discussed in general in chapter 7. It is straightforward to derive the algebra in terms of modes

$$\left[J^a_m, J^b_n\right] = if^{abc}J^c_{m+n} + km\delta^{ab}\delta_{m+n,0} \tag{11.4}$$

Note that for $m = n = 0$ one obtains a subalgebra which is a simple Lie-algebra. Since the modes with $m = n = 0$ do not alter the conformal weight, this algebra takes the states of a given weight into each other. It is usually referred to as the *horizontal algebra.*

If $k = 0$ the algebra is referred to as the *loop algebra.* If $k \neq 0$ one gets strictly speaking only an algebra if we consider k as the eigenvalue of an operator K, which is called the central extension of the loop algebra. This operator commutes with all others. This is analogous to the central extension of the Virasoro algebra.

The algebra (11.4) is called a centrally extended loop algebra, or current algebra. It is often also referred to as an affine Lie algebra, or a Kac-Moody algebra. This is not quite correct. The mathematical definition of an affine Lie-algebra includes in addition to the operators appearing in (11.4) still one more operator called the *derivation* D. This operator satisfies $[D, J^a_n] = nJ^a_n$, and $[K, D] = 0$. Comparing the first expression with (7.2), one finds that it is satisfied by $D = -L_0$; because of (11.4), L_0 commutes with K and hence the second commutator is also satisfied. Since we will only consider the spin-1 current algebras in combination with a Virasoro algebra, the distinction between the two definitions is not essential for us. Note that the current algebra is unaffected if we omit D, since it never appears on the right hand side of a commutator, but from the mathematical point of view it is convenient to introduce it in order to define an invertible Killing form. The mathematical definition of a Kac-Moody algebra is much more general, and includes ordinary as well as affine Lie algebras, and many more. We will nevertheless use the term "Kac-Moody" algebra from here on in a restricted sense, to refer to (11.4).

11.5. Twisted and untwisted affine algebras

Since the current has integral spin, the "natural" mode expansion is in terms of integer modes. One can however also consider fractionally moded operators by introducing twist fields. One find that in many cases the fractionally mode algebras are isomorphic to the integrally moded ones. There is a set of algebras and twistings (related to so-called outer automorphisms of the horizontal Lie-algebra) for which this is not the case. They are known as *twisted affine algebras.* In these lectures we will only encounter the untwisted ones.

11.6. Primary fields

Primary fields are defined by the condition that they should be Virasoro primary fields, and in addition satisfy

$$J^a(z)\Phi^i(w,\bar{w}) = \frac{T^a_{ij}}{z-w}\Phi^j(w,\bar{w}) + \ldots$$

The leading pole is determined as in the general arguments given in chapter 7. Since the field appearing on the right hand side has the same conformal weight as Φ, one can label all the fields with that conformal weights by a label i, and then the operator product inevitably looks like the one above.

This implies that the ground states $|r\rangle$ are rotated into each other by the horizontal algebra, which acts via the matrices T^a_{ij}:

$$J^a_0 \, |r_i\rangle = T^a_{ij}(r) \, |r_j\rangle \ , \tag{11.5}$$

where

$$|r_i\rangle = \Phi^i(0) \, |0\rangle$$

The matrices $T^a_{ij}(r)$ can be shown to satisfy the commutation relations of the horizontal algebra,

$$\left[T^a, T^b\right] = i f^{abc} T^c \ ,$$

by acting with a second generator J^b_0. They are the representation matrices of the horizontal algebra in some representation r determined by Φ^i.

Note that the current itself is not a Kac-Moody primary field, just as the energy momentum tensor is not a conformal field.

11.7. The Sugawara tensor

In addition to the current modes the algebra under consideration consists of Virasoro generators, with definite commutation relations with themselves and the currents. Actually, there is one as yet unknown quantity in the Virasoro algebra, namely its central charge. It turns out that the Virasoro generators can be expressed in terms of the currents in the following way:

$$T(z) = \frac{1}{2(k+g)} : \sum_a J^a(z) J^a(z) : \ , \tag{11.6}$$

where the sum is over all generators of the horizontal algebra. This is called the Sugawara energy-momentum tensor [34]. As usual, normal ordering means subtraction of the singular terms,

$$:\sum_a J^a(z)J^a(z):\equiv \lim_{w\to z}\left[\sum_a J^a(z)J^a(w)-\frac{k\,\mathrm{dim(adj)}}{(z-w)^2}\right]$$

(For $U(1)$ algebras dim(adj) should be interpreted as the number of $U(1)$ generators.) To verify that this is indeed the Virasoro generator, we have to check the operator product with the current, and with $T(w)$. The requirement that $J^a(z)$ is a conformal field of weight 1 fixes the normalization in (11.6). In the computation one uses the relation

$$-f^{acd}f^{bdc} = \mathrm{Tr}\ T^a_{\mathrm{adj}}T^b_{\mathrm{adj}} = 2g\delta^{ab}$$

Then the computation of $T(z)T(w)$ serves as a check, but in addition determines the central charge:

$$c=\frac{k\,\mathrm{dim(adj)}}{k+g}$$

The Virasoro generators can be expressed in terms of the modes of the currents:

$$L_n=\frac{1}{2(k+g)}\sum_{m=-\infty}^{\infty} : J^a_{m+n}J^a_{-m}: ,$$

where normal ordering means that positive modes should appear to the right of negative ones.

11.8. Highest weight representations

Highest weight representations are characterized by a ground state $|r\rangle$ that is annihilated by all positive modes of J_n. This implies automatically that it is annihilated by all positive modes of the (Sugawara) energy-momentum tensor, *i.e.* that it is a Virasoro highest weight.

The only remaining freedom we have in characterizing representations is the action of the zero-mode generator J^a_0. We have already seen before that the ground states form a representation r of the horizontal algebra generated by the zero-modes. Representations of simple Lie algebras are themselves generated by step operators acting on highest weight vectors. This implies that any irreducible unitary representation of a Kac-Moody algebra is completely characterized by a highest weight vector of the horizontal algebra and the eigenvalue of the operator K, called the level (k). Completely, because once we know the horizontal algebra highest weight and k we know the action of all current modes and the Virasoro generators.

In particular we know the conformal weight of the ground state:

$$h = \frac{\sum_a \langle r| J_0^a J_0^a |r\rangle}{2(k+g)} \tag{11.7}$$

The expectation value can be computed using (11.5):

$$\sum_a \langle r_i| J_0^a J_0^a |r_j\rangle = \sum_a (T^a(r)T^a(r))_{ij} = C_2(r)\delta_{ij} \ .$$

Here i and j label the components of the representation r, and $C_2(r)$ is the quadratic Casimir operator. The result is thus

$$h_r = \frac{\frac{1}{2}C_2(r)}{k+g}$$

Note that our normalization is such that in the adjoint representation $C_2(\text{adj}) = 2g$.

The representation r must be an irreducible highest weight representation of the horizontal algebra. What remains to be done is to determine which representations and which values of k are allowed. Rather than attempting to solve this directly in general, we start by looking at the simplest theories.

11.9. $U(1)$ THEORIES

If all structure constants f^{abc} vanish one obtains a product of one or more $U(1)$ factors. Their currents can always be written in terms of free bosons, $J^i = i\partial\Phi^i$. They satisfy the operator product (11.1) with $k = 1$. We have already studied this case in detail, and discuss it only here to show how it fits in.

Since $f^{abc} = 0$, the dual Coxeter number g vanishes. Then the energy-momentum tensor has the standard form for free bosons, $T(z) = -\frac{1}{2}(\partial\Phi(z))^2$. The central charge is equal to the number of free bosons, as expected.

The representations are labelled by the zero-mode momenta of the ground states, usually referred to as charges. The ground states satisfy thus

$$J_0 |q\rangle = q |q\rangle \ ,$$

and they are in fact uniquely labelled by q. Their conformal weight is $\frac{1}{2}q^2$. Note that $J_0 = p$, the momentum operator.

11.10. The $SU(2)$ Kac-Moody algebra

The root system of $SU(2)$ has just one simple root α. This is also the only positive root, and is also equal to the highest root. The Weyl vector ρ is equal to half the sum of the positive roots, and is thus equal to $\frac{1}{2}\alpha$. The dual Coxeter number is easily computed to be 2.

The algebra is generated by three currents $J^a, a = 1\dots 3$. The structure constants are proportional to ϵ^{abc}. The proportionality constant can be determined by (11.3), which reads

$$(-ix\epsilon^{acd})(-ix\epsilon^{bdc}) = 2g\delta^{ab} = 4\delta^{ab} \ ,$$

where x is the proportionality constant. We find thus that $x = \sqrt{2}$. This is a disadvantage of this normalization: $SU(2)$ generators are not normalized in the familiar way. Similarly the generators in the spinor representation are $\frac{1}{2}\sqrt{2}\tau^i$, where τ^i are the Pauli matrices. [An advantage of our normalization is that for any algebra and any representation the quantity $I_2(R)$, defined by $\mathrm{Tr}\ T^aT^b = I_2(r)\delta^{ab}$ is an integer.]

Highest weight representations of the $SU(2)$ Kac-Moody algebra are characterized by $SU(2)$ Lie-algebra representations and the level k; hence they are characterized by k and the $SU(2)$ spin j. A ground state has $2j+1$ components $|j,m\rangle$. Its conformal weight is

$$h = \frac{j(j+1)}{k+2}$$

Here we recognize the $SU(2)$ Casimir eigenvalue $j(j+1)$.

The following argument restricts the values of k. The algebra (11.4) has several interesting sub-algebras. One is the zero-mode algebra,

$$\left[J_0^a, J_0^b\right] = i\sqrt{2}\epsilon^{abc}J_0^c \ .$$

Apart from the normalization this is a standard $SU(2)$ algebra. Since we want to use results from $SU(2)$ representation theory, we have to change the normalization of the generators. Furthermore we go to a basis of raising/lowering operators. Hence we define

$$I^{\pm} = \frac{1}{\sqrt{2}}(J_0^1 \pm iJ_0^2); \quad I^3 = \frac{1}{\sqrt{2}}J^3 \ ,$$

so that $\left[I^+, I^-\right] = 2I^3$. Standard results in $SU(2)$ unitary representation theory tell us now that the eigenvalues of I^3 must be (half)-integers. It is easy to check that the following generators satisfy the same commutation relations:

$$\tilde{I}^{+} = \frac{1}{\sqrt{2}}(J^{1}_{+1} - iJ^{2}_{+1}); \quad \tilde{I}^{-} = \frac{1}{\sqrt{2}}(J^{1}_{-1} + iJ^{2}_{-1});$$
$$\tilde{I}^{3} = \tfrac{1}{2}k - \frac{1}{\sqrt{2}}J^{3} = \tfrac{1}{2}k - I^{3} \ .$$

Hence we conclude that the eigenvalues of $\tilde{I}^3$ must also be (half)-integers, and furthermore since I^3 and $\tilde{I}^3$ commute we can diagonalize them simultaneously. This is only consistent if k is an integer. Furthermore unitarity (positivity of the residue of the propagator) requires it to be a positive integer.

Now we can directly get a further constraint by computing the norm of the state $\tilde{I}^{-}|j,m\rangle$, where $|j,m\rangle$ is one of the components of the ground state

$$\begin{aligned} 0 &\leq \langle j,m|\, \tilde{I}^{+}\tilde{I}^{-}\, |j,m\rangle \\ &= \langle j,m|\, \left[\tilde{I}^{+},\tilde{I}^{-}\right] |j,m\rangle \\ &= \langle j,m|\, 2\tilde{I}^{3}\, |j,m\rangle \\ &= \langle j,m|\, (k-2I^{3})\, |j,m\rangle \\ &= \langle j,m|\, (k-2m)\, |j,m\rangle \end{aligned}$$

Here we used the requirement of unitarity (positivity of the norm), the highest weight property of $|j,m\rangle$, which implies that $J^{a}_{+1}|j,m\rangle = 0$, and the $SU(2)$ commutator $[I^{+},I^{-}] = 2I^{3}$. Clearly m cannot be larger than $\frac{1}{2}k$, and the same follows then for j. It is convenient to label the representations by integers $l = 2j$. They are thus restricted to the values $0 \leq l \leq k$.

11.11. $SU(2)$ at level 1.

For $k = 1$ there are thus precisely 2 representations, with ground state spins $j = 0$ and $\frac{1}{2}$. We have already seen a realization of this theory, namely in the self-dual point of the $c = 1$ circle theory. At this point there are three spin-1 fields, namely $\partial\Phi$ and $e^{\pm i\sqrt{2}\Phi}$. Their operator products are (singular terms only)

$$\partial\Phi(z)e^{\pm i\sqrt{2}\Phi}(w) = \frac{\mp i\sqrt{2}}{(z-w)}e^{\pm i\sqrt{2}\Phi}(w)$$

$$\partial\Phi(z)\partial\Phi(w) = -\frac{1}{(z-w)^2}$$

$$e^{i\sqrt{2}\Phi(z)}e^{i\sqrt{2}\Phi(w)} = \text{non-singular}$$

and

$$e^{i\sqrt{2}\Phi(z)}e^{-i\sqrt{2}\Phi(w)} = \frac{1}{(z-w)^2} + \frac{i\sqrt{2}}{z-w}\partial\Phi(w)$$

These is precisely equal to (11.1) with $k = 1$ provided we define

$$\begin{aligned} J^1(z) = \tfrac{1}{2}\sqrt{2}(e^{i\sqrt{2}\Phi} + e^{-i\sqrt{2}\Phi})\ , \quad & J^2(z) = -\tfrac{1}{2}i\sqrt{2}(e^{i\sqrt{2}\Phi} - e^{-i\sqrt{2}\Phi})\ , \\ J^3 = i\partial\Phi\ . & \end{aligned}$$

Thus we see that this algebra can be realized with a single free boson. We have already seen in the previous chapter that for $R^2 = 2N$ the bosonic theories have $2N$ characters with conformal weights $\frac{m^2}{4N}$, $-N < m < N$. For $N = 1$ this agrees with the $SU(2)$ level-1 description of the same theory.

The primary field corresponding to the only non-identity representation can also be written in terms of the free boson, namely as $\exp(i\frac{1}{2}\sqrt{2}\Phi(z)$. Unfortunately things are less simple at higher levels.

11.12. Generalization to other Kac-Moody algebras

The foregoing results on $SU(2)$ have an immediate generalization to other algebras. This generalization works exactly like the reasoning one follows to derive the Lie-algebra representations from the representation theory of $SU(2)$. The results for $SU(2)$ are valid for any $SU(2)$ sub-algebra of some Kac-Moody algebra, and now it is simply a matter of finding the most suitable one.

Let us first find a suitable basis for the current modes J^a_n. For the zero modes there is a standard basis, the one introduced in section 11.2.

To generalize this to Kac-Moody algebras one simply attaches an extra index n to all operators, and includes the central term. The result is

$$\begin{aligned} \left[H^i_m, H^j_n\right] &= m\delta_{m+n,0}\delta^{ij} \\ \left[H^i_m, E^{\vec\alpha}_n\right] &= \alpha^i E^{\vec\alpha}_{m+n} \\ \left[E^{\vec\alpha}_m, E^{\vec\beta}_n\right] &= \epsilon(\vec\alpha,\vec\beta)E^{(\vec\alpha+\vec\beta)}_{n+m} \\ \left[E^{\vec\alpha}_n, E^{-\vec\alpha}_m\right] &= \vec\alpha\cdot\vec H_{n+m} + Kn\delta_{n+m,0}\ . \end{aligned}$$

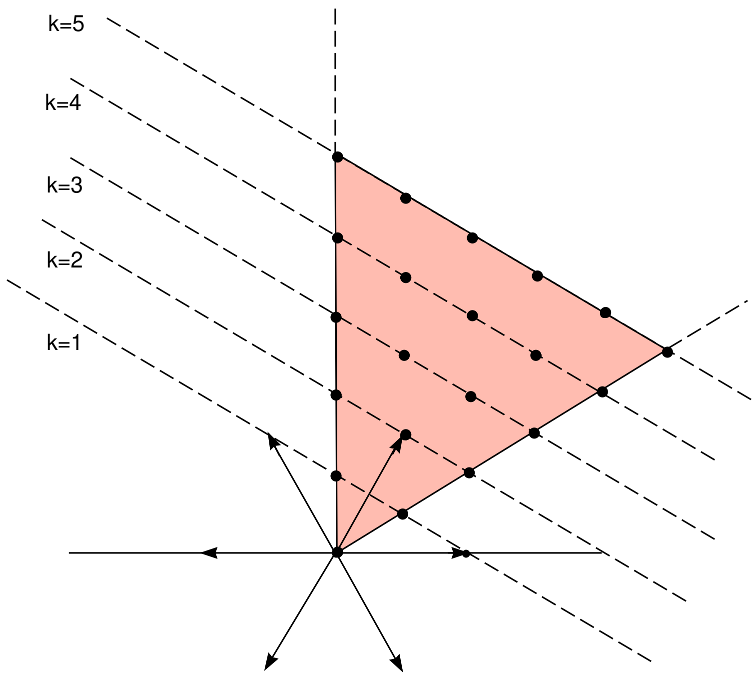
It is easy to see that any root $\vec\alpha$ defines a conventionally normalized $SU(2)$ subalgebra, whose generators are $\tilde I^+ = \sqrt{\frac{2}{\vec\alpha^2}}E^{-\vec\alpha}_n$, $\tilde I^- = \sqrt{\frac{2}{\vec\alpha^2}}E^{\vec\alpha}_{-n}$ and $\tilde I_3 = \frac{1}{\vec\alpha^2}(Kn - \vec\alpha\cdot H_0)$. The normalization of this $SU(2)$ is the traditional one, *i.e.* $\left[I^+, I^-\right] = 2I^3$ etc. By arguments similar to the ones used for $SU(2)$ we conclude that the quantity $2nK/(\vec{\alpha}^2)$ must have integer eigenvalues, for any n and $\vec{\alpha}$. Obviously the strongest constraint comes from $n = 1$. If we have normalized our root system in the canonical way, *i.e.* $\vec{\psi}^2 = 2$, there is always a root with norm 2, and we find that K must have integer eigenvalues k. The norms of other roots that can occur in simple Lie algebras are 1 or $\frac{2}{3}$, in the canonical normalization. This does not impose additional constraints.

One can use the same subalgebra to find constraints on the ground states. We know already that the ground states are representations of the horizontal algebra, and are characterized by a highest weight $\vec{\lambda}$. The ground state has then $\dim(r_{\vec{\lambda}})$ components, where $r_{\vec{\lambda}}$ indicates the representation with highest weight $\vec{\lambda}$.

Take any component $|\mu\rangle$, where μ is any weight in $r_{\vec{\lambda}}$. By requiring positivity of the norm of $\tilde{I}^- |\mu\rangle$ we get now the condition $2\frac{\vec{\alpha}\cdot\vec{\mu}}{\vec{\alpha}\cdot\vec{\alpha}} \leq k$. This condition is most restrictive if we take μ equal to the highest weight of the ground state representation $\vec{\lambda}$, and ψ equal to the highest root. In the canonical normalization we get then

$$\vec{\psi} \cdot \vec{\lambda} \leq k$$

Just as for $SU(2)$ the number of representations satisfying this condition is finite. The following picture shows the allowed highest weights for the algebra A_2 at various levels



The negatively moded currents J^a_{-n} act on these ground states and create the full Kac-Moody representation. Since they are in the adjoint representation of the horizontal algebra, they also change the representation that one finds at higher excitation levels (the excitation level, also called *grade* is defined as the conformal weight of a descendant minus that of the ground state. It should not be confused with the level of the algebra). Naively the representation content at the higher excitation levels can be obtained by selecting all combinations of current modes that produce the desired excitation level, and tensoring the ground state representation with the adjoint representation as many times as required. For example, one might expect the first excited level to contain all representations in the tensor product $r_\lambda \otimes r_\psi$, the latter being the adjoint representation. However, the norms of some of the representations in the tensor product might be 0, just as was the case for Virasoro representations.

Zero-norm states are removed. Nothing in the previous arguments guarantees the absence of negative norm states, which would make the representation non-unitary. The conditions we have satisfied are necessary conditions for the absence of some potential negative norm states, namely those occurring in certain $SU(2)$ subalgebras. One way to show that the representations are indeed unitary is to find an explicit realization of the symmetries in some well-defined field theory.

11.13. The Frenkel-Kac construction

One such realization is the Frenkel-Kac construction [35].

This is a generalization of the level-1 construction which we gave for $SU(2)$. It works for any Lie-algebra whose roots have the same length, which is conveniently normalized to the value 2. Such a Lie-algebra is called *simply-laced*, and the algebras enjoying this property are A_r, D_r and E_6, E_7 and E_8.

The generators of these algebras at level 1 can be written down explicitly in terms of r free bosons, where r is the rank. One simply writes them as

$$E^{\vec{\alpha}}(z) = e^{i\vec{\alpha}\cdot\Phi}(z)$$

and

$$H^i(z) = i\partial\Phi^i(z) \ ,$$

and defines modes in the usual way. This yields the correct operator product for the $SU(2)$ sub-algebras associated to each of the roots (as one may check), and furthermore one gets

$$e^{i\vec{\alpha}\cdot\Phi}(z)e^{i\vec{\beta}\cdot\Phi}(w) = (z-w)^{\alpha\cdot\beta}e^{i(\vec{\alpha}+\vec{\beta})\cdot\Phi}(w) + \dots$$

Inner products between roots of simply laced algebras can be $2, 1, 0, -1$ and -2. In the first case $\vec{\alpha} = \vec{\beta}$, and in the last case $\vec{\alpha} = -\vec{\beta}$. If $\vec{\alpha}\cdot\vec{\beta} = -1$ one finds that $\vec{\alpha}+\vec{\beta}$ is a root. Precisely in that case the operator product has a pole, exactly as required by (11.1).

However, this is not quite the end of the story, because the coefficients $\epsilon(\vec{\alpha},\vec{\beta})$ can have signs. Although many of these signs are merely a conventions, some are essential. To reproduce them one has to introduce so-called co-cycle factors in the definition of the root generators, whose commutators produce the correct signs. We will not discuss this further.

The Frenkel-Kac construction yields thus an explicit realization of level-1 simply laced algebras in terms of free bosons. The lattice on which the momenta of these bosons are quantized is the weight lattice of the simply-laced algebra, which is the dual of the root lattice.

11.14. The WZW-model

Realizations of the other theories can be obtained from the so-called Wess-Zumino-Witten models. These are conformal field theories with a two-dimensional action

$$S = k[S_{WZ} + S_W]$$

The first term is due to Wess and Zumino [36], and has the form

$$S_{WZ} = \frac{1}{16\pi} \int d^2z \ \mathrm{Tr}\ \partial_\alpha g(z) \partial^\alpha g(z) \ ,$$

where $g(z)$ is a map from the two-dimensional surface to a group manifold. In other words, for every point z on the manifold, $g(z)$ is some element of the group G under consideration. Here G can be any compact group belonging to a simple Lie algebra.

The second term was added by Witten [37], and has the bizarre form

$$S_W = \frac{1}{24\pi} \int d^3y \epsilon^{\alpha\beta\gamma} \ \mathrm{Tr}\ g^{-1}(y)\partial_\alpha g(y) g^{-1}(y) \partial_\beta g(y) g^{-1}(y) \partial_\gamma g(y)$$

The strange feature is that the integral is over a three-dimensional surface. However the integral is a total derivative, and hence it can be written as a surface integral over the boundary of the three-surface. The boundary of a three-surface is a two-dimensional manifold, for which we take the one used in the first term, with the boundary condition $g(y)|_{y=z} = g(z)$. The extra term is required to make the theory conformally invariant. Upon quantizing the theory one finds that k must be an integer for the integral to be consistent.

The currents that generate the Kac-Moody algebra for this model are $J(z) = \partial g g^{-1}$ and $\bar{J}(\bar{z}) = g^{-1}\bar{\partial} g$.

11.15. Modular transformation properties

Virasoro characters for representations of Kac-Moody algebras can be defined in the usual way. It is however useful to define a more general quantity, namely

$$\mathcal{X}_{\lambda,k}(\tau, \vec{\theta}) = \ \mathrm{Tr}_{\ V_{\lambda,k}} e^{2\pi i \tau (L_0 - c/24)} e^{2\pi i \vec{\theta} \cdot \vec{H}_0} \ .$$

Here the trace is over all states in the representation with highest weight λ and level k. If we put the variables θ^i to zero this reduces to the Virasoro character.

A general formula for the characters and their transformation properties was given by Kac and Peterson [38]. The result is

$$\mathcal{X}_{\lambda,k}(\tau+1, \vec{\theta}) = e^{2\pi i (h_{\lambda,k} - c_k/24)} \mathcal{X}_{\lambda,k}(\tau, \vec{\theta}) \ , \tag{11.8}$$

with h and c as defined earlier, and

$$\mathcal{X}_{\lambda,k}(-\frac{1}{\tau}, \frac{\vec{\theta}}{\tau}) = e^{\frac{-ik\vec{\theta}^2}{4\pi\tau}} \sum_{\lambda'} S^k_{\lambda\lambda'} \mathcal{X}_{\lambda',k}(\tau, \vec{\theta})$$

A very important feature is that different levels do not mix under modular transformations. This could have been expected on the basis of the WZW-model (which has a definite level and can be defined on the torus).

Formulas for the matrix elements of S can be found in the literature. Many important results on Kac-Moody algebras are due to V. Kac, in collaboration with various other authors. These results are collected in a book [39], but this is not easily accessible. The formulas for S can be found for example in [40] or [8].

11.16. Modular invariant partition functions for $SU(2)$

An as yet unsolved problem is that of finding all modular invariant partition functions for all WZW-models. That is, one wants to find all non-negative integer matrices $M_{\lambda,\lambda'}$ that commute with S and T (the latter is implicitly defined in (11.8)) and with $M_{00} = 1$, so that the vacuum is unique.

The only horizontal algebras for which this problem has been solved are $SU(2)$ and $SU(3)$. For $SU(2)$ the solutions are divided into three types called A, D and E:

- A : These are simply the diagonal invariants, which exist at any level, and for any algebra.
- D : They occur at all even levels. If the level is a multiple of 4, they imply and extension of the chiral algebra. For the other even levels they correspond to automorphisms of the fusion rules.
- E : They occur for level 10, 16 and 28.

The notation is chosen because the solutions resemble the classification of the simply-laced Lie-algebras. The resemblance is more precise than suggested here, but so far there is no deep understanding of the mathematical structure (if any) behind this observation.

The A and D invariants are explicitly

$$\sum_{l=0}^{k} |\mathcal{X}_l|^2$$

$$\sum_{m=0}^{k/4-1} |\mathcal{X}_{2m} + \mathcal{X}_{k-2m}|^2 + 2|\mathcal{X}_{k/2}|^2 \quad (k = 0 \bmod 4)$$

$$\sum_{l=0,\text{even}}^{k} |\mathcal{X}_l|^2 + \sum_{l=0,\text{odd}}^{k} \mathcal{X}_l \mathcal{X}^*_{k-l} \quad (k = 2 \bmod 4)$$

This is called the ADE-classification of the $SU(2)$ modular invariants. It was obtained and shown to be complete by Cappelli, Itzykson and Zuber [41].

11.17. Fusion rules and simple currents

The fusion rules can be derived using Verlinde's formula. There is also a more direct approach which is a modified version of the tensor product rules of the horizontal Lie algebra.

Such a tensor product has the general form

$$r_i \otimes r_j = \sum M_{ij}{}^l r_l \ ,$$

where $M_{ij}{}^l$ gives the multiplicity of the representation r_l in the tensor product of r_i and r_j. For example, in $SU(3)$ one has the rule

$$(8) \otimes (8) = (1) + (10) + (\overline{10}) + 2(8) + (27)$$

Here representations are indicated by their dimension, and the bar indicates the complex conjugate. The coefficients $M_{ij}{}^l$ are somewhat reminiscent of the fusion rule coefficients. Indeed, it is true that $N_{ij}{}^l \leq M_{ij}{}^l$, with equality in limit of infinite level (for fixed i, j and k). For example, these are the results for $SU(3)$ at various level, with $[n]$ indicating a Kac-Moody representations whose ground state is the Lie-algebra representation (n):

$$\begin{aligned}
k = 2: &\quad [8] \times [8] = [1] + [8] \\
k = 3: &\quad [8] \times [8] = [1] + [10] + [\overline{10}] + 2[8] \\
k = 4: &\quad [8] \times [8] = [1] + [10] + [\overline{10}] + 2[8] + [27] \ .
\end{aligned}$$

For higher levels the result is as for $k = 4$. For $k = 1$ the ground state [8] does not exist. One method for finding these results starts with the group theory tensor products, to which certain level-dependent projections are applied.

Most Kac-Moody algebras have simple currents. They are the representations whose ground state highest weight is k times a so-called co-minimal fundamental weights. The only exception is E_8 level 2, which has a simple currents even though it has no fundamental weights at all.

For $SU(2)$ the simple current is the representation with $j = k$. For $SU(N)$ they are all N representations with Dynkin labels $(0, \ldots, 0, k, 0, \ldots, 0)$, etc.

11.18. Modular invariant partition functions for other Kac-Moody algebras

No complete classification exists, although it seems plausible that at least for simple horizontal algebras the present list of solutions is close to complete. The majority of the invariants on that list are simple current invariants. For example for $SU(2)$ all D-type invariants are simple current invariants. Only the three exceptional invariants remain mysterious. This is also the pattern one observes for other algebras.

11.19. Coset conformal field theories

A huge class of rational conformal field theories can be obtained with the *coset construction* [42]. Consider a Kac-Moody algebra G and another Kac-Moody algebra H. Suppose the horizontal sub-algebra of H can be embedded in that of G, Then one can associate a conformal field theory with any such pair G and H. For simplicity we will assume that both horizontal algebras are simple.

The embedding implies that one can write the currents of H in terms of those of G:

$$J_H^i(z) = \sum_a M_a^i J_G^a(z)$$

Substituting this into the operator product (11.1) one finds

$$J_H^i(z)J_H^j(w) = \frac{kM_a^i M_b^j \delta^{ab}}{(z-w)^2} + \frac{iM_a^i M_b^j f^{abc}}{z-w} J_G^c(w)$$

The fact that we have an embedding in the horizontal algebra implies that in the last term the identity $M_a^i M_b^j f^{abc} = f^{ijl} M_c^l$ can be used to get $f^{ijl} J^l(w)$, and that in the first term $M_a^i M_b^j \delta^{ab} \propto \delta^{ij}$. However, in general there is a proportionality coefficient, which is called the Dynkin index of the embedding. This index, which we denote $I(G,H)$ is an integer. We find thus the following relation for the level of G and H:

$$k_H = I(G,H) k_G$$

If H is not simple, one simply attaches a label to H to indicate the simple factors; if G is not simple one does the same, and includes on the right-hand side a sum over the G-labels.

The energy-momentum tensor of the coset conformal field theory is $T_G(z) - T_H(z)$, where T_G and T_H are the Sugawara tensors for G and H, each at the appropriate level.

The currents of H are spin-1 conformal fields with respect to T_H; on the other hand, they are linear combinations of currents of T_G, and hence they are also spin-1 conformal fields with respect to T_G. But that implies that the operator product of $T_G(z) - T_H(z)$ with J^i_H is non-singular, since the singularities cancel. Furthermore, since the Sugawara tensor T_H is constructed completely out of the currents of H, it follows that the operator product $(T_G(z) - T_H(z))T_H(w)$ is non-singular, or in other words $T_G(z)T_H(w) = T_H(z)T_H(w)$ up to non-singular terms. The same is true for $T_H(z)T_G(w)$. Hence we get

$$\begin{aligned}(T_G(z) - T_H(z))(T_G(w) - T_H(w)) &= T_G(z)T_G(w) - T_H(z)T_H(w)\\ &= \frac{c_G - c_H}{(z-w)^4} + 2\frac{T_G(w) - T_H(w)}{(z-w)^2} + \frac{\partial_w(T_G(w) - T_H(w))}{(z-w)^2} + \ldots\end{aligned}$$

This tells us that $T_G - T_H$ is a Virasoro generator with central charge $c_G - c_H$ that is "orthogonal" to T_H in the sense that their operator product is non-singular. Hence the original energy-momentum tensor T_G has been decomposed into two orthogonal pieces

$$T_G = T_{G/H} + T_H \ ,$$

with $T_{G/H} = T_G - T_H$.

Given such a decomposition, any representation of G can be decomposed in terms of H representations,

$$V(\lambda_G) = \sum \oplus_{\lambda_H} V_H(\lambda_H) \otimes V_{G/H}(\lambda_G, \lambda_H) \ .$$

Here λ_H labels all representations of the Kac-Moody algebra H, and $V(\lambda)$ denotes a representation space. Each single state in the G representation is a product of some state in an H-representation times a state in a G/H representation. In this way we define the representation spaces for the coset theory. Note that $T_{G/H} = T_G - T_H$ realizes a unitary representation on this space. This follows from unitarity of the modes of T_G and T_H (in the sense that $L_n^\dagger = L_{-n}$) as well as the fact that the norm of states in G representations are equal to products of norms of states in H and G/H representations. Hence the norms of the latter cannot be negative.

Naively, we can explicitly construct the characters of the coset theory by decomposing any G representation systematically into H representations. This corresponds to the following relation

$$\mathcal{X}_{\lambda_G}(\tau) = \sum_{\lambda_H} b^{\lambda_G}_{\lambda_H}(\tau)\mathcal{X}_{\lambda_H}(\tau) \tag{11.9}$$

The functions $b^{\lambda_G}_{\lambda_H}(\tau)$ are called the branching functions of the embedding. They are sometimes confused with the characters of the coset theory, but in general this is not correct. The relation (11.9) does not give sufficient information to compute the branching functions. To compute them one has to take into account not only the dependence on τ of the characters, but use also the representation content with respect to the horizontal algebra.

11.20. The minimal discrete series as a coset theory

An interesting example is the series

$$\frac{SU(2)_1 \times SU(2)_k}{SU(2)_{k+1}}$$

The central charge is

$$c = 1 - \frac{6}{(k+2)(k+3)} ,$$

which corresponds precisely to the central charges of the minimal Virasoro models if we make the identification $m = k + 2$. Since the minimal models are the only unitary theories with these central charges (apart from non-diagonal modular invariants of these theories) the coset theories must form an explicit realization of the minimal models. This is quite useful, because we had not proved that the minimal models are actually unitary, we just had not been able to rule them out.

Let us compute some of the branching functions. The representations of G are labelled by two integers $0 \le l_1 \le 1$ and $0 \le l_2 \le k$, and those of H by one integer $0 \le l_3 \le k+1$. Let us consider $l_1 = l_2 = 0$. The ground state of the G Kac-Moody representation is then the Lie-algebra representation $(0,0)$. It decomposes to (0) of H. The branching function starts thus at $h_G - h_H = 0$. At the next excitation level we encounter the states $(J^a_{-1})_1 \left|0\right\rangle$ and $(J^a_{-1})_2 \left|0\right\rangle$, generated from the vacuum by the currents of $SU(2) \times SU(2)$. There are six states, and they transform in the representation $(3) + (3)$ of H.

In the vacuum representation of H we will have also a set of states $J^a_{-1} \left|0\right\rangle$, in the (3) of $SU(2)$. This removes one of the (3)'s we found. The other is not a singlet, and hence can not contribute to the branching function $b_0^{0,0}$. It must thus be interpreted as the first term in a new branching function $b_2^{0,0}$, where "2" denotes the representation (3) (in general the dimension is $(l+1)$, since l denotes twice the usual $SU(2)$ spin). The leading conformal weight in that branching function is

$$0 + 0 + 1 - \frac{\frac{1}{2}l_3(\frac{1}{2}l_3 + 1)}{(k+1)+2} = \tfrac{1}{2} ,$$

where the first two terms are the ground state weight in G, the third is the excitation level, and the last is the contribution from the term $-T_H$, with $l_3 = 2$ and $k = 1$. This branching function is seen to correspond to the $h = \frac{1}{2}$ representation of the Ising model. There is no contribution at the first excited level to $b_0^{0,0}$. This agrees with the fact that $L_{-1} \left|0\right\rangle = 0$ on the ground state.

11.21. Field Identification

The complications with interpreting the branching functions as characters start becoming clear as soon as one observes that for example the branching function $b_1^{0,0}$ is identically zero, since the G-representation contains only integer spin representations of $SU(2)$. Closely related, but less obvious, is the fact that several branching functions are in fact identical. Something like this had to happen, since the total number of branching functions one gets for the coset $SU(2)_1 \otimes SU(2)_1/SU(2)_2$ is $2 \times 2 \times 3 = 12$. This exceeds the number of Ising model representations by a factor of 4.

The solution is that only the following branching functions are non-vanishing, and that they are identical in pairs:

$$\begin{aligned} b_0^{0,0} &= b_2^{1,1} \qquad & h &= 0 \\ b_1^{1,0} &= b_1^{0,1} \qquad & h &= \frac{1}{16} \\ b_2^{0,0} &= b_0^{1,1} \qquad & h &= \frac{1}{2} \end{aligned}$$

This phenomenon is called *field identification* [43,44].

In this case it is still true that the branching functions are equal to the characters. However, in other cases it happens that the number of fields that is identified is not always the same. In that case there are non-trivial problems [45]. The solution is beyond the scope of these lectures, and partly beyond the scope of what is presently known. However, it is certainly true that in these cases the characters are not simply equal to the branching functions. This problem occurs frequently, for example in the cosets $SU(2)_k \otimes SU(2)_l/SU(2)_{k+l}$ whenever k and l are both even. For a more detailed discussion of this problem see [22] and [46].

11.22. Other coset models

It should be clear that the set of coset models is huge. Most of them have a central charge larger than 1, and are example of rational conformal field theories with an extended algebra. For example, it was shown that the series

$$\frac{SU(3)_1 \times SU(3)_k}{SU(3)_{k+1}}$$

has a chiral algebra with currents of spin 3, and forms the minimal series of the W_3 algebra (which will not be discussed here further).

The number of coset models is so large that it has even been suggested that in combination with free bosonic theories and orbifolds and perhaps some other ideas it exhausts the set of rational conformal field theories. Unfortunately this "conjecture" has never been made sufficiently precise to disprove it. Any claims that rational conformal field theories have in some – usually vague – sense been classified should be regarded with a great amount of suspicion. In fact it is fairly clear that even rational conformal field theories with a single primary field are essentially unclassifiable.

12. Superconformal algebras

There is still another important class of extensions of the chiral algebra, namely by currents of spin $\frac{3}{2}$. Since these are half-integer spin currents, many of the remarks we made in the section on fermionic currents are valid here as well. In particular there are two sectors, Neveu-Schwarz and Ramond, and there may be square root branch cuts in operator products.

The name "superconformal" refers to the fact that a spin-$\frac{3}{2}$ current can be put in a supermultiplet together with the energy-momentum tensor. The currents of this algebra generate the so-called superconformal transformations, a supersymmetric generalization of conformal transformations. Indeed, one can describe the entire algebra in a manifestly supersymmetric way, but we will write it in terms of components.

12.1. The $N = 1$ algebra

The simplest superconformal algebra is generated by a single spin-$\frac{3}{2}$ current $T_F(z)$ in addition to the Virasoro generator. This is the $N = 1$ superconformal algebra. The complete set of operator products is

$$
\begin{aligned}
T(z)T(w) &= \frac{\frac{1}{2}c}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial T(w) \\
T(z)T_F(w) &= \frac{\frac{3}{2}}{(z-w)^2}T_F(w) + \frac{1}{z-w}\partial T_F(w) \\
T_F(z)T_F(w) &= \frac{\frac{1}{6}c}{(z-w)^3} + \frac{\frac{1}{2}}{z-w}\partial T(w)
\end{aligned}
$$

The first two operator products simply state that $T(z)$ is the energy-momentum tensor and $T_F(z)$ a spin $\frac{3}{2}$ conformal field.

Modes are defined as in section 7. The modes of the supercurrent are traditionally called G_n. The algebra in terms of modes looks like this

$$
\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \\
[L_m, G_r] &= (\tfrac{1}{2}m - r)G_{m+r} \\
\{G_r, G_s\} &= 2L_{r+s} + \tfrac{1}{3}c(r^2 - \tfrac{1}{4})\delta_{r+s,0}
\end{aligned}
$$

In the last term we find an anti-commutator because the left-hand side of the corresponding operator product is odd under the exchange $z \leftrightarrow w$. This is also exactly like the free fermion.

The fermionic currents G can be half-integer-moded (Neveu-Schwarz) or integer moded (Ramond). To emphasize this we have used indices r and s for this current. A new feature, in comparison with the free fermion, is that it is now possible that a Ramond ground state is annihilated by G_0 (because of the anti-commutator $\{b_0, b_0\} = 1$ this is impossible for the free fermion). Because of the last relation, this implies immediately that $h = \frac{c}{24}$ for such a state. These states are often called *chiral states*. Furthermore any state which is not annihilated by G_0 must have $h > \frac{c}{24}$. Note that the latter ground states necessarily come in pairs of opposite fermion number, related by G_0, whereas the ones annihilated by G_0 are unpaired. This also implies that in superconformal theories the trace in the Ramond sector with $(-1)^F$ projection may be non-zero, unlike the free fermion case. In fact this trace clearly receives contributions only from the chiral states with $h - c/24 = 0$, so that the corresponding terms in the partition function are constants. This implies in particular that this contribution to the partition function (which corresponds to the PP-sector) is modular invariant by itself.

In the Neveu-Schwarz sector one should note the relation

$$\{G_r, G_{-r}\} = 2L_0 + \tfrac{1}{3}c(r^2 - \tfrac{1}{4})$$

Since $r^2 \geq \frac{1}{4}$ the left-hand side is positive or zero, with the latter value occurring only for $r = \frac{1}{2}$ and $h = 0$. If the left-hand side is positive we have

$$|G_{-r}\,|x\rangle\,|^2 > 0$$

for ground states. Hence the excitations have positive norm. There is a unique ground state with the property $G_{-1/2}\,|x\rangle = 0$, namely the vacuum (note that ground states in any case satisfy $G_r\,|x\rangle = 0, r \geq \frac{1}{2}$).

The unitary representations of this algebra form a discrete series for $0 \leq c < 3/2$, whereas for larger values of c there are infinitely many representations, just as for the Virasoro algebra. The c-values for this series are

$$c = \tfrac{3}{2}\left[1 - \frac{8}{m(m+2)}\right], \quad m = 3, 4, \ldots$$

The $m = 3$ value is $c = 7/10$, and coincides with a member of the minimal Virasoro series. Obviously superconformal representations are in particular representations of the Virasoro algebra. The second member is on the $c = 1$ boundary of the Virasoro representations.

A concrete realization of this series is given by the coset models

$$\frac{SU(2)_2 \times SU(2)_k}{SU(2)_{k+2}}\,.$$

12.2. The $N = 2$ algebra

There are also superconformal models with extended supersymmetry [47]. The case of most interest is $N = 2$, since it occurs in supersymmetric string theories. In these theories there are two supercurrents. Note that just having two supercurrents is not yet enough, since a tensor product of two $N = 1$ models would also have that property, and one would not expect it to have extended supersymmetry. To get an $N = 2$ algebra the currents need to satisfy a set of operator products. Furthermore it turns out that the algebra must contain one additional current J of spin 1. This current generates a $U(1)$ algebra.

The full algebra is, in terms of modes:

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \\
\left[L_m, G_r^{\pm}\right] &= (\tfrac{1}{2}m - r)G_{m+r}^{\pm} \\
\left\{G_r^-, G_s^+\right\} &= 2L_{r+s} - (r-s)J_{r+s} + \tfrac{1}{3}c(r^2 - \tfrac{1}{4})\delta_{r+s,0} \\
[L_m, J_n] &= -nJ_{m+n} \\
[J_m, J_n] &= \tfrac{1}{3}cm\delta_{m+n,0} \\
\left[J_m, G_r^{\pm}\right] &= \pm G_{m+r}^{\pm}
\end{aligned}$$

This algebra also has a discrete series, with central charges

$$c = 3(1 - \frac{2}{m}) \,, \quad m = 3, 4, \ldots$$

The first member of this series has $c = 1$. It is also in the $N = 1$ series, and can be realized as a circle compactification of a single free boson (with $R^2 = 2N = 12$). The central charges turn out to be identical to those of the $SU(2)$ Kac-Moody algebras, if one substitutes $m = k + 2$. This is related to the fact that the minimal series can be obtained from the following cosets

$$\frac{SU(2) \times SO(2)}{U(1)} \,.$$

Ground states are characterized by a conformal weight h and a $U(1)$ charge q. In addition one can have chiral states both in the Neveu-Schwarz and in the Ramond sector. In the Neveu-Schwarz they have the special property

$$G^+_{-1/2}\,|\phi\rangle = 0 \qquad \text{or} \qquad G^-_{-1/2}\,|\phi\rangle = 0$$

and are called respectively chiral or anti-chiral states. Primary states satisfy the condition $G_r^+\,|\phi\rangle = G_r^-\,|\phi\rangle = 0$ for $r > 0$. Chiral primary (or anti-chiral primary) states satisfy the corresponding combination of these conditions.

Using the algebra (as for $N = 1$ above) it is easy to deduce that for chiral primaries $h = \frac{1}{2}q$, and for anti-chiral primaries $h = -\frac{1}{2}q$. The only state that is both chiral and anti-chiral primary is thus the vacuum. Furthermore it can be shown that any other state in the theory has $h > \frac{1}{2}|q|$, and that the conformal weights of chiral primaries satisfies $h \leq c/6$.

An interesting consequence of the relation between charges and conformal weights is that within the set of chiral primary states conformal weights are "conserved" in operator products just like charges. Consider the operator product of two chiral primary fields ϕ_1 and ϕ_2 (ignoring anti-holomorphic components). Then

$$\phi_1(z)\phi_2(w) = (z-w)^{h_3-h_1-h_2}\phi_3(w) + \text{less singular terms.}$$

The charge of ϕ_3 is q_1+q_2, and therefore $h_3 \geq \frac{1}{2}(q_1+q_2) = h_1+h_2$ (note that chiral primaries have positive charges). Hence the operator product is non-singular. Therefore we can define

$$\phi_{1\times 2}(z) = \lim_{w\to z} \phi_1(z)\phi_2(w) \ .$$

This limit is zero if ϕ_3 is not a chiral primary state, and is equal to ϕ_3 if it is a chiral primary. Hence this defines a closed operation on the chiral primary states. This is called the chiral ring [45]. There is of course also an anti-chiral ring.

In the Ramond sector one defines chiral states as those which are annihilated by both G_0^+ and G_0^-. From the anti-commutator of these two operators one learns that those are precisely the states with $h - c/24 = 0$.

An important property of $N = 2$ algebras is *spectral flow*. This means that there exists an operator $\mathcal{U}_\theta$ that maps the entire algebra to an isomorphic one. It acts on the generators by conjugation, and the mapping has the following effect

$$\begin{aligned}
\mathcal{U}_\theta L_n \mathcal{U}_\theta^{-1} &= L_n + \theta J_n + \frac{c}{6}\theta^2\delta_{n,0} \\
\mathcal{U}_\theta J_n \mathcal{U}_\theta^{-1} &= J_n + \frac{c}{3}\theta\delta_{n,0} \\
\mathcal{U}_\theta G_r^{\pm} \mathcal{U}_\theta^{-1} &= G_{r\pm\theta}^{\pm}
\end{aligned}$$

The interesting feature of this map is that it changes the mode of the supercurrent. Closer inspection shows that for $\theta = \frac{1}{2}$ it maps the Neveu-Schwarz moded algebra to the Ramond moded algebra. It is not difficult to show that chiral primary states are mapped to the chiral Ramond grounds states, while the latter are mapped to the anti-chiral states by the same map. This shows in particular that there is a one-to-one correspondence between chiral Ramond ground states and (anti)-chiral states in the Neveu-Schwarz sector. In string theory this is related to space-time supersymmetry, as the Neveu-Schwarz sector yields space-time bosons and the Ramond sector space-time fermions; one of the conditions for having space-time supersymmetry is $N = 2$ supersymmetry in two dimensions.

REFERENCES

[1] P. Ginsparg, *Applied Conformal Field Theory*, in the Proceedings of the Les Houches school on Fields, Strings and Critical Phenomena, 1988.

[2] J. Cardy, *Conformal Invariance and Statistical Mechanics*, in the Proceedings of the Les Houches school on Fields, Strings and Critical Phenomena, 1988.

[3] W. Lerche, A. Schellekens and N. Warner, Physics Reports 177 (1989) 1.

[4] A. N. Schellekens, introduction to *Superstring Construction*, Current Physics Sources and Comments Vol. 4, North Holland.

[5] M. Green, J. Schwarz and E. Witten, Superstring Theory, Cambridge Univ. Press, (1986).

[6] D. Lüst and S. Theisen, Lectures on String Theory. Springer (1989) (Lecture notes in physics, 346).

[7] P. Goddard and D. Olive, Int. J. of Mod. Phys. *A*1 (1986) 303.

[8] J. Fuchs, *Affine Lie algebras and Quantum Groups*, Cambridge University Press (1992).

[9] A. Belavin, A. Polyakov and A. Zamolodchikov, Nucl. Phys. *B*241 (1984) 333.

[10] M. Virasoro, Phys. Rev. *D*1 (1970) 2933.

[11] B. Feigin and D. Fuchs, Functs. Anal. Pril. 16 (1982) 47.

[12] C. Thorn, Nucl. Phys. *B*248 (1984) 551.

[13] D. Friedan, Z. Qiu and S. Shenker, Phys. Rev. Lett. 52 (1984) 1575; Commun. Math. Phys. 107 (1986) 535.

[14] J. F. Gomes, Phys. Lett. *B*171 (1986) 75.

[15] A.M. Polyakov, Phys. Lett. 103B (1981) 207.

[16] E. Verlinde, Nucl. Phys. *B*300 (1988) 360.

[17] G. Moore and N. Seiberg, Phys. Lett. *B*212 (1988) 451; Nucl. Phys. *B*313 (1988) 16.

[18] G. Anderson and G. Moore, Commun. Math. Phys. 117 (1988) 441.

[19] C. Vafa, Phys. Lett. 206*B* (1988) 421.

[20] A. B. Zamolodchikov and V. A. Fateev, *Sov. Phys. JETP* 62 (1985) 215.

[21] P. Ramond, Phys. Rev. *D*3 (1971) 2415; A. Neveu and J. H. Schwarz Nucl. Phys. *B*31 (1971).

[22] A.N. Schellekens and S. Yankielowicz, Int. J. Mod. Phys. *A*5 (1990) 2903.

[23] F. Gliozzi, J. Scherk and D. Olive, Nucl. Phys. *B*122 (1977) 253.

[24] H. Kawai, D. Lewellen and S. Tye, Phys. Rev. Lett. 57 (1986) 1832; Nucl. Phys. *B*288 (1987) 1.

[25] I. Antoniadis, C. Bachas and C. Kounnas, Nucl. Phys. *B*289 (1987) 87.

[26] K. Narain, Phys. Lett. 169*B* (1986) 41.

[27] L. Dixon, J. Harvey, C. Vafa and E. Witten, Nucl. Phys. $\underline{B261}$ (1985) 678; Nucl. Phys. $\underline{B274}$ (1986) 285.

[28] L. Dixon, D. Friedan, E. Martinec and S. Shenker, Nucl. Phys. $\underline{B282}$ (1987) 13.

[29] S. Hamidi and C. Vafa, Nucl. Phys. $\underline{B279}$ (1987) 465.

[30] R. Dijkgraaf, E. Verlinde and H. Verlinde, Commun. Math. Phys. $\underline{115}$ (1988) 649.

[31] K. Bardakci, E. Rabinovici and B. Säring, Nucl. Phys. $\underline{B299}$ (1988) 151.

[32] P. Ginsparg, Nucl. Phys. $\underline{B295}$ (1988) 153.

[33] A. Zamolodchikov, Theor. Math. Phys. $\underline{65}$ (1986) 1205.

[34] H. Sugawara, Phys. Rev. $\underline{D170}$ (1968) 1659.

[35] I.B. Frenkel and V.G. Kac, Inv. Math. $\underline{62}$ (1980) 23; G. Segal, Commun. Math. Phys. $\underline{80}$ (1981) 301.

[36] J. Wess and B. Zumino, Phys. Lett. $\underline{37}$ (1971) 95.

[37] E. Witten, Commun. Math. Phys. $\underline{92}$ (1984) 455.

[38] V.G. Kac and D.H. Peterson, In *Anomalies, Geometry, Topology*, Argonne 1985, W. A. Bardeen and A. R. White eds.

[39] V.G. Kac, *Infinite Dimensional Lie Algebras*, second edition, Cambridge University Press (1985).

[40] D. Gepner and E. Witten, Nucl. Phys. $\underline{B278}$ (1986) 493.

[41] A. Cappelli, C. Itzykson and J.-B. Zuber, Nucl. Phys. $\underline{B280}$ (1987) 445; Commun. Math. Phys. $\underline{113}$ (1987) 1.

[42] P. Goddard, A. Kent and D. Olive, Phys. Lett. $\underline{152B}$ (1985) 88; Commun. Math. Phys. $\underline{103}$ (1986) 105.

[43] G. Moore and N. Seiberg, Phys. Lett. $\underline{B220}$ (1989) 422.

[44] D. Gepner, Phys. Lett. $\underline{B222}$ (1989) 207.

[45] W. Lerche, C. Vafa and N. Warner, Nucl. Phys. $\underline{B324}$ (1989) 427.

[46] J. Fuchs, A.N. Schellekens and C. Schweigert, *The resolution of field identification fixed points in diagonal Coset Theories* NIKHEF-95/52.

[47] G. Waterson, Phys. Lett. $\underline{B171}$ (1986) 77; P. Di Vecchia, J.L. Petersen and M. Yu, Phys. Lett. $\underline{B172}$ (1986) 211; W. Boucher, D. Friedan and A. Kent, Phys. Lett. $\underline{B172}$ (1986) 316; S. Nam, Phys. Lett. $\underline{B172}$ (1986) 323; P. Di Vecchia, J.L. Petersen, M. Yu and H. B. Zheng, Phys. Lett. $\underline{B174}$ (1986) 280; A. Schwimmer and N. Seiberg, Phys. Lett. $\underline{B184}$ (1987) 191.