Caputcollege string theory 2005

First lecture

1.1 Introduction

String theory is a theory in which the elementary particles are one dimensional objects (strings) instead of (zero dimensional) point particles. The theory originated in the 1960's, but the real interest in string theory began in the 1980's. Even though string theory is not at all a complete theory yet, there are quite a few physicists who believe that string theory will form the basis of some ultimate Theory of Everything. Whether this is really the case or not is a question that we cannot answer yet; what is clear is that the theory has many very interesting new aspects. In these lectures, we will try to present as many of these aspects as possible.

Before we introduce the mathematics behind the theory, we want to scetch the reader an intuitive picture that shows why string theory is so interesting. One of the most important reasons for this is that string theory contains much less (and much weaker) infinities than quantum field theory does. In quantum field theory, we can calculate physical quantities by calculating expressions corresponding to Feynman diagrams. An example of such a diagram is drawn in figure 1.1a. The readers who have never seen a Feynman diagram before can think of a motion picture of an interaction, drawn in Minkowski space. This diagram (in which the vertical direction is time-like) shows two particles that join and form another particle, which at a later time falls apart and forms two new particles. The diagram contains two points of interaction that both correspond to a delta function in the calculations. In this calculation, we have to integrate over the positions of the points of interaction, and hence we will encounter a product of two delta functions that have the same argument. These products lead to inacceptable infinities in the theory.

In some cases, these infinities can be removed by using a renormalization procedure, but often this is not possible. A very important example of a theory in which this does not work is the theory of quantum gravity. This means we have a major problem: it seems to be impossible to unite quantum mechanics and gravity!

Now take a look at diagram 1.1b. This is the same interaction diagram as figure 1.1a, but with little loops (strings) replacing the point particles. The important thing about this diagram is that it is not possible to bring the points of interaction together without changing the topological properties of the diagram! In fact, it is impossible to define a Lorentz invariant point of interaction: to every observer, the strings seem to join and split at a different point in space-time. The interaction does not take place at a specific point, but in a certain area.

The consequence of all this is that string theory contains a lot less divergencies than ordinary quantum field theory. We want to stress that this is just an intuitive argument: if we regard

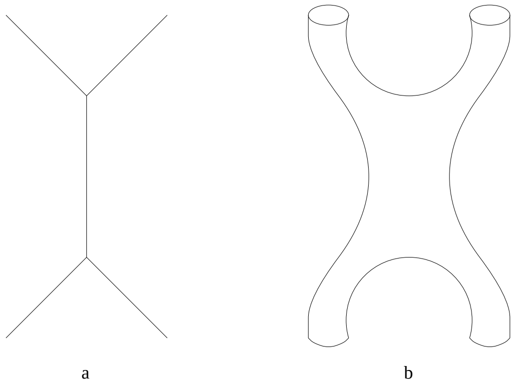

Figure 1.1: A Feynman diagram in quantum field theory and in string theory.

diagram 1.1b as a diagram in Minkowski space, this means that certain points of the string reach infinite velocities! However, the outcome of the argument turns out to be correct: string theory conains far less infinities than quantum field theory. It even turns out to be possible (and actually inevitable) to include gravity in string theory!

Of course, all this does not mean that we have found a unifying theory. Ironically, untill now gravity is the only force of nature that we have been able to describe using string theory. It is not at all impossible that the other forces can be described as well, but before we can do this, many problems will have to be overcome.

To summarize: string theory is a theory that has many possibilities, but also many problems. In these lectures, we will try to show both sides of the theory.

To conclude this introduction, we present a list of the literature that is available on the subject.

- The standard work on string theory is 'Superstring Theory' by Green, Schwarz and Witten (Cambridge 1987). This two-volume book was written in 1987 and therefore does not contain all the recent developments. However, it is a very good and thorough introduction to the subject.

- The readers who feel somewhat uncertain about their knowledge of quantum field theory should consult 'Quantum Field Theory of Point Particles and Strings' by Hatfield (Addison-Wesley 1992). The main part of this book discusses quantum field theory; the final chapters introduce the reader to (bosonic) string theory.

- A book that had been eagerly awaited was published this summer: 'String Theory' by Polchinski. This book also contains more recent developments, and may therefore be the successor of the book by Green, Schwarz and Witten.

- Some other books that are often used are 'Introduction to superstrings' by Kaku (Springer 1988) (especially the first few chapters are a good introduction) and 'Lectures on String Theory' by Lüst and Theisen (Springer 1989).

- The electronic pre-print archive hep-th (http://xxx.lanl.gov/archive/hep-th) contains many good articles on string theory. We present two titles: 'What is String Theory?' by Polchinski (hep-th/9411028) and 'Lectures on Strings and Dualities' by Vafa (hep-th/9702201).

The pre-prints in this archive (including the reviews) are usually of a higher level than the introducory books, but it may be useful to look at some of them as extra reading material.

1.2 The relativistic point particle

For a good understanding of string theory, it is useful to study the theory of the classical (i.e. non-quantized) relativistic point particle first. As we know, such a particle moves along a geodesic, meaning that the relativistic length of its world line is minimal. It is therefore natural to choose the length of this world line as an action for the relativistic point particle. Parametrizing this world line by $x^\mu(\tau)$, we find the following action:

$$S = -m \int d\tau \sqrt{|\dot{x}^\mu \dot{x}_\mu|}. \tag{1.1}$$

We choose units in which $\hbar = c = 1$, so the arbitrary constant m has the dimension of $[\text{length}]^{-1} = [\text{mass}]$. Moreover, we work in the Minkowski metric $\eta^{\mu\nu} = \text{diag}(-1, 1, \ldots, 1)$. This means a real particle has negative length, which explains the minus sign in (1.1). Finally, the number of dimensions of the theory is arbitrary ($\mu, \nu = 0, 1, \ldots, D$).

Note that there is an important difference between this action and the types of action we usually encounter in classical mechanics and field theory. The parameters that we integrate the Lagrangian over are usually space-time coordinates, such as t, x, y and z. In our case, the integration is over the arbitrary parameter τ, and the coordinates in space-time are the fields.

Exercise 1.1 *Show that the action is invariant under reparametrizations of the world line by substituting $\tau' = f(\tau)$.*

The square root in equation (1.1) makes this action very hard to work with. However, this problem can be solved by introducing an auxiliary field $e(\tau)$. We now consider the action

$$S = \frac{1}{2} \int d\tau \left\{ \frac{\dot{x}^2}{e} - em^2 \right\}, \tag{1.2}$$

Exercise 1.2 *Write down the Euler-Lagrange equation for the field e, and show that the action (1.1) is equivalent to the action (1.2).*

We would like to preserve the property of reparametrization-invariance that the original action had. We can do this because we have not decided how e transforms under reparametrizations yet. From the action, we see that the substitution $\tau \to \tilde{\tau} = f(\tau)$ implies that the e-field should transform as

$$\tilde{e}(\tilde{\tau}) = \left(\frac{df}{d\tau}(\tau) \right)^{-1} e(\tau). \tag{1.3}$$

Exercise 1.3 *Show that in the infinitesimal case ($\tau \to \tilde{\tau} = \tau - \xi(\tau)$, with ξ small) this leads to the following transformations:*

$$\begin{aligned} \delta x &= \xi \dot{x} \\ \delta e &= \frac{d(\xi e)}{d\tau} \end{aligned} \tag{1.4}$$

We can now use reparametrization-invariance to put the field e into any form we like. Of course, we choose a simple form: e=1. This leads to the action

$$S = \frac{1}{2}\int d\tau \left\{\dot{x}^2 - m^2\right\}. \tag{1.5}$$

The equations of motion following from this action are

$$\ddot{x}^\mu = 0. \tag{1.6}$$

Note that the particle moves along a straight line – i.e. along a geodesic in Minkowski space. Of course, we should not forget about the equation of motion for e. When we substitute $e = 1$ into this equation, we find the condition

$$\dot{x}^2 + m^2 = 0. \tag{1.7}$$

Since $p^\mu = \dot{x}^\mu$ is the momentum conjugated to x^μ, this is the mass-shell equation $p^2 + m^2 = 0$. This means we can interpret m as the mass of the particle.

1.3 The classical bosonic string

Instead of looking at a point particle, we will now describe a one dimensional object propagating through space-time: a string. We need two variables to parametrize the string: a time-like parameter, analogous to the parameter τ in the previous section, and a variable that parametrizes the length of the string. We will call these parameters τ and σ or σ^0 and σ^1. Instead of a world line, we now speak of a world sheet.

The most natural choice is now to take the (relativistic) area of the world sheet as the action for the string. This leads to the action

$$S = -\frac{1}{2\pi\alpha'}\int d\sigma d\tau\,\sqrt{|\det \partial_\alpha X^\mu \partial_\beta X_\mu|}. \tag{1.8}$$

In this equation, $X^\mu(\tau,\sigma)$ are the coordinates of the world sheet points, and α and β label the world sheet parameters. The reader should be aware of the difference between world sheet indices ($\alpha,\beta = 0,1$) and space-time indices ($\mu,\nu = 0,1,\ldots,D$). The notation ∂_α is shorthand for $\partial/\partial\sigma^\alpha$. The determinant in the formula is also over the world sheet indices. The constant T has dimension $[\text{mass}]^2$, and will later on be interpreted as the string tension or energy per unit length. Like in the point particle case, this action contains a square root, so we would like to find another action. Again, we introduce an auxiliary field – or more precisely a matrix $h^{\alpha\beta}(\tau,\sigma)$ of auxiliary fields:

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\,\sqrt{h}\,h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu, \tag{1.9}$$

where h is the determinant of the matrix $h_{\alpha\beta}$ (note the lower indices!). It is possible to derive this action from the embedding of a general (curved) surface with metric $h^{\alpha\beta}$ into the $(D+1)$-dimensional space-time. We will not do this here, but we will call $h^{\alpha\beta}$ the world sheet metric. In the next exercise, we will prove the equivalence of the action (1.9) to the action (1.8).

Like in the point particle case, we now want to find the equations of motion corresponding to the auxiliary field $h^{\alpha\beta}$.

Exercise 1.4 *Show that these equations of motion are the following:*

$$T_{\alpha\beta} \equiv \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2} h_{\alpha\beta} h^{\gamma\delta} \partial_\gamma X^\mu \partial_\delta X_\mu = 0. \tag{1.10}$$

Use these equations to prove the equivalence of (1.9) and (1.8).

Again, the action (1.9) is invariant under reparametrizations. Since we now have a two dimensional world sheet, we also have two degrees of freedom: ξ^0 and ξ^1. If we reparametrize the X-fields, $\delta X^\mu = \xi^\alpha \partial_\alpha X^\mu$, the h-fields have to transform in the following way:

$$\delta h^{\alpha\beta} = \xi^\gamma \partial_\gamma h^{\alpha\beta} - \partial_\gamma \xi^\alpha h^{\gamma\beta} - \partial_\gamma \xi^\beta h^{\alpha\gamma}. \tag{1.11}$$

Those who like a challenge can try to derive these transformations. Unlike the point particle case, there is now another, third invariance of the theory. We can rescale the world sheet metric by an arbitrary function:

$$\delta h^{\alpha\beta} = \zeta(\tau, \sigma) h^{\alpha\beta}. \tag{1.12}$$

This rescaling is called the *Weyl-rescaling.*

Exercise 1.5 *Show that the Weyl-transformation is a symmetry of the action by checking how the different factors transform under such a scaling.*

Since a two dimensional metric has three degrees of freedom (remember a metric is a symmetric tensor), we can use the three degrees of freedom we found to put $h^{\alpha\beta}$ in any form we like. The most natural choice is the Minkowski-metric:

$$h^{\alpha\beta} = \eta^{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.13}$$

This simplifies the string action considerably:

$$S = -\frac{1}{2\pi\alpha'} \int d^2\sigma \left\{ X'^2 - \dot{X}^2 \right\}, \tag{1.14}$$

where a prime means differentiation to σ and a dot differentiation to τ.

Exercise 1.6 *Use the argument above to explain why it is mathematically much easier to describe strings than higher dimensional objects.*

The gauge we picked also simplifies the equations of motion corresponding to $h^{\alpha\beta}$:

$$\begin{aligned} \dot{X} \cdot X' &= 0 \\ \dot{X}^2 + X'^2 &= 0. \end{aligned} \tag{1.15}$$

The equations of motion corresponding to X^μ are now also easy to derive:

$$\partial^\alpha \partial_\alpha X^\mu = X'' - \ddot{X} = 0. \tag{1.16}$$

Open strings

Until now, we treated the world sheet is a very general way. However, if we want to solve the equations of motion, we have to make some choices. One option is to discuss open strings, i.e. strings with two endpoints. This means the parameter σ runs from (for example) 0 to π.

Exercise 1.7 *Show that the open strings have to satisfy the boundary conditions*

$$X'^{\mu}(\tau,0) = X'^{\mu}(\tau,\pi) = 0. \tag{1.17}$$

Hint: derive the equations of motion from a variational principle.

As we know from classical mechanics, these boundary conditions lead to standing waves. Besides, the string as a whole can move in an inertial way. This means that the general solutions to the equations of motion for the open string are

$$X^{\mu} = x^{\mu} + (2\alpha')p^{\mu}\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^{\mu}e^{-in\tau}\cos n\sigma, \tag{1.18}$$

The various constants have been chosen to have canonical commutation relations when we quantize the theory; the real degrees of freedom are x^{μ}, p^{μ} and α_n^{μ}. Since we want the X-coordinates to be real, we have to pick the α-coordinates in such a way that

$$\alpha_{-n}^{\mu} = (\alpha_n^{\mu})^{*}. \tag{1.19}$$

Closed strings

A different choice is to consider closed strings: strings that do not have endpoints, but are topologically equivalent to a circle. We parametrize these strings by a parameter σ running from 0 to 2π. Note the difference by a factor of two with respect to the open strings! The reason for this choice will become clear later. We can make the strings closed by demanding that

$$X^{\mu}(\tau,0) = X^{\mu}(\tau,2\pi). \tag{1.20}$$

As is well-known, arbitrary solutions to the wave equation (1.16) are of the form

$$X^{\mu}(\tau,\sigma) = X_R^{\mu}(\tau-\sigma) + X_L^{\mu}(\tau+\sigma). \tag{1.21}$$

In other words: the solutions can be written as a right-moving part plus a left-moving part. To satisfy the condition (1.20), we split these two parts into Fourier components, and add a term for the center-of-mass motion:

$$\begin{aligned} X_R^{\mu} &= \frac{1}{2}x^{\mu} + \frac{\alpha'}{2}p^{\mu}(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^{\mu}e^{-in(\tau-\sigma)}, \\ X_L^{\mu} &= \frac{1}{2}x^{\mu} + \frac{\alpha'}{2}p^{\mu}(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^{\mu}e^{-in(\tau+\sigma)}, \end{aligned} \tag{1.22}$$

The main difference with the open strings is that now we have two sets of oscillator coordinates: α_n^{μ} and $\tilde{\alpha}_n^{\mu}$. Both sets should again satisfy the reality condition (1.19).

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Second lecture

2.1 Quantizing the point particle

Now we want to quantize the theory that was introduced in the first chapter. Once again, it is instructive to study the point particle before proceeding to the string case. First of all, we have to make a choice: besides quantising the point particle, we should also still impose the constraint equation (1.7). We can either impose this constraint on the classical solutions of the theory and then quantize, or quantize the unconstrained theory and then impose the constraint.

We will choose the second option; the so-called *covariant quantization*. The reason for this name is that if we would impose the constraint at this stage, we would lose manifest Lorentz covariance. Of course, we will have to consider the constraint later, but this is no concern to us yet.

To quantize, we turn the degrees of freedom of the classical solutions of the theory into operators. Since the particles move along straight lines in Minkowski space (cf. (1.6)), the classical solutions of the theory are simply

$$x^\mu(\tau) = x_0^\mu + p^\mu \tau, \tag{2.1}$$

where $p^\mu = \dot{x}^\mu$ is the conjugate momentum introduced in section 1.2.

Quantizing the theory means turning x_0^μ and p^μ into operators. We know from quantum field theory that a field and its conjugate momentum should satisfy the equal-time (or in this case equal-τ) commutation relation

$$[x^\mu(\tau), p^\nu(\tau)] = i\eta^{\mu\nu}. \tag{2.2}$$

Inserting (2.1), we see that this implies

$$[x_0^\mu, p^\nu] = i\eta^{\mu\nu}. \tag{2.3}$$

As is well-known from quantum mechanics, we can now introduce a Fock space in which x_0^μ and p^μ are operators. We can work in coordinate or momentum representation. Choosing the last option, our Fock space consists of states $|k\rangle$ (with k a 4-vector) that satisfy

$$p^\mu|k\rangle = k^\mu|k\rangle. \tag{2.4}$$

The constraint equation now turns into an operator equation: in the end, we will only keep states for which

$$p^2|k\rangle = k^2|k\rangle = -m^2|k\rangle. \tag{2.5}$$

2.2 Quantizing the bosonic string

Open strings

We will now quantize the bosonic string, starting with the open version of the theory. Apart from the center-of-mass degrees of freedom, the solutions (1.18) also contain the oscillator degrees of freedom α_n^μ, which will now become operators. Like in the point particle case, we will quantize covariantly, meaning that we will ignore the constraints (1.15) for the moment.

From the string action, we see that the momentum conjugate to X^μ is $P^\mu = \frac{1}{2\pi\alpha'}\dot{X}^\mu$. Now we impose the usual commutation relations:

$$\begin{aligned} [X^\mu(\tau,\sigma'), P^\nu(\tau,\sigma)] &= i\delta(\sigma-\sigma')\eta^{\mu\nu} \\ [P^\mu(\tau,\sigma), P^\nu(\tau,\sigma')] &= 0 \\ [X^\mu(\tau,\sigma), X^\nu(\tau,\sigma')] &= 0. \end{aligned} \tag{2.6}$$

Exercise 2.8 *Show that these commutation relations imply the commutation relations*

$$\begin{aligned} [x^\mu, p^\nu] &= i\eta^{\mu\nu} \\ [\alpha_m^\mu, \alpha_n^\nu] &= m\delta_{m+n}\eta^{\mu\nu} \end{aligned} \tag{2.7}$$

for the open string degrees of freedom.

The first of these relations is similar to the one we encountered for the relativistic point particle. The second relation will also look familar to the reader: except for the factor of m (which can be absorbed into the operators if we want to), these are the raising and lowering operators encountered in quantizing the harmonic oscillator. To see this, recall the equation (1.19).

This immediately tells us what to do to quantize the theory: introduce a ground state, and act on it with the raising operators to create the other states. Moreover, like in the point particle case, each state has a momentum label k. This gives us an infinitely degenerated ground state $|0,k\rangle$, defined by

$$\begin{aligned} \alpha_m^\mu|0,k\rangle &= 0 \qquad\qquad (m>0) \\ p^\mu|0,k\rangle &= k^\mu|0,k\rangle. \end{aligned} \tag{2.8}$$

To generate the excited states, we let an arbitrary number of raising operators $\alpha_{-m}^\mu = (\alpha_m^\mu)^\dagger$ (for $m>0$) operate on this state.

Before discussing these excited states, we must say a bit more about the constraints (1.15) that we ignored so far.

Exercise 2.9 *Show that classically these constraints imply*

$$L_m \equiv \frac{1}{2}\sum_{n=-\infty}^{\infty} \alpha_{m-n}^\mu \alpha_{n\mu} = 0, \tag{2.9}$$

where we defined

$$\alpha_0^\mu = \sqrt{2\alpha'}p^\mu. \tag{2.10}$$

The operators L_m are called the *Virasoro-operators.* Note the word ‘classically’ in the exercise: the reader should check where the calculation goes wrong in the quantum mechanical case. For now, we simply state that in the quantum mechanical case, the only difference is in the operator L_0: it should be defined by

$$L_0 \equiv \sum_{n=1}^{\infty} \alpha_{-n}^{\mu} \alpha_{n\mu} + \frac{1}{2}\alpha_0^2 - 1. \tag{2.11}$$

In the next chapter, we will prove this statement.

We can use this definition to derive an important formula. Remember that $\frac{1}{2\alpha'}\alpha_0^{\mu}$ is the momentum p^{μ} of the string center-of-mass. Using the mass-shell equation, this means that

$$\begin{aligned} M^2 &= -p^{\mu} p_{\mu} \\ &= -\frac{1}{2\alpha'}\alpha_0^{\mu}\alpha_{0\mu} \\ &= \frac{1}{\alpha'}\left(\sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n - 1\right). \end{aligned} \tag{2.12}$$

Exercise 2.10 *Using the above formula, calculate M^2 for the excited state $\alpha_{-n}^{\mu}|0, k\rangle$. What is the mass if we apply several raising operators to the ground state?*

Using the above exercise, we see that the operator M^2 does a weighed counting of the excitations of the string. The ground state of open string theory has mass squared $(-\frac{1}{\alpha'})$. Of course, this is hard to interpret physically, so later on we will try to get rid of this state. States with negative mass squared are called *tachyons*.

The lightest excited states are of the form

$$\alpha_{-1}^{\mu}|0, k\rangle, \tag{2.13}$$

and are massless. Notice that these states form a vector (index μ) under Lorentz-transformations. Massless vector particles are usually called photons, in analogy with the photon of electromagnetism.

Intermezzo: Classically we found the equation $L_m = 0$. This cannot be imposed just like that in the quantum theory. Indeed, if we would impose $L_m|\psi\rangle = 0$ for a physical state, no states would survive. The solution is similar to the so-called Gupta-Bleuler formalism of electrodynamics. We don't require $L_m = 0$ on physical states, we only require that for any pair of physical states $|\psi\rangle, |\chi\rangle$ the matrix elements of L_m vanish, so that

$$\langle\psi|L_m|\chi\rangle = 0. \tag{2.14}$$

In other words, $L_m = 0$ in the weak operator topology. Because $L_m^{\dagger} = L_{-m}$, it is sufficient to impose

$$L_m|\psi\rangle = 0, \qquad m \geq 0 \tag{2.15}$$

as condition on the physical states, as this will guarantee that the matrix elements will all vanish. Thus, only the positive frequency modes of L_m annihilate physical states. However, this is not yet the full story. Typically, what we can measure in this theory are matrix elements like $\langle\chi|\psi_1\rangle$ and $\langle\chi|\psi_2\rangle$ of physical states. Now suppose that

$$|\psi_1\rangle - |\psi_2\rangle = L_{-k}|\eta\rangle \tag{2.16}$$

for some $k > 0$ and and an arbitrary (not necessarily physical state) $|\eta\rangle$. Then one easily sees that

$$\langle\chi|\psi_1\rangle = \langle\chi|\psi_2\rangle \tag{2.17}$$

for all physical states $|\chi\rangle$. In other words, no measurement can distinguish the two physical states $|\psi_1\rangle$ and $|\psi_2\rangle$: we should identify these states. The final definition of physical states is thus all states annihilated by L_m, $m \geq 0$, where we identify states that differ by the L_{-k} of an arbitrary state with $k > 0$. It is a good exercise to check that states of the form $\epsilon_\mu \alpha^\mu_{-1}|0, k^\nu\rangle$ are physical if the polarization ϵ_μ obeys $k^\mu \epsilon_\mu = 0$, where two polarizations are identified if the difference is proportional to k. The physical state condition corresponds to the Lorentz gauge of electrodynamics, the identification to the residual gauge invariance. We see that string theory gives us naturally electrodynamics and also selects a prefered gauge choice, the Lorentz gauge.

Closed strings

For the closed string theory, the techniques we use are all the same, but the results are slightly different. First of all, in the closed string case, there are two sets of oscillators: α and $\tilde{\alpha}$. This leads to two extra commutation relations, which are the expected ones:

$$\begin{aligned} [\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n] &= m\delta_{m+n}\eta^{\mu\nu} \\ [\tilde{\alpha}^\mu_m, \alpha^\nu_n] &= 0, \end{aligned} \tag{2.18}$$

The $\tilde{\alpha}$ are therefore also raising and lowering operators, and hence we can define a ground state analogous to the one in the open string case. Again, the constraints imply that the Virasoro operators vanish; the only difference is that now we have to define

$$\alpha^\mu_0 = \tilde{\alpha}^\mu_0 = \sqrt{\frac{\alpha'}{2}} p^\mu. \tag{2.19}$$

Notice that with these conventions $[x^\mu, p^\nu] = i\eta^{\mu\nu}$. Of course, now we also have Virasoro operators $\tilde{L}_m$. The factor of two in the above formula leads to some factors of two in other formulae. In particular, the mass formula for closed strings picks up two of these factors and becomes

$$M^2 = \frac{2}{\alpha'}\left(\sum_{n=1}^{\infty}(\alpha_{-n}\cdot\alpha_n + \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n) - 2\right). \tag{2.20}$$

Note that since $\alpha^\mu_0 = \tilde{\alpha}^\mu_0$, the two terms in this equation coming from the left- and right-moving sector are in fact equal. In other words: the weight of the left- and right-moving oscillations is equal. This is a constraint to the states we create: we can apply any combination of raising operators to the ground state, as long as the weights of the left- and right-moving operators match.

Exercise 2.11 *Write down the states at the lightest three mass levels of the closed string. What is the number of different states at each level?*

Notice that the states at the massless level can be viewed as a rank 2 $D \times D$ tensor. We can split this tensor into its trace, an antisymmetric part and a traceless symmetric part. This seperation can easily be shown to be invariant under Lorentz transformations. Using quantum field theory, it can be proved that the graviton should be a traceless symmetric rank 2 particle. This is a first sign of the fact that gravity can be included in string theory. In later chapters, the traceless symmetric rank 2 state we found will indeed turn out to play the role of a graviton.

The light cone gauge

Until now, we have quantized covariantly – i.e. we have quantized the unconstrained solutions (1.18) and (1.22) to the field equations. As we said at the beginning of this section, there is also another option: we can deal with the constraints (1.15) first and then quantize the theory. To do so, we switch to so-called light cone coordinates:

$$\begin{aligned} X^+ &= \frac{1}{2}\sqrt{2}(X^0 + X^D) \\ X^- &= \frac{1}{2}\sqrt{2}(X^0 - X^D). \end{aligned} \tag{2.21}$$

Note that at this point, we lose manefest Lorentz invariance: the choice for X^D instead of another X^μ is completely arbitrary. This means that we cannot take Lorentz invariance for granted anymore – if we were to proceed with this theory after quantizing, we would have to prove its Lorentz invariance first.

Now, look back at equations (1.11) and (1.12). We used these symmetries to turn the world sheet metric into the Minkowskian metric. However, we have not completely fixed the gauge by doing so: it is easily seen that a combined transformation with

$$\partial^\alpha \xi^\beta + \partial^\beta \xi^\alpha = \zeta \eta^{\alpha\beta} \tag{2.22}$$

leaves the Minkowski metric intact.

It can be shown that in terms of τ and σ, this corresponds to the transformations

$$\begin{aligned} \tau + \sigma &\rightarrow f(\tau + \sigma) \\ \tau - \sigma &\rightarrow g(\tau - \sigma), \end{aligned} \tag{2.23}$$

with f and g arbitrary functions.

Exercise 2.12 *Write down what this means for τ and σ separately; call the new coordinates $\tilde{\tau}$ and $\tilde{\sigma}$. What familiar differential equation does $\tilde{\tau}$ satisfy in terms of τ and σ?*

From this exercise, we see that we can use the gauge invariance to make τ equal to any of the X^μ. We choose X^+, but leave the coordinates x^+ and p^+ intact:

$$X^+(\sigma, \tau) = x^+ + p^+ \tau. \tag{2.24}$$

In other words: we have used the leftover gauge symmetry to gauge away the oscillator coordinates α_n^+.

Now we are ready to impose the constraint equations. Writing them as $(\dot{X} \pm X')^2 = 0$ and inserting (2.24), we see that we can express X^- in terms of the other X^i:

$$(\dot{X}^- \pm X'^-) = \frac{(\dot{X}^i \pm X'^i)^2}{2p^+}, \tag{2.25}$$

where $i = 1, 2, \ldots, D-1$. This means we can eliminate the α_n^- as well, and quantize using only α_n^i.

There is an easy physical explanation for the fact that after satisfying the constraints, there are only $D-1$ sets of oscillator coordinates left to quantize. Remember that we are embedding a two dimensional world sheet into a $(D+1)$-dimensional space-time. The world sheet can oscillate in space-time, but an oscillation of an interior point of the string in one of the directions of the world sheet will not be visible from the outside: it is nothing but a reparametrization of the world sheet. Since our theory is supposed to be reparametrization invariant, we do not expect these oscillations to be physical degrees of freedom, and hence we only expect $D-1$ sets of physical oscillator coordinates.

In the next lecture, we will see how to impose the constraints after having covariantly quantized. Of course, for consistency we should also show that both methods lead to the same theory. The proof of this statement is quite complicated, and will not be given in these lectures. We refer the interested reader to the book by Green, Schwarz and Witten. Note that once this equivalence is proven, we have also regained Lorentz invariance.

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Third lecture

3.1 Interactions and vertex operators

In the previous two lectures, we have studied the non-interacting string. Nevertheless, what we have been studying is some kind of perturbation theory around a ‘vacuum’. This vacuum, however, is not a state without strings, but simply the state of a single string that is not excited. The excited states are oscillating states of a single string. In other words: all we have done so far is consider the internal degrees of freedom of a single string.

Of course, in a physically interesting theory we want to have much more than this. We want to be able to describe states consisting of an arbitrary number of strings, and even more importantly: we want to be able to describe interactions.

The first thing we should realize is that it is not possible – like we would do in quantum field theory – to introduce interactions by writing down higher order terms in the Lagrangian. If we would do that, we would obtain some complicated field theory on the string world sheet. However, the world sheet would still be the world sheet of one single string, so this would mean that we were describing a much more complicated but still single and free string.

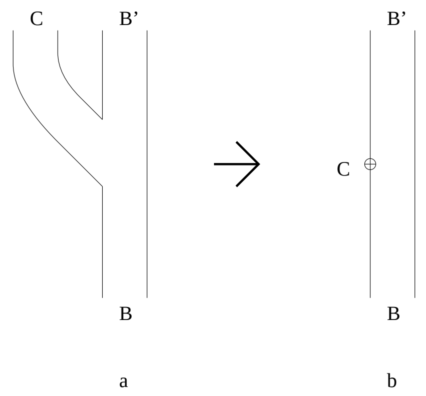

Figure 3.1: Conformal transformation of a tree-diagram world sheet.

In figure 3.1a, we have drawn the world sheet of the simplest kind of interacting string: a single string state emitting another string. What we could do is try to parametrize this world sheet, find appropriate boundary conditions, and quantize the theory we obtain. However, there turns out to be a much easier way to describe interacting strings – especially when we only consider tree diagrams.

The key to this description is the property of conformal invariance. We already mentioned this in the previous lecture: it is the leftover gauge freedom that assures us we can reparametrize

$$\begin{aligned} \sigma^+ &= f(\sigma^+) \\ \sigma^- &= g(\sigma^-) \end{aligned} \tag{3.1}$$

where we called $\sigma^+ = \tau + \sigma$ and $\sigma^- = \tau - \sigma$; f and g are arbitrary functions. Reparametrizing means changing the coordinates and changing the metric, so we can think of this as 'reshaping the world sheet'. It turns out that very drastic changes are possible; for example, we can always reshape a three-particle tree diagram into the shape of figure 3.1b. Here, the external state C is contracted into the world sheet of the other state. The point C in the contracted state corresponds to the points at infinity of the original external state C. We can do the same trick to tree diagrams with more external states: all external states can be contracted to points on the boundary.

Now that we know this, we want to take the next step, and calculate probability amplitudes of certain scattering processes. In quantum mechanics, a probability ampitude is calculated by taking the inner product of two states:

$$A = \langle \psi_{\text{in}} | \psi_{\text{out}} \rangle. \tag{3.2}$$

We want to do the same thing in string theory. However, there is the external state we have to deal with: if we are calculating the amplitude of figure 3.1b, we can not simply substitute $|\psi_{\text{in}}\rangle$ by $|B\rangle$ and $|\psi_{\text{out}}\rangle$ by $|B'\rangle$. The state before and after the emission of the state C will be different, and hence we would like to find some operator V_C that turns the state after the emission into the state before the emission[1]. The probability amplitude will then be

$$A = \langle B | V_C(\tau, \sigma) | B' \rangle. \tag{3.3}$$

Note that we expect V_C to be dependent on the position (τ, σ) of the external state. From now on, we will be considering open strings, which can only emit states from their boundary. Hence σ will be 0 or π; when no distinction is necessary, we will usually pick $\sigma = 0$.

Of course, the precise form of the vertex operator will also depend on the type of state that is emitted. The easiest state to be emitted is the tachyon. Since it has no degrees of freedom except for momentum, it can hardly do anything more than change the momentum of the propagating state B. The momentum of this state will be decreased by the amount of momentum of the external state.

Exercise 3.13 *Calculate the following commutators:*

- $[p^\mu, (x^\nu)^n]$
- $[p^\mu, (X^\nu)^n]$
- $[p^\mu, exp(ik_\nu X^\nu)]$

What is the momentum of the state $e^{ik_1 \cdot X}|0, k_2^\mu\rangle$*?*

[1] Of course, it would sound more natural to try and find an operator that turns the state *before* emission into the state *after* emission, but since we let V_C oprate on $|B'\rangle$, this is the correct order. The operator describing the opposite order is simply the Hermitian conjugate of V_C.

The exercise shows that the operator $e^{ik_C \cdot X(\tau,\sigma)}$ is likely to be the operator V_C we are looking for. In fact, since the exponential is defined as a power series, there is an ordering ambiguity in this choice. We define the vertex operator to be normal ordered; i.e. we write all lowering operators to the right and all raising operators to the left. We denote normal ordering by colons, and obtain the following vertex operator for emission of a tachyonic open string state:

$$\begin{aligned} V_C(\tau,\sigma) &= \; : e^{ik^\mu X_\mu} : \\ &= e^{ik^\mu(x_\mu + (2\alpha')p_\mu \tau)} \exp\left[-\sqrt{2\alpha'} k^\mu \sum_{n<0} \frac{1}{n} \alpha_{n\mu} e^{-in\tau} \cos n\sigma \right] \\ & \quad \times \exp\left[-\sqrt{2\alpha'} k^\mu \sum_{n>0} \frac{1}{n} \alpha_{n\mu} e^{-in\tau} \cos n\sigma \right]. \end{aligned} \tag{3.4}$$

Note that for operators, the equation $e^A e^B = e^{A+B}$ does not hold!

Now, we have all the necessary ingredients to obtain probability amplitudes. The amplitude of figure 3.1b can be calculated by

$$A = \langle A | V_C(0,0) | B \rangle. \tag{3.5}$$

Note thate we put $\sigma = 0$ and $\tau = 0$. We already discussed the first choice; the second is a consequence of translational invariance along the τ-direction: we can always shift this variable so that the vertex is at $\tau = 0$.

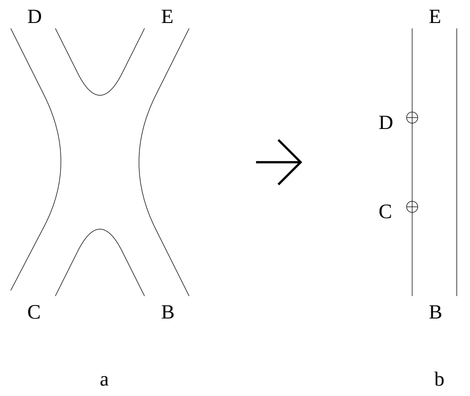

Figure 3.2: Scattering of two open strings.

When we have more external particles, we can only fix one of them at $\tau = 0$; we have to integrate over the position of the other. For example, the diagram in figure 3.2 has amplitude

$$A = \int_0^\infty d\tau \, \langle B | V_C(0,0) V_D(\tau,0) | E \rangle. \tag{3.6}$$

Finally, we want to say some words about the vertex operators of excited states, such as the open string photon. It is clear that these states have the same effect on the momentum as the tachyon, so again we expect a factor $e^{ik\cdot X}$. However, the photon also has a polarization, which we expect to find back in the vertex operator. This means we want to find a vector of vertex operators, which we can contract with the polarization vector of the photon. It turns out (but we will not prove this here) that the right choice is also the easiest one: it is the vector $\dot{X}^\mu e^{ik\cdot X}$. After normal ordering, this means the vertex operator for a photon of polarization η^μ is

$$V_{ph} =: \eta^\mu \dot{X}_\mu e^{ik\cdot X} : \tag{3.7}$$

Note that this method of contracting the world sheet to a strip with vertex operators only works for tree diagrams. In the next lecture, we will see how we can deal with loop diagrams.

3.2 The 2-tachyon scattering amplitude

As an example, we will now calculate the two-tachyon scattering amplitude in open string theory. The expression for this amplitude is

$$A = \int_0^\infty d\tau \, \langle 0, k_1| : e^{ik_2^\mu X_\mu(0,0)} :: e^{ik_3^\nu X_\nu(\tau,0)} : |0, -k_4\rangle, \tag{3.8}$$

corresponding to the Feynman diagram in figure 3.2. We called the final momentum $-k_4$ since we want to consider incoming momenta as positive. Note that we chose the situation where both tachyons are on the $\sigma = 0$ edge of the open string world sheet; the reader should check that the results will be the same for all other choices.

Inserting the normal ordered expression (3.4) for the vertex operators, we get a product of six exponentials between the bra and the ket. However, the rightmost of these factors has only annihilation operators in the exponent, so the exponent vanishes when operating on the ket and the factor is equal to one. In the same manner, the leftmost exponential operating on the bra gives a factor of one. We are therefore left with

$$\begin{aligned} A = \int_0^\infty d\tau \, \langle 0, k_1| \quad & e^{ik_2^\mu x_\mu} \exp\left[-k_2^\mu \sum_{n>0} \frac{1}{n}\alpha_{n\mu}\right] \exp\left[-k_3^\nu \sum_{m<0} \frac{1}{m}\alpha_{m\nu} e^{-im\tau}\right] \\ & \times e^{ik_3^\nu(x_\nu + p_\nu \tau)} |0, -k_4\rangle, \end{aligned} \tag{3.9}$$

where we used the fact that the third and the fourth factor commute to interchange these factors. We put $\alpha' = \frac{1}{2}$ so we will not have to include factors of α' which will only obscure the calculation.

Our next step is to evaluate the results of the first and fourth factor in this expression operating on the bra and ket respectively. Note that in the fourth factor we have another ordering problem: we cannot simply split e^{A+B} into $e^A e^B$. We should use the Baker-Cambell-Hausdorff formula which states that if $[A, B]$ is a c-number then

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}.$$

If we also use the fact that for a tachyon $k^2 = 2$, we obtain

$$e^{ik\cdot(x+p\tau)} = e^{ik\cdot x} e^{i(k\cdot p+1)\tau}. \tag{3.10}$$

We substitute this expression for the fourth factor in (3.9). The exponential containing the p-operator is easily evaluated since the out-state is an eigenstates of this operator. Note that $e^{ik\cdot x}$ is the operator that changes the momentum of a state by an amount $-k$. We can therefore absorb these operators in the states:

$$\begin{aligned} A = \int_0^\infty d\tau \quad & e^{i(-k_3^\nu k_{4\nu}+1)\tau} \langle 0, k_1 + k_2| \exp\left[-k_2^\mu \sum_{n>0} \frac{1}{n}\alpha_{n\mu}\right] \\ & \times \exp\left[-k_3^\nu \sum_{m<0} \frac{1}{m}\alpha_{m\nu} e^{-im\tau}\right] |0, -k_3 - k_4\rangle. \end{aligned} \tag{3.11}$$

Now we want to evaluate the two final factors within the bra and ket. Since the first one contains only lowering operators and the second one only raising operators, we would like to interchange their order. Of course, the two operators do not commute, but we can use the Campbell-Baker-Hausdorff formula

$$e^A e^B = e^B e^A e^{[A,B]}. \tag{3.12}$$

The commutator of the two operators can be evaluated as follows:

$$\begin{aligned}[-k_2^\mu \sum_{n>0} \frac{1}{n}\alpha_{n\mu}, -k_3^\nu \sum_{m<0} \frac{1}{m}\alpha_{m\nu} e^{-im\tau}] &= +k_2^\mu k_3^\nu \sum_{n>0,m<0} \frac{e^{-im\tau}}{mn}[\alpha_{n\mu}, \alpha_{m\nu}] \\ &= -k_2^\mu k_{3\mu} \sum_{n>0} \frac{e^{in\tau}}{n}. \end{aligned} \tag{3.13}$$

For the sum over n, we can use the Taylor-expansion

$$\sum_{n>0} \frac{x^n}{n} = -\ln(1-x). \tag{3.14}$$

By now, we have removed all operators from the integral. Using orthogonality of the states, we can remove them as well, and are left with

$$A = \int_0^\infty d\tau\, e^{i(-k_3\cdot k_4+1)\tau}(1-e^{i\tau})^{k_2\cdot k_3}\delta(k_1-k_2-k_3+k_4). \tag{3.15}$$

The delta-function expresses momentum conservation; we will not include it in our final formula. Using the delta function and the fact that for a tachyon, $k^2=2$, we can write the second factor in terms of the first:

$$\begin{aligned} k_3\cdot k_4 &= -\frac{1}{2}(k_3-k_4)^2+2 \\ &= -\frac{1}{2}(k_1-k_2)^2+2 \\ &= k_1\cdot k_2 \end{aligned} \tag{3.16}$$

We now do a Wick-rotation $i\tau \to -\tau$ and change the variable of integration to $x = e^{-\tau}$ (so $d\tau = -\frac{dx}{x}$) to obtain our final result

$$A = i\int_0^1 dx\, x^{-k_1\cdot k_2}(1-x)^{k_2\cdot k_3}\delta(k_1-k_2-k_3+k_4). \tag{3.17}$$

Exercise 3.14 *We want to calculate the amplitude (3.17) in an alternative way. First of all, calculate the expectation value*

$$\langle 0|\tilde{X}^\mu(0,0)\tilde{X}^\nu(\tau,0)|0\rangle, \tag{3.18}$$

where $\tilde{X}$ means the part of X that does not contain the zero modes $x+p\tau$. Now find an appropriate current $J_\mu(\tau)$ and use Wick's theorem

$$\begin{aligned}\langle 0|T\left\{\exp\left(i\int_{-\infty}^\infty d\tau\, J_\mu(\tau)\tilde{X}^\mu(\tau)\right)\right\}|0\rangle &= \exp\Big(-\int_{-\infty}^\infty d\tau \int_{\tau'<\tau} d\tau'\, J_\mu(\tau')J_\nu(\tau) \\ &\quad \times\langle 0|\tilde{X}^\mu(\tau')\tilde{X}^\nu(\tau)|0\rangle\Big) \end{aligned} \tag{3.19}$$

to arrive at the formula (3.17). The operator T is a time-ordering operator: it rearranges the subsequent operators in increasing order of time.

We can put the amplitude (3.17) in a very interesting form. We can use the Mandelstam variables $s = (k_1 - k_2)^2$ and $t = (k_2 + k_3)^2$ to write it in the form

$$A = i \int_0^1 dx\, x^{\frac{1}{2}s-2}(1-x)^{\frac{1}{2}t-2}, \tag{3.20}$$

where we for convenience dropped a delta-function of the momenta. If we substitute $a = \frac{1}{2}s - 1$ and $b = \frac{1}{2}t - 1$, this is the integral representation of the Euler beta-function $B(a, b)$. It also has a representation in terms of Γ-functions so that we get

$$A = i \frac{\Gamma(\frac{s}{2} - 1)\Gamma(\frac{t}{2} - 1)}{\Gamma(\frac{s}{2} + \frac{t}{2} - 2)}. \tag{3.21}$$

One of the properties we can immediately read off from this form of the amplitude is that it is symmetric under the exchange of t and s. It was the search for amplitudes with this property that in the 1960's led Veneziano to this amplitude, which was the starting point of modern string theory.

Notice that the amplitude has a pole when $s = 2 - 2N$ for some nonnegative integer N. Similarly, there are poles for suitable values of t. Translated back into the original momenta, this means that there are poles when $(k_1 - k_2)^2 + 2N - 2 = 0$. These are precisely the momenta for which there are physical open string states. Thus we see that the four-point function has a pole when the intermediate state is on-shell, a phenomena well-known in quantum field theory.

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Fourth lecture

4.1 Complex coordinates

In this lecture, we develop the subject of conformal field theory. By doing this, we will obtain some basic mathematical tools that we will use in the rest of the lectures.

Until now, we have been working with a Minkowskian metric: a metric which contains a minus sign. Since this minus sign leads to many computational difficulties, we will remove it by defining the *Euclidean time* to be

$$\sigma^0 = i\sigma^2 = i\tau. \tag{4.1}$$

This turns the world-sheet metric into a positive definite one:

$$ds^2 = (d\sigma^1)^2 + (d\sigma^2)^2. \tag{4.2}$$

Throughout this lecture, we will pick the spatial coordinate to be $\sigma^1 = \sigma \in [0, 2\pi]$, so we work with the closed string. Now, we introduce the complex world-sheet coordinates

$$w = \sigma^1 + i\sigma^2, \qquad \bar{w} = \sigma^1 - i\sigma^2. \tag{4.3}$$

Using these coordinates, we can describe many useful transformations in a natural way. For example, if we want to map the open string world sheet (i. e. the cylinder) to the complex plane, we can do this by the transformation

$$z = e^{-iw} = e^{-i\sigma^1 + \sigma^2} = z^1 + iz^2. \tag{4.4}$$

This map (see figure 4.1) sends lines of equal σ^2 to circles around the origin $\{z = 0\}$. The origin itself corresponds to $\{\tau \to -\infty\}$. In other words: time moves radially and outward.

The derivative operators in the new coordinates are

$$\begin{aligned} \partial = \partial_z &= \partial/\partial_z = \frac{1}{2}(\partial_1 - i\partial_2) \\ \bar{\partial} = \partial_{\bar{z}} &= \partial/\partial_{\bar{z}} = \frac{1}{2}(\partial_1 + i\partial_2). \end{aligned} \tag{4.5}$$

Tensors with upper indices transform to the new coordinates like the coordinates themselves:

$$A^\alpha = (A^1, A^2) \to (A^z, A^{\bar{z}}) = (A^1 + iA^2, A^1 - iA^2) \tag{4.6}$$

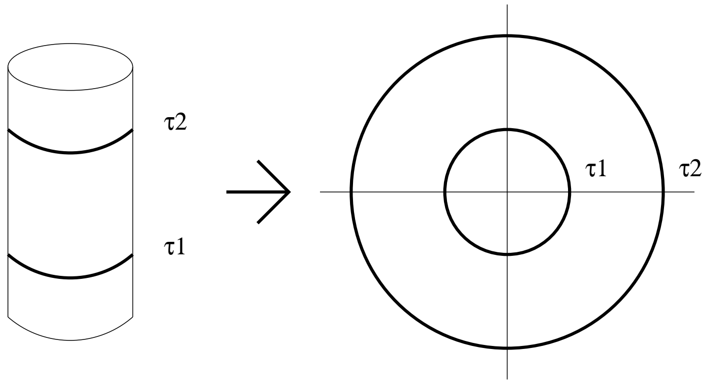

Figure 4.1: The conformal map from the closed string world sheet into the plane.

Note that $A^{\bar{z}} = (A^z)^*$. Tensors with lower indices transform to the new coordinates like the derivatives:

$$A_\alpha = (A_1, A_2) \to (A_z, A_{\bar{z}}) = \frac{1}{2}(A_1 - iA_2, A_1 + iA_2) \tag{4.7}$$

The two derivatives in the old coordinates can be expressed in terms of the new coordinates as follows:

$$\begin{aligned} H &\equiv \partial/\partial_\tau = z\partial_z + \bar{z}\partial_{\bar{z}} \\ P &\equiv \partial/\partial_\sigma = z\partial_z - \bar{z}\partial_{\bar{z}} \end{aligned} \tag{4.8}$$

The τ-derivative corresponds to the Hamiltonian (compare the energy oprator in quantum mechanics); the σ-derivative to the momentum along the string.

4.2 The free scalar field

We have seen that the bosonic string in d dimensions can be formulated in terms of d free scalar fields on the two-dimensional world-sheet. Furthermore it was very important that this field theory was conformal invariant: because of that reason the scale of the metric decoupled and we could use local coordinate transformations to choose a locally flat metric. (We will see later how on surfaces of more complicated topology there are extra subtleties.)

With the string theory action in mind, we will now study the free Euclidean scalar field $x(z, \bar{z})$ using the complex notation. In this the action reads

$$S = \frac{1}{2\pi\alpha'} \int d^2z\, \partial x \bar{\partial} x. \tag{4.9}$$

Note that this corresponds to the action for one of the components X^μ of the free string, written in complex coordinates. From now on, we will set $\alpha' = 2$ (?). The equation of motion derived from this action is

$$\bar{\partial}\partial x = 0. \tag{4.10}$$

This holds if x is a function of z or $\bar{z}$ only. In fact, the most general solution is a combination of the two:

$$x = x_L(z) + x_R(\bar{z}). \tag{4.11}$$

In Minkowskian signature this means x_L is a function of $\tau+\sigma$; x_R is a function of $\tau-\sigma$, justifying the indices L and R. We will use the terms left-movers for holomorphic fields, and right-movers for anti-holomorphic fields. Since $\bar{\partial}(\partial x) = 0, \partial x = \partial x_L$ is holomorphic, i.e. it is a function of z only. Hence it can be expressed as a Laurent power series

$$i\partial x = \sum_{n\in\mathbb{Z}} \alpha_n z^{-n-1}. \tag{4.12}$$

Using Cauchy's formula, we can calculate the coefficient α_n by the line integral

$$\alpha_n = \oint \frac{dz}{2\pi i} z^n \, i\partial x(z). \tag{4.13}$$

where the contour is a closed loop around the origin that does not circumscribe any other singularities. Taking the Hermitean conjugate of the above equation, we find that the coefficients α_n satisfy the Hermiticity condition

$$\alpha_n^\dagger = \alpha_{-n} \tag{4.14}$$

In the same way as we did before we obtain the canonical commutation relations

$$[\alpha_n, \alpha_m] = n\delta_{n+m,0}. \tag{4.15}$$

The same argument for the right-moving side of the theory leads to the mode expansion

$$-i\bar{\partial} x = \sum_{n\in\mathbb{Z}} \bar{\alpha}_n \bar{z}^{-n-1}. \tag{4.16}$$

In analogy with string theory, we define the momentum operator to be

$$\hat{p} = \alpha_0 = \bar{\alpha}_0. \tag{4.17}$$

From the commutation relations (4.15) we see that these are the only operators that cannot be interpreted as raising and lowering operators by a rescaling. Thus, again in complete analogy with the string case, we arrive at the notion of a ground state, labelled by the eigenvalue $p \in \mathbb{R}$ of $\hat{p}$:

$$\alpha_n|0,p\rangle = \bar{\alpha}_n|0,p\rangle = 0, \qquad n > 0. \tag{4.18}$$

$$\alpha_0|0,p\rangle = p|0,p\rangle \tag{4.19}$$

By acting on these states with the creation operators α_{-n} we obtain the Hilbert space $\mathcal{H}$. A general Fock space state has the form

$$|\phi\rangle = \alpha_{-n_1} \cdots \alpha_{-n_r} \bar{\alpha}_{-m_1} \cdots \bar{\alpha}_{-m_s}|p\rangle, \qquad n_i, m_j > 0. \tag{4.20}$$

4.3 Conformal transformations

In complex coordinates, the (Euclidean) flat metric is $ds^2 = 2dzd\bar{z}$. We define *conformal transformations* to be transformations that rescale this metric by a $z, \bar{z}$-dependent positive factor:

$$ds^2 \to e^{\phi(z,\bar{z})} ds^2 \tag{4.21}$$

These transformations turn out to be the holomorphic mappings

$$z \to z' = f(z), \qquad \bar{z} \to \bar{z}' = \bar{f}(\bar{z}) \tag{4.22}$$

Exercise 4.15 *Show that in d Euclidean dimensions a vector field ξ^μ generates a conformal transformation if*

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \frac{2}{d} \partial_\lambda \xi^\lambda \delta_{\mu\nu} = 0. \tag{4.23}$$

Show that for $d = 2$ this becomes the Cauchy-Riemann equation

$$\partial_1 \xi_1 = \partial_2 \xi_2, \qquad \partial_1 \xi_2 = -\partial_2 \xi_1. \tag{4.24}$$

Write this equation in terms of $\xi = \xi^1 + i\xi^2$ as $\bar{\partial}\xi = 0$.

Remark: in d Euclidean dimensions the conformal group of $\mathbb{R}^d$ is $SO(d+1,1)$. This contains the Poincaré group $SO(d) \rtimes \mathbb{R}^d$. So for $d = 2$ we would expect $SO(3,1) \cong SL(2,\mathbb{C})$. However "by accident" this group is enlarged to infinite-dimensional group of all holomorphic maps. We will see in a moment how $SL(2,\mathbb{C})$ appears as the group of global conformal transformations of the Riemann sphere (the conformal compactification of $\mathbb{R}^2$).

Infinitesimally, the conformal transformations are obtained by adding holomorphic vector fields $\xi^z(z), \xi^{\bar{z}}(\bar{z})$ to the original coordinates:

$$\delta z = \xi(z), \qquad \partial_{\bar{z}} \xi = 0, \tag{4.25}$$

and similarly for $\bar{z}$. Again, this means we can do a Laurent expansion of $\xi(z)$ on $\mathbb{C} - \{0\}$:

$$\xi(z) = \sum_{n \in \mathbb{Z}} \xi_n z^{-n+1} \tag{4.26}$$

From this infinitesimal expansion, we see that as a basis of the holomorphic vector fields we can take

$$l_n(z) = z^{n+1} \partial_z, \qquad \bar{l}_n(\bar{z}) = \bar{z}^{n+1} \partial_{\bar{z}} \tag{4.27}$$

A simple calculation shows that these operators satisfy the commutation relations

$$[l_m, l_n] = -(m-n) l_{m+n}, \qquad [\bar{l}_m, \bar{l}_n] = -(m-n) \bar{l}_{m+n}, \tag{4.28}$$

whereas the holomorphic and anti-holomorphic vector fields commute

$$[l_m, \bar{l}_n] = 0. \tag{4.29}$$

The algebra defined by these commutation relations is called the *Witt algebra.*

We define a primary field $\phi(z,\bar{z})$ of weight $(h,\bar{h})$ as a field that under conformal transformations transforms as

$$\phi(z,\bar{z}) \to (\partial f)^h (\bar{\partial} \bar{f})^{\bar{h}}\, \phi(f(z), \bar{\phi}(\bar{z})). \tag{4.30}$$

By writing ∂f as $\partial f / \partial z$ we see that this is equivalent to the statement that

$$\phi(z,\bar{z}) (dz)^h (d\bar{z})^{\bar{h}} \tag{4.31}$$

is invariant under conformal maps.

Infinitesimally, this implies that under a vector field $\xi(z)$ the primary field transforms as

$$\delta_\xi \phi(z,\bar{z}) = (\xi(z)\partial_z + h \partial_z \xi(z)) \phi(z,\bar{z}). \tag{4.32}$$

The sum and difference of the conformal weights both have a natural interpretation. $\Delta = h + \bar{h}$ is the scaling dimension: under rescalings of z the fields transform as

$$\phi(z) = e^{\lambda\Delta}\phi(e^{\lambda} \cdot z), \qquad \lambda \in \mathbb{R}_{+}. \tag{4.33}$$

Δ can be any real number, but for unitary theories we will find that it is positive for any other field than the identity. For the identity, of course, $\Delta = 0$.

The quantity $s = h - \bar{h}$ is the two-dimensional spin: under a rotation of z the fields transform as

$$\phi(z) = e^{is\theta}\phi(e^{i\theta}z), \qquad \theta \cong \theta + 2\pi. \tag{4.34}$$

Since fields should be single-valued under a 2π rotation, the spin is necessarily an integer:

$$s = h - \bar{h} \in \mathbb{Z}. \tag{4.35}$$

If $\bar{h} = 0$ this means h is an integer and the field satifies

$$\phi(z, \bar{z})(dz)^h = \phi(w, \bar{w})(dw)^h, \qquad z = z(w). \tag{4.36}$$

Later, we will see that a field that satisfies this condition is necessarily holomorphic:

$$\partial_{\bar{z}}\phi = 0 \Leftrightarrow \phi = \phi(z). \tag{4.37}$$

Once again, we can expand these holomorphic fields:

$$\phi(z) = \sum_{n\in\mathbb{Z}} \phi_n z^{-n-h} \tag{4.38}$$

Here the normalization is chosen such that in the coordinate $w = \sigma^1 + i\sigma^2$ with $z = e^{-iw}$ on the cylinder

$$\phi(w) \propto \sum_{n\in\mathbb{Z}} \phi_n e^{in\sigma^1 - n\sigma^2}. \tag{4.39}$$

and so ϕ_n are the usual Fourier modes on the circle. Note that $n < 0$ corresponds to a mode with positive energy $-n$.

As an example, the reader can check that ∂x is a primary field of weight $(1, 0)$; $\bar{\partial} x$ is a primary field of weight $(0, 1)$.

In any two-dimensional field theory we can define the stress tensor $T_{\mu\nu}$. It is symmetric: $T_{\mu\nu} = T_{\nu\mu}$, and hence has three independent components which we denote by

$$T_{zz} \equiv T, \qquad T_{z\bar{z}}, \qquad T_{\bar{z}\bar{z}} \equiv \bar{T}. \tag{4.40}$$

Conformal invariance implies that the trace $T^{\mu}_{\mu} = 0$. In terms of the components, this means that

$$T_{z\bar{z}} = 0. \tag{4.41}$$

Moreover, we would like our theories to be translationally invariant (which means momentum is conserved), so we impose

$$\partial^{\mu}T_{\mu\nu} = 0 \Rightarrow \bar{\partial}T = 0, \ \partial\bar{T} = 0. \tag{4.42}$$

Here we reach an important conclusion: for a two-dimensional conformal field theory (CFT) the stress-tensor $T = T_{zz}$ is a holomorphic field. Since $T(z)$ is holomorphic we can write it in a Laurent expansion:

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \qquad L_n^{\dagger} = L_{-n}. \tag{4.43}$$

Using Cauchy's formula, we can calculate the modes L_n:

$$L_n = \oint \frac{dz}{2\pi i} z^{n+1} T(z). \tag{4.44}$$

where the contour should satisfy the same properties as the contour in (4.13).

Of course, for the antiholomorphic field we have a similar expansion:

$$\bar{T}(\bar{z}) = \sum_{n\in\mathbb{Z}} \bar{L}_n \bar{z}^{-n-2}. \tag{4.45}$$

By using Noethers theorem, we can show that an infinitesimal diffeomorphism with a vector field $\xi(z)$ is implemented by an operator

$$L_\xi = \oint \frac{dz}{2\pi i} \xi(z) T(z) \tag{4.46}$$

$$L_n = L_{l_n}, \tag{4.47}$$

where l_n are the operators defined in (4.27). We would now like to calculate the commutators of the $L_n, \bar{L}_n$. Naively, one would expect the commutator to be $[L_\xi, L_\eta] = -L_{[\xi,\eta]}$, which would mean

$$[L_m, L_n] = (m-n)L_{m+n}, \text{ etc.} \tag{4.48}$$

Using more sophisticated mathematical techniques one can show that the most general answer is the Virasoro algebra: a central extension[2] of holomorphic vector fields given by

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0} \\ [\bar{L}_m, \bar{L}_n] &= (m-n)\bar{L}_{m+n} + \frac{\bar{c}}{12} m(m^2-1)\delta_{m+n,0} \\ [L_n, \bar{L}_m] &= 0 \end{aligned} \tag{4.49}$$

We will see later that central charges $c, \bar{c} \in \mathbb{R}$ will satisfy:

1. $c = \bar{c}$ as a result of 2d Poincaré invariance
2. $c > 0$ in a unitary CFT.
3. for $0 < c < 1$, c is of the form

$$c = 1 - \frac{6}{k(k+1)}, \qquad k \geq 3. \tag{4.50}$$

A free scalar field has $c = 1$. A free (Majorana) fermion field has $c = 1/2$. So roughly we can think of c as the number of fields, although c is not necessarily integer.

We already know that the stress tensor should implement the conformal transformation on the primary fields, so using (4.32) we find

$$[L_\xi, \phi(z,\bar{z})] = (\xi(z)\partial_z + h\partial_z\xi(z))\phi(z,\bar{z}). \tag{4.51}$$

For the components L_n, this implies

$$[L_n, \phi(z,\bar{z})] = z^{n+1}\partial_z\phi(z,\bar{z}) + h(n+1)z^n\phi(z,\bar{z}) \tag{4.52}$$

[2]A central extension of a Lie algebra with basis $\{\xi_a\}$ and with Lie bracket $[\xi_a, \xi_b] = f_{ab}{}^c\xi_c$ is a new algebra with basis $\{\xi_a, \mathbf{1}\}$ with brackets $[\xi_a, \xi_b] = f_{ab}{}^c\xi_c + c_{ab}\mathbf{1}$, $[\xi_a, \mathbf{1}] = 0$. The new element $\mathbf{1}$ is called central, since it commutes with everything. The Jacobi-identity imposes conditions on the coefficients $c_{ab} = -c_{ba}$. A representation of the central extension of a Lie algebra is the same as a projective representation of the original Lie algebra. A projective representation is a representation up to phase factors.

Exercise 4.16 *We want to see whether the stress tensor is a primary field of weight (2,0) (it is not). Show that the Virasoro algebra gives*

$$\delta_\xi T(z) = (\xi\partial + 2\partial\xi)T(z) + \frac{c}{12}\partial^3\xi. \tag{4.53}$$

Show that this is the infinitesimal version of:

$$T(z) \to (\partial f)^2 T(f(z)) + \frac{c}{12}\{f, z\}, \tag{4.54}$$

where we used the Schwarzian derivative

$$\{f, z\} = \frac{\partial f \partial^3 f - \frac{3}{2}(\partial^2 f)^2}{(\partial f)^2}. \tag{4.55}$$

Check the composition rule under two conformal maps, i. e. check that applying the map g first and then the map f gives the same result as applying the map $f \circ g$.

Now consider the map that takes the cylinder to the plane: $w \mapsto z = e^{-iw}$. Prove that

$$T_{\text{cyl}}(w)(dw)^2 = \left(T_{\text{plane}}(z) - \frac{c}{24}\right)(dz)^2. \tag{4.56}$$

Use this relation to show that the Hamiltonian on the cylinder

$$H = \int \frac{d\sigma}{2\pi} T_{\tau\tau} \tag{4.57}$$

is given by

$$H = L_0 + \bar{L}_0 - \frac{c + \bar{c}}{24}. \tag{4.58}$$

There are some special types of conformal transformations we want to mention. First of all, infinitesimal conformal transformations $\mathbb{C} \to \mathbb{C}$ correspond to vector fields $\xi(z)$ that are regular at the origin $z = 0$:

$$\xi(z) = \sum_{n \leq 1} \xi_n z^{-n+1}. \tag{4.59}$$

Secondly, maps from the Riemann sphere $\mathbb{P}^1 = \mathbb{C} \cap \{\infty\}$ are generated by vector fields of the form

$$\xi(z) = \xi_{-1}z^2 + \xi_0 z + \xi_1 \tag{4.60}$$

Exercise 4.17 *Perform a transformation $z \to z' = 1/z$ and show that regularity at $\{z = \infty\} = \{z' = 0\}$ implies $\xi_n = 0$ for $n < -1$. [Hint: to find the right transformation rule for $\xi(z)$ you should use that for a holomorphic vector field the combination $\xi(z)\partial/\partial z$ is invariant.]*

Finally, the *finite transformations* are the so-called Möbius transformations

$$z \to \frac{a + bz}{c + dz}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C}). \tag{4.61}$$

These transformations are easily seen to be generated by the operators L_1, L_0, L_{-1}. In particular, $L_1 = \partial_z$ generates translations $z \to z + a$ and $L_0 = z\partial_z$ generates dilatations $z \to bz$.

¿From the Virasoro algebra one easily shows that the algebra generated by these three operators is the $sl(2)$ Lie algebra

$$[L_{\pm 1}, L_0] = \pm L_{\pm 1}, \qquad [L_1, L_{-1}] = 2L_0 \tag{4.62}$$

Note that the central charge c drops out.

We claim, but will not prove, that the Hilbert space $\mathcal{H}$ of any two-dimensional unitary CFT contains a unique vacuum state $|0\rangle$ that is $SL(2,\mathbb{C})$ invariant, and satisfies

$$L_n|0\rangle = 0, \qquad n \geq -1. \tag{4.63}$$

$$\langle 0|L_n = 0, \qquad n \leq 1. \tag{4.64}$$

4.4 Correlation functions

In any CFT we have a set of local operators $\phi_i(z,\bar{z})$ which are operators on the Hilbert space of the field theory. *Correlation functions* of such operators are defined by

$$\langle \phi_1(z_1,\bar{z}_1)\cdots\phi_n(z_n,\bar{z}_n)\rangle = \langle 0|\mathsf{T}\left(\phi_1(z_1,\bar{z}_1)\cdots\phi_n(z_n,\bar{z}_n)\right)|0\rangle. \tag{4.65}$$

In this equation, we introduced the time-ordering operator T for bosonic operators:

$$\mathsf{T}(\phi_1(z)\phi_2(w)) = \begin{cases} \phi_1(z)\phi_2(w), & |z| > |w|, \\ \phi_2(w)\phi_1(z), & |z| < |w|. \end{cases} \tag{4.66}$$

Exercise 4.18 *Use the mode expansion of $\partial x(z)$ to prove that*

$$\langle \partial x(z)\partial x(w)\rangle = -\frac{1}{(z-w)^2}. \tag{4.67}$$

Exercise 4.19 *Use the mode expansion of $T(z)$ to show that*

$$\langle T(z)T(w)\rangle = \frac{c/2}{(z-w)^4}. \tag{4.68}$$

Exercise 4.20 *If ϕ_i are primary fields then there are relations which allow us to calculate the correlation functions. In particular, we will calculate the two-point function*

$$G(z,w) = \langle \phi_1(z,\bar{z})\phi_2(w,\bar{w})\rangle. \tag{4.69}$$

Use the fact that $L_{0,\pm1}$ annihilate $|0\rangle$ and $\langle 0|$, to obtain the following relations:

- $L_{-1} \Rightarrow G(z,w) = G(z-w)$.
- $L_0 \Rightarrow G(z) \propto z^{-(h_1+h_2)}$
- $L_1 \Rightarrow G(z) \propto \delta_{h_1,h_2} z^{-2h_1}$.

Putting these facts together, we obtain

$$\langle \phi_1(z,\bar{z})\phi_2(w,\bar{w})\rangle = \begin{cases} (z-w)^{-2h_1}(\bar{z}-\bar{w})^{-2\bar{h}_1}, & \text{if } \phi_2 = \phi_1, \\ 0, & \text{otherwise.} \end{cases} \tag{4.70}$$

Exercise 4.21 *(Optional) Similarly, show that the three-point function is given by*

$$\langle \phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\phi_3(z_3,\bar{z}_3)\rangle \propto (z_1-z_2)^{-h_1-h_2+h_3}(z_2-z_3)^{-h_2-h_3+h_1}(z_3-z_1)^{-h_3-h_1+h_2} \times c.c. \tag{4.71}$$

Exercise 4.22 *(Optional) Finally, we calculate the four-point function. Show that conformal invariance implies that*

$$G = \langle \phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2)\phi_3(z_3, \bar{z}_3)\phi_4(z_4, \bar{z}_4)\rangle \tag{4.72}$$

is of the form

$$G = f(x, \bar{x}) \cdot \prod_{i<j} (z_i - z_j)^{-h_i - h_j + \sum_k h_k/3} \times c.c. \tag{4.73}$$

with f an arbitrary function of the "cross ratio"

$$x = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}. \tag{4.74}$$

4.5 Operator-state correspondence

For any local operator $\phi(z, \bar{z})$ we can define a state

$$|\phi\rangle = \lim_{z,\bar{z}\to 0} \phi(z, \bar{z})|0\rangle \tag{4.75}$$

Vice versa, we can define the local operator inserted at $z \neq 0$ by

$$\phi(z, \bar{z})|0\rangle = e^{zL_{-1} + \bar{z}\bar{L}_{-1}}|\phi\rangle. \tag{4.76}$$

For a primary field of weight $(h, \bar{h})$ the transformation rules give

$$[L_0, \phi(0)] = h\phi(0), \qquad [L_n, \phi(0)] = 0,\ n > 0, \tag{4.77}$$

Acting with these identities on the vacuum $|0\rangle$ we obtain

$$L_0 = h|\phi\rangle, \qquad L_n|\phi\rangle = 0,\ n > 0. \tag{4.78}$$

These very important relations show that $|\phi\rangle$ is a "highest weight" state of the Virasoro algebra. More physically, only the operators L_n with $n > 0$ can act non-trivially on the state $|\phi\rangle$. Acting with L_n raises the energy L_0 from h to $h + n$. So $|\phi\rangle$ is the state with lowest energy among all states obtained by acting with the Virasoro algebra.

Exercise 4.23 *Evaluate $\langle\phi|[L_n, L_{-n}]|\phi\rangle$ and show that for a unitary theory where the inner product is hermitean and positive (1) $c > 0$ and (2) $h \geq 0$.*

Exercise 4.24 *Show that $h = \bar{h} = 0 \iff |\phi\rangle = |0\rangle$.*

Exercise 4.25 *(optional) Show that*

$$\langle\phi| = \lim_{z,\bar{z}\to\infty} z^{2h}\bar{z}^{2\bar{h}}. \tag{4.79}$$

For the free scalar field we have two kinds of operators. First we can consider expressions like

$$\partial x, \partial^2 x, \ldots \tag{4.80}$$

Then we can make (normal ordered) composite of these fields like

$$:\partial x\, \partial^2 x: \tag{4.81}$$

Finally we can introduce the vertex operators

$$: e^{ipx} : \tag{4.82}$$

and products of vertex operators and derivatives of x such as

$$: (\partial x)^2 e^{ipx} : \tag{4.83}$$

In fact, because of the operator-state correspondence, for any state in the Hilbert space $\mathcal{H}$ of the scalar field we have a corresponding operator. The identification is simple, to the state

$$|\varphi\rangle = \alpha_{-n_1} \cdots \alpha_{-n_s} |p\rangle. \tag{4.84}$$

the operator-state correspondence relates the operators

$$\varphi(z) =: \partial^{n_1} x \cdots \partial^{n_s} x e^{ipx} : \tag{4.85}$$

Let us check this in a few examples. We have seen that the field $\partial x(z)$ has a mode expansion

$$\partial x(z)|0\rangle = \sum_{n \in \mathbb{Z}} -iz^{-n-1} \alpha_n |0\rangle \tag{4.86}$$

If we put $z = 0$ we see why the modes α_n with $n \geq 0$ are annihilation modes. Otherwise the field $\partial x(z)$ would not be regular in the origin. Therefore we have the correspondence

$$\partial x(0) \quad \sim \quad \alpha_{-1}|0\rangle \tag{4.87}$$

One easily checks that similarly we have

$$\partial^n x(0) \quad \sim \quad \alpha_{-n}|0\rangle \tag{4.88}$$

Finally, for the vertex operator we have

$$: e^{ipx(0)}|0\rangle = e^{ipx_0}|0\rangle = |p\rangle, \tag{4.89}$$

with x_0 defined by the mode expansion

$$x(\sigma, \tau) = x_0 + p\tau + osc. \tag{4.90}$$

4.6 Operator products

There is a "theorem" that states that any product of two local operators in a quantum field theory can be again re-expressed in terms of local operators. In two-dimensional CFT such an expansion can be written as a power series in the distance between the two points where the operators are evaluated:

$$\phi_i(z, \bar{z})\phi_j(w, \bar{w}) \sim \sum_k c_{ijk}(z-w)^{-h_{ijk}}(\bar{z}-\bar{w})^{-\bar{h}_{ijk}}\phi_k(w, \bar{w}). \tag{4.91}$$

Here the ϕ_k are some basis of all local operators. Such an expansion is called an *Operator Product Expansion (OPE)*. We will now use this notation to rewrite some of the relations we have found earlier.

Consider for example the formula for the change of an operator under arbitrary conformal reparametrizations:

$$\partial_\xi \phi(w,\bar{w}) = [\oint_{\mathcal{C}} \frac{dz}{2\pi i}\xi(z)T(z), \phi(w,\bar{w})] = (\xi(w)\partial_w + h\partial_w\xi(w))\phi(w,\bar{w}). \tag{4.92}$$

We would like to know how to interpret the expression

$$[\oint_{\mathcal{C}} \frac{dz}{2\pi i}\xi(z)T(z), \phi(w,\bar{w})] \tag{4.93}$$

and how to choose the contour $\mathcal{C}$. Use time ordering T we can show that

$$\partial_\xi \phi(w,\bar{w}) = \left(\oint_{|z|>|w|} \frac{dz}{2\pi i} - \oint_{|z|<|w|} \frac{dz}{2\pi i}\right) \xi(z)\mathsf{T}\left(T(z)\phi(w,\bar{w})\right) \tag{4.94}$$

Now we can deform the contour to show that it equals

$$\partial_\xi \phi(w,\bar{w}) = \oint_{\mathcal{C}_w} \frac{dz}{2\pi i}\mathsf{T}\left(T(z)\phi(w,\bar{w})\right) \tag{4.95}$$

with $\mathcal{C}_w = \{|z-w| = \epsilon\}$ for $\epsilon > 0$ small enough, the contour encircling w.

From this we can now, using Cauchy's theorem[3], read off the singular terms in the OPE of the stress tensor and the primary field ϕ:

$$T(z)\phi(w,\bar{w}) \sim \frac{h\,\phi(w,\bar{w})}{(z-w)^2} + \frac{\partial\phi(w,\bar{w})}{z-w}. \tag{4.96}$$

This particular simple form is the "signature" of a primary field.

Exercise 4.26 *Show repeating all the steps above that the OPE of T with itself takes the form*

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}. \tag{4.97}$$

Exercise 4.27 *(optional) Show that for the free scalar field*

$$\partial x(z)\cdot\partial x(w) \sim -\frac{1}{(z-w)^2}. \tag{4.98}$$

4.7 OPE's of the free scalar field

We define the stress tensor operator for the free scalar field as

$$T(z) = -\frac{1}{2} : \partial x(z)\partial x(z) := \lim_{z\to w} -\frac{1}{2}\left(\partial x(z)\partial x(w) + (z-w)^{-2}\right) \tag{4.99}$$

Here we introduced a new definition of the normal ordered product; it can be shown to be equivalent to the definition we gave in the previous case. We now want to compute the central charge in the Virasoro algebra. This can be done by finding the singular behaviour

$$T(z)T(w) \sim \frac{1}{2}\frac{1}{(z-w)^4} + \ldots, \tag{4.100}$$

which gives the result $c = 1$.

[3] $\oint_{\mathcal{C}_0} \frac{dz}{2\pi i} f(z) z^{-n-1} = \frac{1}{n!} f^{(n)}(0)$.

Exercise 4.28 *Show that for d free scalar fields with stress tensor $T = -\frac{1}{2} : \partial x^\mu \partial x_\mu :$ the central charge is $c = d$.*

In the previous lecture, We have already introduced vertex operators. Wick's theorem can now be written as

$$e^{ipx(z)} \cdot e^{iqx(w)} \sim |z-w|^{2pq} e^{ipx(z)+iqx(w)} \tag{4.101}$$

We now want to compute the conformal dimension of the vertex operators.

Exercise 4.29 *Use that*

$$\partial x(z) \cdot e^{ipx(w)} \sim \frac{ip}{z-w} e^{ipx(w)}. \tag{4.102}$$

to show that

$$T(z) \cdot e^{ipx(w)} \sim \frac{\frac{1}{2}p^2}{(z-w)^2} e^{ipx(w)} + \frac{ip\partial_w x}{z-w} e^{ipx(w)} \tag{4.103}$$

From this we derive the conformal dimension of the vertex operators:

$$h = \frac{1}{2}p^2. \tag{4.104}$$

Note that these vertex operators are not real:

$$\left(e^{ipx}\right)^\dagger = e^{-ipx} \tag{4.105}$$

This explains the form of the two-point function:

$$\langle e^{ipx(z)} e^{iqx(w)} \rangle = \frac{1}{|z-w|^{2p^2}} \delta_{p+q,0}. \tag{4.106}$$

Exercise 4.30 *Use Wick's theorem to prove*

$$\left\langle \prod_i e^{ip_i x(z_i)} \right\rangle = \prod_{i<j} |z_i - z_j|^{2p_i p_j} \delta_{\sum_i p_i, 0}. \tag{4.107}$$

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Fifth lecture

5.1 The Polyakov path integral

In the third lecture, we have seen how to calculate probability amplitudes for scattering processes using an operator-state formalism. In quantum field theories for point particles, there is another way to calculate these quantities: using a path integral formalism and a partition function

$$Z = \int D\phi \, e^{-S[\phi]}. \tag{5.1}$$

where $\int D\phi$ stands for a path integral over all possible configurations of all the fields. Note that we used a Euclidean formalism, so there are no factors of i in the exponent.

In this lecture, we want to look at the equivalent of (5.1) for the string. The reason for this is that this formalism gives us a much more powerful tool to evaluate amplitudes in string theories than the operator-state formalism does. In fact, when we go beyond one loop, we cannot use the latter formalism anymore.

We will start from the general form of the action, i. e. the action including the metric h:

$$S[h, X] = -\frac{T}{2} \int d^2\sigma \, \sqrt{h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu. \tag{5.2}$$

To calculate amplitudes, we want to use the partition function

$$Z = \int DhDX e^{-S[h,X]}. \tag{5.3}$$

However, because of the gauge freedom of the theory, this partition function will lead to all kinds of infinities. Particularly, since we have reparametrization invariance and Weyl-invariance, many different h-fields correspond to the same physical configuration. This means we do an enormous over-counting in the partition function above. We would like to solve this problem by taking into account only one configuration of h for each physical configuration, i. e. by fixing the gauge.

The way in which we fix the gauge will depend on the topological properties of the wold sheet. We will only look at closed strings for the moment, so the topologically different world sheets are the sphere, the torus, and more generally the closed Riemann surface with n handles (see figure 5.1). In general, there will be vertex operators on these surfaces, but for now we will restrict ourselves to the surfaces without external states, and hence without vertex operators. In this section we consider the path integral over the sphere; in the next two sections we will discuss the torus.

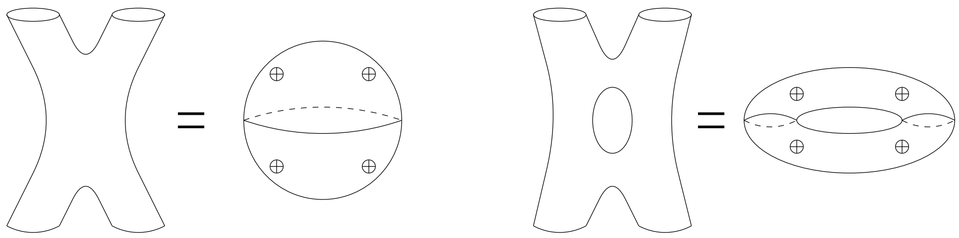

Figure 5.1: Scattering diagrams of closed strings are conformally equivalent to closed Riemann surfaces.

It can be shown that on a sphere, it is alway possible to gauge the world sheet metric to the Minkowskian metric

$$h^{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{5.4}$$

in every point σ.

Exercise 5.31 *What does this correspond to in light-cone coordinates? In other words, what are the values of h_{++}, h_{+-} and h_{--}? Call $h_{+-} = \phi$; after we fix the gauge for the other two components, the string action is no longer dependent on ϕ. Why not?*

From this exercise, we see that what we have to do is put $h_{++} = h_{--} = 0$. The remaining integral over ϕ can then simply be discarded, since all ϕ correspond to the same physical configuration.

The simplest way to make the two h-components vanish would be to insert delta functions $\delta(h_{++})$, $\delta(h_{--})$ into the path integral. Of course, we are not allowed to do this but what we can do is insert 1 into the integral in a shape that contains these integrals. The simplest shape would be

$$1 = \int Dh_{++}\, Dh_{--}\, \delta(h_{++})\delta(h_{--}), \tag{5.5}$$

but this is of no direct help to us. First of all, we already have an integration over the h-components in our original integral; secondly, it is not clear how we could discard the new integrations afterwards. We can however rewrite this expression in a way that is more useful to us. Remember that we have a continuous group of symmetries – reparametrizations and Weyl-scalings – that transform h, but leave the action invariant. Instead of integrating over components of h, we could 'change variables' and integrate over the elements of this group. Since any metric can be gauged to the Minkowskian metric, integrating a delta function over all possible metrics is the same as integrating it over all group elements. Of course, if we change variables, we have to include the Jacobian[4]:

$$1 = \int Dg\, \delta(h_{++})\delta(h_{--}) \det\left(\frac{\delta h_{++}}{\delta g}\right) \det\left(\frac{\delta h_{--}}{\delta g}\right). \tag{5.6}$$

[4]Mathematically, we are being quite sloppy here. Since we are talking about path integrals instead of normal integrals, we are dealing with all kinds of infinities. For example, the Jacobian is usually the determinant of a finite matrix; here it is the determinant of a transformation that can be viewed as an infinite matrix. However, our goal is not to give a mathematically perfect derivation of the Polyakov path integral, but to make intuitively clear why it has the shape it has.

If we insert this identity in the partition function (5.3) we obtain

$$Z = \int Dg\, Dh\, DX\, e^{-S[h,X]} \delta(\hat{h}_{++})\delta(\hat{h}_{--}) \det\left(\frac{\delta \hat{h}_{++}}{\delta g}\right) \det\left(\frac{\delta \hat{h}_{--}}{\delta g}\right). \tag{5.7}$$

Note that we put a hat on some of the h in this equation. The reason is that these h are not the h that we integrate over, but the h that is obtained from that by acting on it with the symmetry group element g. Now there is an important observation: since the action is invariant under the symmetry group, we can write

$$S[h,X] = S[\hat{h},X], \tag{5.8}$$

and then all the h's in the partition function are the same. Now if we change the variable of integration from h to $\hat{h}$ (which is a simple one-to-one mapping for every g, and hence is allowed), we see that nothing in the expression for the partition function depends on g anymore.

Now we have achieved our goal: we have completely seperated the over-counting that was a result of our gauge freedom: it is expressed in the integrals over g and ϕ (which is part of the integration over h), which do nothing more than multiplying by an (infinite) factor. We can now discard this factor, and arrive at

$$Z = \int Dh_{++}\, Dh_{--}\, DX\, e^{-S[h,X]} \delta(h_{++})\delta(h_{--}) \det\left(\frac{\delta h_{++}}{\delta g}\right) \det\left(\frac{\delta h_{--}}{\delta g}\right). \tag{5.9}$$

The integral over h_{++} and h_{--} can now be carried out, and the delta functions can be removed. Finally, we want to obtain an expression for the determinants that are in this formula. To calculate these, we have to check how h_{++} and h_{--} transform under world sheet reparametrizations $\sigma^{\pm} \to \sigma^{\pm} + \xi^{\pm}$.

Exercise 5.32 *Use equation (1.11) to show that these transformations are*

$$\begin{aligned} \delta h_{++} &= 2\nabla_{+}\xi_{+} \\ \delta h_{--} &= 2\nabla_{-}\xi_{-}, \end{aligned} \tag{5.10}$$

where ∇ stands for the covariant derivative

$$\nabla_\alpha \xi_\beta = \partial_\alpha \xi_\beta - \Gamma^{\gamma}_{\alpha\beta}\xi_\gamma \tag{5.11}$$

and Γ are the Christoffel symbols

$$\Gamma^{\gamma}_{\alpha\beta} = \frac{1}{2} h^{\gamma\delta} \left\{\partial_\alpha h_{\beta\delta} + \partial_\beta h_{\alpha\delta} - \partial_\delta h_{\alpha\beta}\right\}. \tag{5.12}$$

From this exercise, we see that we have to evaluate the determinants of the operators $\nabla_{\pm}$. To do this, we use a fact that we know from quantum field theory: the determinant of an operator can be calculated as a Gaussian integral over two *anticommuting* variables:

$$\begin{aligned} \det\left(\frac{\delta h_{++}}{\delta g}\right) &= \int Dc^{-} Db_{--} \exp\left\{-\frac{1}{\pi}\int d^2\sigma c^{-}\nabla_{+} b_{--}\right\}. \\ \det\left(\frac{\delta h_{--}}{\delta g}\right) &= \int Dc^{+} Db_{++} \exp\left\{-\frac{1}{\pi}\int d^2\sigma c^{+}\nabla_{-} b_{++}\right\}. \end{aligned} \tag{5.13}$$

The labels on the anticommuting variables (especially on b) may seem a bit strange, but we adapt to the literature; it turns out to be useful to regard these variables as components of two tensors.

Putting all the pieces together, we arrive at the final form of the partition function:

$$Z = \int DX\, Dc\, Db\, e^{-S[X,b,c]}, \tag{5.14}$$

where S is the action

$$S = \frac{T}{2} \int d^2\sigma \left\{ \partial_\alpha X^\mu \partial^\alpha X_\mu + 2c^- \nabla_+ b_{--} + 2c^+ \nabla_- b_{++} \right\}. \tag{5.15}$$

5.2 Loop amplitudes

On the sphere, we were able to use conformal invariance to completely remove the dependence of the partition function on the shape (i. e. the metric) of the sphere. For the torus, this is no longer possible, since not all tori are conformally equivalent. In particular, what remains is a complex parameter τ that determines the shape of the torus.

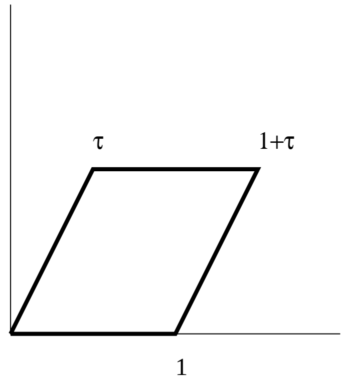

Figure 5.2: A torus is created by identifying opposite edges of a parallellogram.

This parameter τ can be thought of as follows: a torus is created by taking a parallellogram, and gluing opposite edges together. By rescaling and rotating (which are both conformal transformations), we can embed this parallellogram in the complex plane in such a way that one of the corners is the origin, and the other corners are at $1, \tau$ and $1 + \tau$ (see figure 5.2), where τ is in the upper right quarter of the plane. In other words, the parallellogram is defined by the relations

$$\begin{aligned} w &\equiv w + 1 \\ w &\equiv w + \tau \end{aligned} \tag{5.16}$$

If we conformally map w to $z = e^{2\pi i w}$, the first equivalence is automatically satisfied. The second equivalence translates into

$$z \equiv qz, \qquad q = e^{2\pi i \tau}. \tag{5.17}$$

In a picture, this looks like figure 5.3. We can take an annulus as the fundamental region, and identify the inner and outer circle. However, in this identification, there is a twist: the point 1 is identified with the point q.

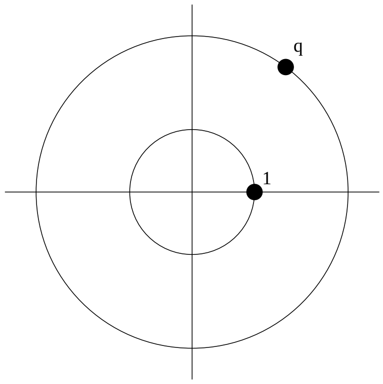

Figure 5.3: A torus as an annulus in the z-plane.

To keep things clear, in this section we will calculate the partition function Z for a single free scalar field $X(\tau)$ on the torus. As we said above, Z will depend on τ. In general, Z will not be a holomorphic function, so we write $Z(\tau,\bar{\tau})$. We also write $\tau=\tau_1+i\tau_2$.

Once again, look at figure 5.3. If we view this picture as a string diagram, we see that what happens between the inner and the outer circle are two things. First of all, (remember time runs radially), there is a propagation in time. The duration of this propagation is $2\pi\tau_2$. Second of all, there is a translation along the spatial direction (i. e. along the circle) of length $2\pi\tau_1$. The operator that achieves both these facts is

$$\begin{aligned} 2\pi i\tau_1 P - 2\pi\tau_2 H &= 2\pi L_0(i\tau_1-\tau_2)+2\pi\bar{L}_0(-i\tau_1-\tau_2)+\frac{\pi\tau_2}{24} \\ &= 2\pi i L_0\tau - 2\pi i\bar{L}_0\bar{\tau}+\frac{\pi\tau_2}{24}, \end{aligned} \tag{5.18}$$

where we used $P=L_0-\bar{L}_0$ and $H=L_0+\bar{L}_0-\frac{1}{12}$. The latter constant is the central charge for a single free bosonic field.

It can be shown, that calculating the partition function is equivalent to exponentiating (5.18) and taking the trace of this operator over the Hilbert space[5]:

$$Z(\tau,\bar{\tau})=(q\bar{q})^{-1/24}\mathrm{Tr}(q^{L_0}\bar{q}^{\bar{L}_0}). \tag{5.19}$$

Taking the trace over the Hilbert space means sandwiching an operator between all basis states and summing (or if necessary integrating) over these states.

Exercise 5.33 *Show that for a finite dimensional Hilbert space, this prescription corresponds to the familiar definition of the trace of a matrix.*

In evaluating this expression, we use the fact that L_0 on a certain state counts the right-moving oscillations and adds the square of the momentum; $\bar{L}_0$ does the same on the right-moving side (compare (2.11)).

Exercise 5.34 *Show that this means that*

$$Z(\tau,\bar{\tau})=C(q\bar{q})^{-1/24}\int dk\,\exp(-\pi\tau_2\alpha' k^2)\prod_n\sum_{N_n,\bar{N}_n=0}^{\infty} q^{nN_n}\bar{q}^{n\bar{N}_n}, \tag{5.20}$$

[5]We will not prove this here; for an example, the reader is referred to *Evaluation of the one loop string path integral* by Polchinski; Commun. Math. Phys. 104 (1986), pp. 37-47

where C is a normalizaton constant.

The Gaussian integral is ealily evaluated; for the sums we use the identity

$$\sum_{N=0}^{\infty} a^N = \frac{1}{1-a}. \tag{5.21}$$

Inserting this, we arrive at the final expression

$$Z(\tau,\bar{\tau}) = C(4\pi^2\alpha'\tau_2)^{-1/2}\left|\prod_{n=1}^{\infty}\frac{q^{-1/24}}{(1-q^n)}\right|^2. \tag{5.22}$$

5.3 Modular invariance

In the previous section, we have seen that intgegration over all tori comes down to integration over a parameter τ. However, not all different τ correspond to different tori. As an example, look at figure 5.4. If we want to construct a torus, we take some fundamental parallellogram in the plane, and impose periodicity – which is the same as gluing the opposite sides of the parallellogram together. However, note that there is no difference between taking the parallelogram in solid lines as the fundamental one and taking the parallellogram in dashed lines; both correspond to the same torus.

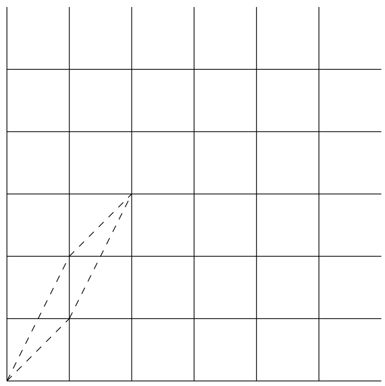

Figure 5.4: Two equivalent fundamental regions for a torus.

It can be shown that two fundamental regions with sides l_1, l_2 and l_1', l_2' correspond to the same torus if and only if

$$\begin{pmatrix} l_1' \\ l_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} l_1 \\ l_2 \end{pmatrix}, \tag{5.23}$$

where the matrix is an element of $SL(2,\mathbb{Z})$, i. e. a, b, c, d are integers and the determinant $ad - bc = 1$. The proof of this fact is not very difficult; it consists of the following three observations:

- The endpoints of l_1' and l_2' should be identified with their starting points (i. e. the origin). This means these points have to be lattice points in the original lattice. In other words: a, b, c, d have to be integers.

- From elementary vector calculus, we know that the absolute value of the determinant of the transformation matrix (the Jacobian) gives the change in area of the parallellogram. Therefore, $ad - bc = \pm 1$. By changing the order of l_1' and l_2' (i. e. interchanging two rows), we can always choose the determinant to be $+1$.

- From the above two points, we see that any transformation that gives the same torus has to be an $SL(2,\mathbb{Z})$-transformation. To prove the converse, note that we can cover the entire plane with parallellograms with sides l_1', l_2'. Since the area of the new parallelogram is equal to the area of the old one, there cannot be any lattice points of the old lattice inside the fundamental area of the first one. The converse of course also holds, since the inverse of an $SL(2,\mathbb{Z})$-transformation is again an $SL(2,\mathbb{Z})$-transformation. Therefore, the two lattices consist of the same points, and hence the tori are the same.

Exercise 5.35 *Translate the equivalence between the sides of the parallellogram to an equivalence relation for the values of* τ.

All of this means that we should not integrate τ over the entire space $\mathbb{C}$, but over a part of this space, which can be indicated by $\mathbb{C}/SL(2,\mathbb{Z})$. For higher-loop amplitudes, a similar reasoning holds, and we obtain some parameters in a relatively simple space (called the *Teichmüller space*), where we should integrate only over part of this space (called the *Moduli space*).

Caputcollege string theory 1998-99

Sixth lecture

6.1 Dualities

A very important concept in string theory is the concept of a *duality*. In general, a duality between two theories (not necessarily string theories) is an invertible map between the theories that sends states to states, and preserves the interactions (i. e. the amplitudes) and the symmetries of the theory. If two theories are dual to one another, we can in a sense identify them, because mathematically they lead to the same results.

To see why dualities can be very useful, we have to realize what 'doing quantum field theory' generally means. Usually, we start from some action, and obtain equations of motion from this action. However, only in a few cases it is possible to exactly solve these equations of motion. In general, we have to apply some kind of perturbation theory to obtain our results. In quantum field theory, this corresponds to drawing Feynman diagrams: the readers who have taken a course in quantum field theory will know that the number of loops in a Feynman diagram corresponds to the order of $\hbar$ (the expansion parameter) in a perturbative series.

Now in general, perturbation theory does not give us all the possible information about a certain theory. This can be compared to Taylor expanding ordinary functions: not every function has a Taylor expansion. The famous example is

$$f(x) = e^{-\frac{1}{x^2}}, \tag{6.1}$$

which has all derivatives equal to 0 at the origin. In the same way, not all phenomena in a field theory can be described using a perturbative series.

Notice that which phenomena can be described depends crucially on the point in which we make a perturbative expansion. For example, if we would have Taylor expanded (6.1) around $x = 1$ instead of $x = 0$, we would have obtained a non-zero Taylor expansion.

This is where dualities come into our story. A duality maps one theory into another theory, but in general, it does not map the point around which we perturb in one theory onto the point around which we perturb in the other theory. This has the great advantage that we are able to 'see' things in one description which are invisible in the other. When applying this to string theory, we will obtain some remarkable results.

Two kinds of dualities have a special importance. In figure 6.1a, we have drawn a general duality. The fact that we can only 'see' part of the theory is indicated by the colored circle. In general, a duality maps the perturbative region of one theory into some region of another theory, and vice versa. One special situation arises when the theory on both sides of the map is the same theory; in this case we speak of a *self-duality* (figure 6.1b). Another very important case

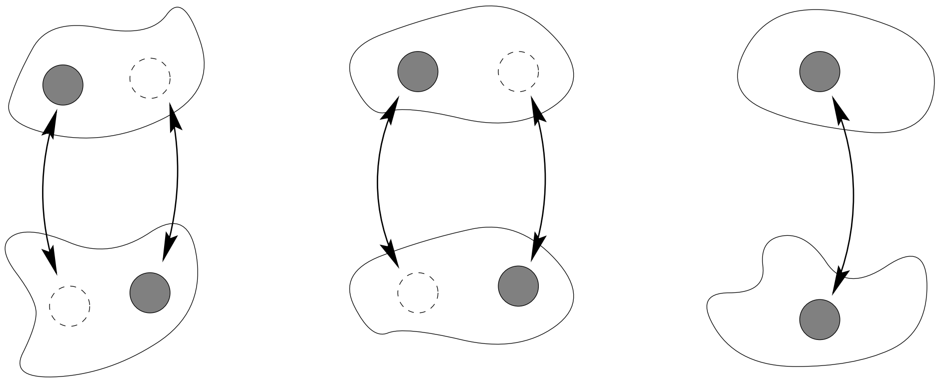

Figure 6.1: A general duality, a self-duality and a T-duality.

is the case where a duality maps the perturbative region of one theory onto the perturbative region of another theory. This kind of duality is called *T-duality* (where the T stands for 'Target space' – the reason for this will become clear in this lecture), and it is depicted in figure 6.1c.

6.2 Compactifying closed string theory on a circle

In a previous lecture, we already mentioned the fact that a bosonic string theory can only be consistent in 26 space-time dimensions. Of course, this does not correspond to our everyday experience, in which we observe only four dimensions of space-time. Somehow, to obtain a physical theory, we will have to get rid of the 22 extra dimensions.

The way in which this can be done was invented by Kaluza and Klein in the beginning of this century. Suppose we have a two-dimensional sheet of paper. If we make this sheet into a tube by rolling it up, and make the rolling tighter and tighter, in the end we wil have a very small, long tube. If we roll tight enough, it wil hardly be possible to distinguish this tube from a truly one-dimensional object. In the same way, we can get rid of the 22 extra dimensions of bosonic string theory: instead of describing this part of the theory on $\mathbb{R}^{22}$, we can take a closed 22-dimensional manifold (such as a 22-torus), and make this manifold very small. Effectively, what we are left with will then be a 4-dimensional theory. This procedure is called *compactification* of the theory.

To see how this works in practice, let us not compactify 22 of the space-time dimensions, but only one – say X^{25}. Compactifying this dimension on a circle means that we make the identification

$$X^{25} \sim X^{25} + 2\pi R, \tag{6.2}$$

where R is the radius of the circle on which we compactify. What this equation says is simply: if I walk in the X^{25}-direction for a distance $2\pi R$, I am back at my starting point. Of course, this means that X^{25} is also identified with $X^{25} + 2\pi m R$ for any $m \in \mathbb{Z}$.

Now let us look at a theory of closed strings. In the first lecture, we wrote down the solutions to the equations of motion for a closed string. There, we used the fact that going around the string once should bring us back to the same point in space-time; i. e.

$$X^\mu(\tau, \sigma + 2\pi) = X^\mu(\tau, \sigma). \tag{6.3}$$

However, now we also have (6.2), which tells us that the point on the right hand side of this equation is identified to many other points. Therefore, for $\mu = 25$ we need to replace this condition by

$$X^{25}(\tau, \sigma + 2\pi) = X^{25}(\tau, \sigma) + 2\pi m R. \tag{6.4}$$

It is not hard to give a physical interpretation for the number m. The above equation tells us that going around the string once, we go around the compactified dimension m times. In other words: m tells us how often the string winds around the compactified dimension; it is the *winding number* of the closed string.

The solutions to the equations of motion with (6.4) replacing (6.3) are easily given. For $\mu = 0, 1, \ldots, 24$, nothing changes. For $\mu = 25$, we expect an extra term linear in sigma in the solutions. Recall the closed string mode expansion

$$\begin{aligned} X_R^{25} &= \frac{1}{2}x^{25} + l\alpha_0^{25}(\tau - \sigma) + \text{oscillators} \\ X_L^{25} &= \frac{1}{2}x^{25} + l\tilde{\alpha}_0^{25}(\tau + \sigma) + \text{oscillators}. \end{aligned} \tag{6.5}$$

Originally, there was no term linear in σ, and we obtained $\alpha_0^{25} = \tilde{\alpha}_0^{25} = \frac{1}{2}lp^{25}$. We can now introduce the term linear in σ by allowing α_0^{25} and $\tilde{\alpha}_0^{25}$ to be different, in such a way that

$$\tilde{\alpha}_0^{25} - \alpha_0^{25} = \frac{mR}{l}. \tag{6.6}$$

Now that we have compactified X^{25}, there is another thing we should realize. This is something we know from quantum mechanics: if we go around a compactified dimension once, the quantum mechanical wave function should come back to itself. Since the wavelength in the 25-direction of such a wave function is $\lambda = 2\pi/p^{25}$, this means that $2\pi n/p^{25} = 2\pi R$, for some $n \in \mathbb{Z}$, or equivalently $p^{25} = n/R$. Since $l^2 p^\mu$ is the coefficient of τ in the mode expansion, we deduce that

$$\tilde{\alpha}_0^{25} + \alpha_0^{25} = \frac{ln}{R}. \tag{6.7}$$

We now have two relations for $\tilde{\alpha}_0^{25}$ and α_0^{25}, so we can express them in terms of m and n. We find

$$\begin{aligned} \alpha_0^{25} &= \frac{ln}{2R} - \frac{mR}{2l} \\ \tilde{\alpha}_0^{25} &= \frac{ln}{2R} + \frac{mR}{2l}. \end{aligned} \tag{6.8}$$

These are the new constraints we have to impose on our quantum mechanical states to obtain a theory that is compactified on a circle of radius R.

It is instructive to calculate the mass spectrum for the compactified theory. Since we do not wish to 'see' the compactified dimension anymore (i. e. we want to interpret it as an internal degree of freedom of the string; not as a dimension of space-time), we still have the relation $M^2 = -p^\mu p_\mu$, but from a 25-dimensional point of view we should now let μ run form 0 to 24. To say it in other words: by doing this, we create a mass that is invariant under the 25-dimensional Lorentz group $SO(24,1)$, which is the symmetry group of the flat, uncompactified part of space-time.

On the other hand, the relation $L_0 - 1 = 0$ still holds, where L_0 is defined in the usual way: containing a sum over *all* dimensions. Using these observations, we see that the mass spectrum of the closed string on a torus of radius R becomes

$$\begin{aligned} M^2 &= -\sum_{\mu=0}^{24} p^\mu p_\mu \\ &= \frac{4}{\alpha'}\left((\alpha_0^{25})^2 - \sum_{\mu=0}^{25}(\alpha_0^\mu)^2\right) \\ &= \frac{4}{\alpha'}\left((\alpha_0^{25})^2 + 2N - 2\right) \end{aligned} \tag{6.9}$$

We see that there is an extra contribution to the mass as a result of the compactification. Of course, we could have just as well expressed this formula in terms of the left-moving oscillators.

Finally, we can of course use the same procedure to compactify d dimensions on a torus T^d. Actually, this is not the only d-dimensional manifold on which we can compactify. For example, if we want to compactify two dimensions, why not take a very small sphere instead of a very small torus?

As it turns out, it is indeed possible to compactify on other manifolds than just tori. However, not any manifold will do – the allowed manifolds are subject to some very important restrictions. Yet still, there are lots of different manifolds we can consider. Unfortunately, we will not have the time to discuss all the interesting aspects of compactification within these lectures.

6.3 T-duality of closed strings

The compactified theory we created has a very special feature. Look at the constraint equations (6.8). They contain the numbers m and n, both multiplied by a certain factor. In fact, we can see a kind of 'symmetry': if we exchange m and n, and at the same time replace R by l^2/R, we obtain similar expressions. To be precise, this transformation sends

$$\begin{aligned} \alpha_0^{25} &\rightarrow -\alpha_0^{25} \\ \tilde{\alpha}_0^{25} &\rightarrow \tilde{\alpha}_0^{25}, \end{aligned} \tag{6.10}$$

Note the quotation marks we put around the word 'symmetry': we are not dealing with a symmetry of a certain theory here, but with a symmetry between two *different* theories. This observation leads us to the conjecture that the following two theories are related by a duality:

1. Bosonic string theory, compactified on a circle of radius R, describing a string of momentum quantum number n and winding number m.

2. Bosonic string theory, compactified on a circle of radius l^2/R, describing a string of momentum quantum number m and winding number n.

The precise prescription for the map is deduced from the relations (6.10): it can be generalized to the one-sided parity transformation

$$\begin{aligned} X_R^{25} &\rightarrow -X_R^{25} \\ X_L^{25} &\rightarrow X_L^{25}. \end{aligned} \tag{6.11}$$

The conjecture can indeed be proven to be right: both theories give us exactly the same results. We will not give a complete proof here, but one thing is easy to check: the mass spectrum for both theories is the same. This can immediately be seen from the mass formula (6.9): it is invariant uder the transformation (6.10).

It is now clear why this type of duality is called target space duality: it changes a property of the target space (i. e. the space in which the fields X^μ live), or in other words: it is a duality between two theories of the same type, but with different target spaces. That this duality is a perturbative one is obvious: the parity operation (6.11) maps perturbative states to perturbative ones.

We will end this section with an almost philosophical statement, which will turn out to be quite important in the next section. Because of the duality transformation, we see that a theory with compactification radius R can equally well be described as a theory with compactification radius l^2/R. This means that in a sense there is a 'minimal radius'

$$R_{\text{min}} = l. \tag{6.12}$$

A theory that is compactified on a circle of smaller radius is equivalent to one that lives on a circle of larger radius. This means that in the limit $R \to 0$, the circle that we want to remove in a way reappears! Phenomenologically, this is not a problem, since a natural l is of the order of the Planck length (remember we are using Planck units, so this is what $l \sim 1$ corresponds to), which is small enough to make the dimension invisible in everyday life. However, mathematically we will never be able to completely remove the extra dimension. But what happens when we look at open strings? These have no winding number, so the entire story that we told above does not hold for these objects. If we let $R \to 0$ in a theory of open strings, we still expect the extra dimension to disappear.

But if that is the case, what happens if we have a theory that contains both open and closed strings? Does the compactified dimension really disappear if we let $R \to 0$, or does it not? To resolve this apparent paradox, we will have to look at T-duality of open strings.

6.4 T-duality of open strings

We have seen that closed string theory on a circle of radius R is equivalent to the same theory on a circle with the inverse radius, ℓ^2/R. This can be rephrased as: closed strings on a circle (or higher dimensional torus) are T-dual to the same theory on the circle of inverse radius. Essential in establishing the invariance of the mass spectrum under the T-duality was that closed strings can wind around the compact dimension. The winding modes have energy proportional to R; this is precisely what is needed to provide the T-dual of the momentum states, of mass $1/R$.

One now may ask whether *open* strings have a T-duality as well, or alternatively, what theory emerges when we take the limit of small compactification radius. Since open strings have no winding modes (an open string winding around a circle can always be contracted to a point, hence the winding is not conserved), following the above argument we can conclude that open strings cannot be T-dual to themselves: there are no states in the spectrum that are T-dual to the momentum modes.

If we do not obtain conventional open string theory, let us then analyse what kind of theory we do get upon acting with the T-duality transformation on open string theory. Recall the open string mode expansion,

$$X^\mu = X^\mu_L(\sigma_2 - i\sigma_1) + X^\mu_R(\sigma_2 + i\sigma_1), \tag{6.13}$$

with

$$X^\mu_L = \frac{x^\mu + c}{2} + i\ell^2 p^\mu(\sigma_2 - i\sigma_1) + \frac{i\ell}{2}\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n e^{-(\sigma_2 - i\sigma_1)n}, \tag{6.14}$$

$$X^\mu_R = \frac{x^\mu - c}{2} + i\ell^2 p^\mu(\sigma_2 + i\sigma_1) + \frac{i\ell}{2}\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n e^{-(\sigma_2+i\sigma_1)n}, \tag{6.15}$$

where we included an arbitrary constant c that drops out of the sum $X_L + X_R$, so that x^μ is indeed the centre of mass position of the string at $\tau = 0$.

Now again assume that the direction X^{25} is periodic, $X^{25} \sim X^{25} + 2\pi R$. We carry out the prescription for T-duality, $X^{25}_R \to -X^{25}_R$. The resulting field $\tilde{X}^{25}$ is

$$\tilde{X}^{25} = X^{25}_L - X^{25}_R = c + 2\ell^2 p^{25}\sigma_1 + \ell\sum\frac{\alpha^{25}_n}{n}e^{-\sigma_2 n}\sin n\sigma_1. \tag{6.16}$$

All the other (uncompactified) coordinates have the same expansion as before the transformation.

¿From the $\tilde{X}^{25}$-expansion, we see first of all that the string carries no momentum in this direction (i.e. $\dot{X}^{25} = 0$), so that its centre of mass is fixed in this direction. Looking at the oscillator part of the expansion, the cosines associated to the Neumann boundary conditions we had are exchanged for sines. The positions of the string endpoints at $\sigma_1 = 0, \pi$ are fixed, so the boundary condition for the open string X^{25}-coordinate has changed from Neumann to Dirichlet. This is in line with the fact that T-duality exchanges the σ_1 and σ_2 coordinates: for the boundary conditions, this implies

$$\partial_{\sigma_1}X^{25} = 0 \to \partial_{\sigma_2}\tilde{X}^{25} = 0, \tag{6.17}$$

i.e. $\tilde{X}_{25}$ is constant at the boundary.

The zero-mode term no longer has the interpretation of momentum, since it is proportional to the space coordinate σ_1. Inserting $p^{25} = n/R$, we see that the endpoint at $\sigma = \pi$ is located at $\tilde{x}^{25} = c + 2\pi n\frac{\ell^2}{R} = 2\pi n\tilde{R}$. This kind of expression is familiar from the discussion of closed strings: this is a winding mode around the dual circle. Although it is now an open string that winds, the winding is stable since the endpoints of the string are stuck to the position $\tilde{X}^{25} = c$ on the circle: the string cannot unwind without breaking.

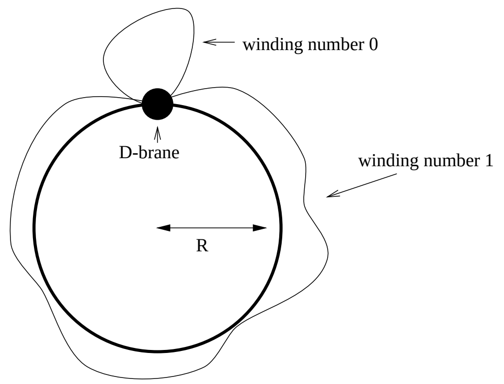

Figure 6.2: Transverse view of the D-brane on the (X^{25}) circle, with strings of winding numbers 0 and 1

Again T-duality has taken us to a theory defined on a circle with the dual radius $\tilde{R} = \ell^2/R$. The difference with closed string theory is that the open string theory we began with has only momentum around the circle, no winding, while the open string theory we ended up with has no momentum, but only winding: the string endpoints are stuck in the x^{25} direction, on the hyperplane $x^{25} = c$, but can wrap around the circle an integer number of times. The hyperplane $x^{25} = c$ is termed a *Dirichlet-brane*, or *D-brane*. We therefore learnt that if one compactifies open strings on a circle and lets the radius shrink below the string scale, we do not lose a dimension: although the open strings end on a 25-dimensional space, the D-brane, they probe the full 26 dimensions via their winding modes.

Instead of using T-duality, we could to begin with have imposed Dirichlet boundary conditions instead of Neumann ones (check that these are also consistent with the vanishing of the boundary contribution in the derivation of the equations of motion). A reason why these boundary conditions were ignored for a long time, is that the resulting theory, with the D-brane positioned at a specific hyperplane in spacetime, is not invariant under the target space Poincare symmetry: translations perpendicular to the D-brane change the value of c. One reason why this does not disqualify D-branes is that the Poincare-invariance of the uncompactified spacetime (which in the end we hope to be our four-dimensional spacetime) is not broken; it is only the symmetry in the compact directions which is affected. Another reason, which we will come to later, is that we may interpret D-branes as dynamical extended objects present in a Poincare-invariant background. Putting an extended D-brane in spacetime is then analogous to putting a particle or string in Minkowski space; this is not translationally invariant, but nevertheless that is not worrisome.

The D-brane we constructed is a $d-1$-dimensional plane, with d the total spacetime dimension (in this case, $d = 26$); unfortunately, it is called a D24-brane, with 24 the number of *spacelike* dimensions. The open strings cannot move off the D24-brane in the perpendicular, x^{25}-direction, due to the Dirichlet conditions, while they have Neumann conditions in the remaining $d-1$ directions. If we compactify another direction, x^{24}, and T-dualise in this direction, the same procedure of course will give a theory where both X^{24} and X^{25} will satisfy Dirichlet boundary conditions. The open strings in this theory can move freely in only $d-2$ directions, and are confined to a D$(d-3)$-brane. If we compactify on a large enough torus, we can therefore, by applying an appropriate number of T-dualities, create any Dp-brane we wish, with p running from 0 to $d-1$. In fact, the conventional open string theory we started out with, may in this light be considered as strings living on a spacetime filling D25-brane (D$(d-1)$-brane, in general).

6.5 Multiple branes

Open string theory is a theory of strings ending on a D$(d-1)$-brane, and by compactifying and T-dualising we can transform it in a theory on any given Dp-brane. However, in every case we only have one D-brane. To make things more interesting it is appropriate to look at a slight generalisation of open string theory.

Note that an open string has two special points, $\sigma_1 = 0, \pi$; for an oriented open string these two points are distinct. We now associate to both these points a label, i and j, running from 1 to N, and let a group $U(N)$ act on these labels, in the fundamental representation (N) on the label at $\sigma = 0$, and in the anti-fundamental representation $(\overline{N})$ at the endpoint $\sigma = \pi$. The original motivation for these *Chan-Paton* degrees of freedom came from the application of string theory as a description of the strong interactions: one can view the string as a gluon flux tube, with a quark and an anti-quark at the endpoints, on which the colour group $SU(3)$ acts. All open string states are now promoted to N by N matrices, transforming in the N^2-dimensional adjoint representation of $U(N)$. We write the states as

$$|\psi\rangle \otimes \lambda_{ij} \tag{6.18}$$

with ψ representing the oscillator content. The 'quarks' at the ends are not dynamical in the world sheet theory; an interaction of open strings now just contains an extra factor expressing that the indices should remain the same on a given boundary, as depicted in the scattering diagram figure(6.3).

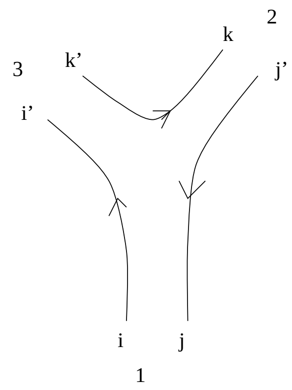

Figure 6.3: In a string scattering diagram, the Chan-Paton labels should not change along a boundary; hence we should have $i = i', j = j', k = k'$.

For the three-point scattering amplitude in the diagram this gives the extra factor

$$\delta^{ii'}\delta^{jj'}\delta^{kk'}\lambda^1_{ij}\lambda^2_{j'k}\lambda^3_{k'i'} = \mathrm{Tr}\lambda^1\lambda^2\lambda^3. \tag{6.19}$$

The amplitudes therefore become invariant under the $U(N)$ symmetry. In the target space theory, the $U(N)$ has the interpretation of a gauge symmetry; the massless vectors in the string spectrum are the gauge bosons of the non-abelian gauge symmetry $U(N)$.

The discussion so far did not mention D-branes. We will now see, however, that the Chan-Paton indices have a very natural interpretation as labels of different D-branes. To identify these we of course again make use of T-duality, transforming the space-filling D25-branes in D24-branes.

Again we identify X^{25} with $X^{25} + 2\pi mR$, compactifying on a circle. Now, however, we have to specify what happens to the Chan-Paton indices when we take a state around the circle: it need not necessarily come back to itself, but may be transformed by $U \in U(N)$,

$$\lambda_{ij} \to U_{ii'}\lambda_{i'j'}U^\dagger_{j'j}. \tag{6.20}$$

This so-called *Wilson line* breaks the gauge symmetry in the resulting 25-dimensional theory to the subgroup of $U(N)$ commuting with U. Let us pick a basis where U is diagonal,

$$U = \exp\mathrm{diag}\,(-i\theta_1, \ldots, -i\theta_N). \tag{6.21}$$

If all θ_i are distinct, the commuting subgroup is just $U(1)^N$ consisting of diagonal matrices; if two θ's are the same, there is one $U(2)$ factor, etc.

Let us look at the mode expansion of the strings with different Chan-Paton labels. The state $\psi \otimes \lambda_{ij}$ acquires a phase factor $\exp i(\theta_j - \theta_i)$ when transported around the circle. But this means that the momentum is no longer quantised in units of $1/R$: we need

$$\exp ip_{25}2\pi R = \exp -i(\theta_i - \theta_j), \text{ so } p_{25} = \frac{n}{R} + \frac{\theta_j - \theta_i}{2\pi R}, \tag{6.22}$$

so the momentum number n acquires a fractional addition $(\theta_j - \theta_i)/2\pi$. Now, when we are T-dualising, we know that momentum numbers become winding numbers, so we expect fractional winding. Concretely, we find for the zero-mode part of $\tilde{X}^{25}_{ij}$:

$$\tilde{X}^{25}_{ij} = c_{ij} + 2\tilde{R}\sigma_1(n + \frac{\theta_j - \theta_i}{2\pi}) + \dots \tag{6.23}$$

Diagonal string states ($i = j$) still wind an integer number of times around the circle before ending, off-diagonal ones wind over a fraction of the circle. These facts can be explained by assuming N D-branes to be present, located along the circle at angular intervals $\theta_i - \theta_j$. If in fact we choose the arbitrary constants c_{ij} to equal $\tilde{R}\theta_i$, then we have $X_{ij}(\sigma = 0) = \tilde{R}\theta_i, X_{ij}(\sigma = \pi) = \tilde{R}\theta_j$. The situation is drawn, for $N = 2$, in figure (6.4)

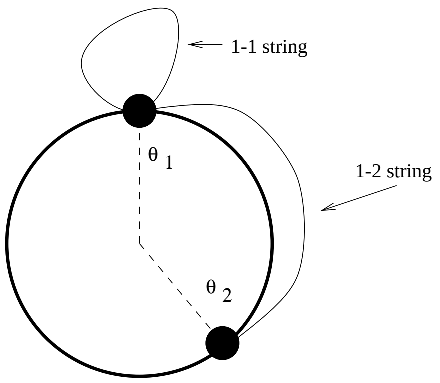

Figure 6.4: Two D-branes at transverse positions $\theta_1\tilde{R}$ and $\theta_2\tilde{R}$

In D-brane language, Chan-Paton indices therefore label the different branes strings can end on. There are N^2 different kinds of strings, characterised by the D-brane they start at and the one they end on (remember that we took oriented strings).

It is interesting to look at the spectrum of the theory: if all θ_i are different, only the vector states coming from strings beginning and ending on the same D-brane are massless; the off-diagonal string states have a mass contribution of $T(\theta_j - \theta_i)\tilde{R}$ from the winding zero mode. There are thus only N different massless vectors, each a gauge boson for a different $U(1)$ gauge group, consistent with the picture we had before dualising. If two θ_i are the same, two extra off-diagonal states become massless as well, and we have a massless vector transforming in the adjoint representation of $U(2)$. This implies that two D-branes on top of each other have a larger, non-abelian gauge symmetry! If the branes are moved apart, we say that the gauge symmetry is broken to the abelian subgroup $U(1) \times U(1)$, with the off-diagonal, noncommuting part of the gauge bosons acquiring a mass (the Higgs mechanism).

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Seventh lecture

7.1 Introduction

Consistency in string theory is a very important and very delicate subject. For example, we have mentioned several times that a bosonic string theory can only be consistent in $D = 26$ dimensions. But even the bosonic string theories in 26 dimensions are not completely consistent: we have seen that they contain a tachyon – a particle of negative mass squared. Since

$$M^2 = -k^2, \tag{7.1}$$

this means that such a particle has a space-like momentum, i. e. it moves faster than the speed of light and can do all kinds of unphysical things like going backwards in time. In a completely consistent string theory, we would like to get rid of this tachyon state.

Another ingredient that our string theories do not have is a collection of fermions. In nature, we clearly observe fermionic particles, but in our theories so far they have been excluded. It turns out that both of these problems can be solved at once: as soon as we introduce fermions into our theory, we obtain a way to project out the tachyon from our theory. Moreover, the theory will turn out to have a very special symmetry called *supersymmetry*. In this lecture and the following ones, we will see how all of this this comes about.

7.2 The supersymmetric point particle

Before discussing the superstring, we will discuss the one-dimensional analogue: the supersymmetric point particle. Its action is a generalization of the action of the massless bosonic point particle:

$$S = \int d\tau \left\{ -\frac{\dot{x}^2}{2e} + \frac{i\dot{x}^\mu \psi_\mu \chi}{e} - i\psi^\mu \dot{\psi}_\mu \right\}. \tag{7.2}$$

In this action, x^μ and e are the fields we already encountered in describing the bosonic point particle: x^μ is the embedding of the point particle world line into space-time, and e is an auxiliary field that can be interpreted as a metric along the world line. The field ψ^μ is the fermionic analogue of x^μ; the fact that it is fermionic is expressed by the fact that classically it is an *anticommuting* variable:

$$\psi^\mu \psi^\nu = -\psi^\nu \psi^\mu. \tag{7.3}$$

The field χ is the fermionic analogue of e; it is also an anticommuting variable.

Exercise 7.36 *Just like in the bosonic case, this action has reparametrization invariance. Show that if we let x and ψ transform in the expected way (i. e. $\delta x^\mu = \xi \dot{x}^\mu$ and $\delta \psi^\mu = \xi \dot{\psi}^\mu$), the transformation*

$$\begin{aligned} \delta e &= \frac{d}{d\tau}(\xi e) \\ \delta \chi &= \frac{d}{d\tau}(\xi \chi) \end{aligned} \tag{7.4}$$

is a symmetry of the action.

Since we inserted x and ψ in such a symmetric way, we may expect a symmetry between the two. In fact there turns out to be such a symmetry. Since x is a commuting field and ψ an anticommuting one, the parameter $\epsilon(\tau)$ that labels this symmetry will be an anticommuting parameter.

Exercise 7.37 *Show that the transformation*

$$\begin{aligned} \delta x^\mu &= i\psi^\mu \epsilon \\ \delta e &= i\chi\epsilon \\ \delta\chi &= \dot{\epsilon} \\ \delta\psi^\mu &= \frac{1}{2e}(\dot{x}^\mu + i\chi\psi^\mu)\epsilon \end{aligned} \tag{7.5}$$

is a symmetry of the action.

As in the bosonic gauge, we can use the invariances to gauge away the auxiliary fields. A nice choice is $e = 1$, $\chi = 0$. If we choose this gauge, we can make two important observations that clarify the fermionic nature of the variables ψ.

Exercise 7.38 *What is the equation of motion for the field χ? Express this equation in terms of the fields and their conjugate momenta in the chosen gauge. What familiar equation for ψ do we get?*

Secondly, observe that the momentum conjugate to ψ^μ is $i\psi^\mu$, so up to a factor of i the field ψ is its own conjugate momentum. When we quantize the theory, this will mean that

$$\{\psi^\mu, \psi^\nu\} = -\eta^{\mu\nu}. \tag{7.6}$$

Note that up to a factor of 2, these are the defining relations of the Clifford algebra, i. e. the defining algebra of the Dirac Γ-matrices.

7.3 The superstring action

Now, we want to add fermions to the bosonic string theory we have. First of all, we should realize what we mean when we speak of fermions. Usually, a fermion is introduced as an object whose wave-function obtains a minus sign when we rotate it over 2π. But in string theory, this immediately raises a question. What do we mean by rotating over 2π? In string theory, it is more or less ambiguous what our 'world' is: is it the two-dimensional world sheet, or is it the D-dimensional space time? Choosing which of these two is more fundamental has not been very important up till now, but when we speak of fermions, we have to make a decision. We can create objects that obtain a minus sign after a full rotation in space-time, or after a full rotation in the world sheet.

The fermions we observe in nature are clearly fermions in the space-time sense. However, since our Lagrangian is given in terms of world sheet fields, it is not immediately clear how we should describe these objects. It is mathematically much easier to describe world sheet fermions, so this is what we will do in this section. Fortunately, this will turn out to be more than just a useful exercise: in the end, the objects we create will turn out to be space-time fermions as well!

From field theory, we know there is a standard prescription that tells us how to incorporate massless[6] fermions in our theory. Just like in the point particle case, we should introduce so-called *Grassman variables* ψ_A into our theory which have the property that they anticommute, i. e.

$$\psi_A \psi_B = -\psi_B \psi_A. \tag{7.7}$$

These objects have to satisfy the Dirac equation of motion, which follows from the action

$$S = \int d^2\sigma i \bar{\psi} \rho^\alpha \partial_\alpha \psi, \tag{7.8}$$

where $\bar{\psi}$ means $\psi^\dagger \rho^0$, and the ρ^α have to satisfy the Clifford algebra:

$$\{\rho^\alpha, \rho^\beta\} = -2\eta^{\alpha\beta}. \tag{7.9}$$

In general, the smallest representation of this algebra (i. e. a set of d matrices ρ^α satisfying these equations) consists of $2^{d/2} \times 2^{d/2}$ matrices if the number of dimensions D is even. In our case, this means we should be able to find 2×2 matrices having this properties. Indeed, such matrices are not very hard to find; an example is

$$\rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{7.10}$$

We see that since ρ^α_{AB} is a matrix, the spinors ψ^A have to consist of two components. For reasons that will become clear in the following exercise, we will choose A to be $+$ or $-$ instead of 1 or 2.

Exercise 7.39 *Show that the action (7.8) leads to the massless Dirac equation for the fermions. Show that with the given ρ^α, this means that ψ_+ is a function of σ^+ and ψ_- is a function of σ^- only.*

The action (7.8) is the action for a *single* fermion. Of course, at the moment we have the freedom to add as many fermions to the theory as we like. However, we would like to have some symmetry between the bosonic and the fermionic fields. The bosonic fields X^μ have an index that tells us how the fields transform under space-time Lorentz transformations: as a covariant vector. To make the action as symmetric as possible, we impose the same thing for our fermionic fields, i. e. we take as many ψ-fields as we had X-fields, and impose that they transform under Lorentz transformations as a vector ψ^μ.

Note that this is somewhat counterintuitive: if we introduce spinors, we do not expect them to behave as vectors under space-time rotations! However, for the moment we have chosen to

[6]We want our fermions to be massless, just like the bosonic fields X^μ are. Again, this is in the world-sheet sense, meaning that a term like $m^2\psi\bar{\psi}$ is not in he Lagrangian. We have seen that the space-time objects that are described by our theory can certainly have masses!

describe *world sheet spinors*, and so their transformations under the *space-time* Lorentz group are not restricted in any way.

Introducing the spinors in the gauge-fixed action, we obtain the action

$$S = -\frac{1}{2\pi}\int d^2\sigma \left[\partial_\alpha X^\mu \partial_\alpha X_\mu - i\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu\right]. \tag{7.11}$$

It may seem a bit strange to add te fermionic fields directly to the gauge-fixed action. If we would do things properly, we would add the fields to the unfixed form of the action and then fix the gauge. This would give us some gauge constraints, just like in the bosonic gauge, which we would have to impose to the quantum mechanical states we obtain. We will come back to the gauge constraints later; for now we note that if we proceed in this way, we will end up with the action (7.11).

This action has, as we already mentioned, a very special symmetry, which is called *supersymmetry.* Very generally speaking, supersymmetry is a symmetry that transforms bosons into fermions and vice versa. In this particular case, the infinitesimal version of the supersymmetry is

$$\begin{aligned}\delta X^\mu &= \bar{\epsilon}\psi^\mu \\ \delta\psi^\mu &= -i\rho^\alpha \partial_\alpha X^\mu \epsilon,\end{aligned} \tag{7.12}$$

where ϵ_A is an infinitesimal spinor. We see that this symmetry indeed mixes the bosonic fields X^μ and the fermionic fields ψ^μ.

Exercise 7.40 1. *Show that for Majorana spinors*

$$\bar{\phi}\psi = \bar{\psi}\phi. \tag{7.13}$$

2. *Show that in the representation (7.10)*

$$\bar{\rho}^\alpha \bar{\rho}^\beta = \rho^\beta \rho^\alpha. \tag{7.14}$$

3. *Show that the action (7.11) is invariant under the transformation (7.12).*

Finally, we mention the fact that, whereas the bosonic string theory can only be consistent in 26 dimensions, a superstring theory can only be consistent in 10.

7.4 Boundary conditions

It is very instructive to look at the boundary conditions that arise from the superstring action (7.11). For the bosonic fields, we have the same possibilities as in the purely bosonic case. Varying the fields ψ gives the Dirac equation and a boundary term

$$\delta S_b = \int d\tau \left[\psi_+^\mu \delta\psi_{\mu+} - \psi_-^\mu \delta\psi_{\mu-}\right]_{\sigma=0}^{\sigma=\pi}. \tag{7.15}$$

This is a collection of four terms, so we have different ways to make this expression vanish. One way that has a physical interpretation is to make the two terms for $\sigma = 0$ vanish and to make the $\sigma = \pi$ terms vanish independently. This can be interpreted as the boundary equation for an open string, since here the two endpoints of the string are independent.

So for an open string, we impose

$$\psi_+^\mu \delta\psi_{\mu+} = \psi_-^\mu \delta\psi_{\mu-} \tag{7.16}$$

at both endpoints. Note that this is equivalent to

$$\delta((\psi_{\mu+})^2) = \delta((\psi_{\mu-}^2)), \tag{7.17}$$

and so (since $\psi = 0$ is also a solution to the massless Dirac equation) we have $\psi_+^\mu = \pm\psi_-^\mu$ at both ends. The overall sign of the components can be chosen at will, so we can use the + sign at one endpoint (say $\sigma = 0$) and are left with two options:

$$\begin{array}{rll} \text{R:} & \psi_+^\mu(\tau,0) = \psi_-^\mu(\tau,0) & \psi_+^\mu(\tau,\pi) = \psi_-^\mu(\tau,\pi) \\ \text{NS:} & \psi_+^\mu(\tau,0) = \psi_-^\mu(\tau,0) & \psi_+^\mu(\tau,\pi) = -\psi_-^\mu(\tau,\pi) \end{array} \tag{7.18}$$

Solutions to the field equations with the first type of boundary conditions are said to be in the *Ramond (R) sector* of the theory. Solutions with boundary conditions of the second type are in the *Neveu-Schwarz (NS) sector.*

In theory, we could pick a different boundary condition (R or NS) for every dimension μ. However, since we are mainly interested in theories that have the full D-dimensional rotational invariance, we will usually choose the same boundary condition for all of the μ.

Just like in the bosonic case, we can write down a Fourier expansion for the solutions of the field equations. We already know that $\psi_\pm$ should be a function of $\sigma^\pm$ only. The expansions are only slightly different for the two sectors, but the differences will turn out to be very important later.

- In the Ramond sector, we find the expansion

$$\psi_+^\mu = \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{\infty} d_n^\mu e^{-in(\tau+\sigma)} \tag{7.19}$$

$$\psi_-^\mu = \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{\infty} d_n^\mu e^{-in(\tau-\sigma)} \tag{7.20}$$

  where n runs over the integers. This should not be very surprising: it is more or less the same expansion as in the bosonic case; the only difference being that now the d_n^μ are *anticommuting* objects.

- In the Neveu-Schwarz sector, there is a second difference with the bosonic case. Since we want the ψ to have opposite signs at the $\sigma = \pi$ end, we have to take *half-integer moded* oscillators only. This leads to the expansion

$$\psi_+^\mu = \frac{1}{\sqrt{2}} \sum_{r=-\infty}^{\infty} b_r^\mu e^{-ir(\tau+\sigma)} \tag{7.21}$$

$$\psi_-^\mu = \frac{1}{\sqrt{2}} \sum_{r=-\infty}^{\infty} b_r^\mu e^{-ir(\tau-\sigma)}, \tag{7.22}$$

  where r runs over the 'half-integers' $n + 1/2$, with n an integer.

This conludes our discussion of the open string case. There is however another natural way in which we can make (7.15) vanish. Instead of the choice made above, we can also choose the terms for ψ_+^μ and ψ_-^μ to vansih independently. This should also sound familiar to the reader: if left and right moving modes are independent, we are talking about closed strings. So for closed strings, we impose the boundary conditions

$$\begin{aligned} \psi_+^\mu(\tau,0)\delta\psi_{\mu+}(\tau,0) &= \psi_+^\mu(\tau,\pi)\delta\psi_{\mu+}(\tau,\pi) \\ \psi_-^\mu(\tau,0)\delta\psi_{\mu-}(\tau,0) &= \psi_-^\mu(\tau,\pi)\delta\psi_{\mu-}(\tau,\pi). \end{aligned} \tag{7.23}$$

This leads to four different options, since now we can impose R- and NS-boundary conditions on both the left and the right moving modes. Moreover, like in the bosonic case, the left and right moving oscillator coefficients are independent, so we end up with the four possibilities

$$\begin{aligned} R-R: &\quad \{d_m^\mu, \tilde{d}_n^\nu\} \quad m,n\in\mathbb{Z} \\ R-NS: &\quad \{d_m^\mu, \tilde{b}_s^\nu\} \quad m,(s+\frac{1}{2})\in\mathbb{Z} \\ NS-R: &\quad \{b_r^\mu, \tilde{d}_n^\nu\} \quad (r+\frac{1}{2}),n\in\mathbb{Z} \\ NS-NS: &\quad \{b_r^\mu, \tilde{b}_s^\nu\} \quad (r+\frac{1}{2}),(s+\frac{1}{2})\in\mathbb{Z} \end{aligned} \tag{7.24}$$

7.5 Heterotic string theories

In the previous sections, we have seen many examples of the fact that the left and right moving modes of a closed string are essentially independent. For example, we have used this fact to create states for which the right moving modes are in the Ramond sector and the left moving modes in the Neveu-Schwarz sector.

Actually, there is nothing that prevents us from introducing a much more drastic asymmetry between the left and right moving modes: it turns out to be possible to make the left moving modes supersymmetric and the right moving modes non-supersymmetric!

To see how this comes about, look at the standard superstring action, but with only the left moving fermionic coordinates included:

$$S=-\frac{1}{2\pi}\int d^2\sigma\,\left\{\partial_\alpha X^\mu\partial^\alpha X_\mu - 2i\psi_-^\mu\partial_+\psi_{\mu-}\right\}. \tag{7.25}$$

Like in the superstring case, the left-moving side of this theory is only consistent in $D=10$ dimensions. However, on the right moving side of the theory we now only have 10 bosonic fields, and we know that a theory of purely bosonic fields can only be consistent if we have 26 of those. So somehow, we will have to add something on the right moving side of the theory.

We cannot use ordinary bosonic fields for this, since these would contain left moving modes as well, and the left hand side of our theory would be ruined. So we have to take right moving fermionic fields. However, we will not give these fermions a vector index, since this would take us back to our starting point: the closed superstring. Instead, we will take Lorentz scalars; it turns out that for consistency we need 32 of them:

$$S=-\frac{1}{2\pi}\int d^2\sigma\,\left\{\partial_\alpha X^\mu\partial^\alpha X_\mu - 2i\psi_-^\mu\partial_+\psi_{\mu-} - 2i\lambda_+^A\partial_-\lambda_+^A\right\}, \tag{7.26}$$

where $A=0,\ldots,31$.

Now what boundary conditions do we take on the λ-fields? Again, consistency plays an important role, and there turn out to be only two possibilities:

- We can impose the same boundary condition (periodic or anti-periodic) on all of the λ-fields, giving us a theory with two sectors. Since these fields are now invariant under rotations, we obtain a theory with an internal $SO(32)$-symmetry group, called the $SO(32)$ heterotic string.

- We can divide the 32 fields in two groups of 16, on which we impose the same boundary conditions. Each set has two possible boundary conditions, so we obtain a theory with four sectors. One would now expect the internal symmetry group to be $SO(16) \times SO(16)$, but in fact the actual symmetry group turns out to be a different (though closely related) group called $E_8 \times E_8$. This string is of course called the $E_8 \times E_8$ heterotic string.

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Eighth lecture

8.1 The open superstring mass formula

Just like in the bosonic string case, we can derive a mass formula for the superstring. The procedure is the same: start with an ungauged action; gauge the metric $h^{\alpha\beta}$ to $\eta^{\alpha\beta}$; derive the gauge constraints in the form of a vanishing stress tensor, and look at the zero-mode L_0 coming from this tensor. We will not explicitly carry out this procedure again, but simply give the results. For the moment, we will look at the open superstring. In the NS-sector of this string theory, we find

$$M^2 = \sum_{m=1}^{\infty} \alpha_{-m} \cdot \alpha_m + \sum_{r=1/2}^{\infty} r b_{-r} \cdot b_r - \frac{1}{2}. \tag{8.1}$$

The $-1/2$ is again a normal ordering constant. Note that the operator $\sum r b_{-r} \cdot b_r$ does the same thing as the bosonic operator $\sum \alpha_{-m} \cdot \alpha_m$: it counts the excited states, and weighs them with a factor of r. In the fermionic case, the weighing factor r is explicit in the mass formula; in the bosonic case the factor of m was already in the commutator and hence is not in the mass formula.

As we see, the ground state in the NS-sector is once again a tachyon; in this case of mass squared $-1/2$. The massless state is a vector particle, obtained by operating on the vacuum with $b^{\mu}_{-1/2}$. The other mass levels are evenly spread with mass difference $1/2$.

In the R-sector, we obtain an analogous formula:

$$M^2 = \sum_{m=1}^{\infty} \alpha_{-m} \cdot \alpha_m + \sum_{n=0}^{\infty} n d_{-n} \cdot d_n. \tag{8.2}$$

In this case, the normal ordering constant vanishes. This means that our groundstate is already the massless state. However, this state is not a scalar anymore, since we can operate on it with d_0^{μ} without changing its mass.

Now observe that the anticommutation relations for the operators d_0^{μ} are

$$\{d_0^{\mu}, d_0^{\nu}\} = \eta^{\mu\nu}. \tag{8.3}$$

Up to a factor, this means that these operator satisfy the anticommutation relations of the ten-dimensional Γ-matrices. Therefore, it is natural to choose these Γ-matrices (up to the same factor) as a representation for our operator d_0^{μ}. For a general even dimension D, the Γ-matrices are of size $2^{D/2} \times 2^{D/2}$, so in this case we obtain 32×32-matrices.

What does this mean physically? To transform under the chosen representation of the d_0^μ, our wave functions have to be 32-component objects. In fact, since the operators that transform them are the Γ-matrices, we have created space-time spinors! Since the above reasoning holds for any mass-level in the R-sector, we conclude that the R-sector of open string theory consists of fermions.

There is one last remark we want to make here. In general, Γ-matrices are complex matrices. However, in some dimensions it is possible to pick them in such a way that they are all real or all complex. For example, we saw that this was possible in $d = 2$ dimensions, and it turns out to be possible in $D = 10$ dimensions as well[7]. This is a very nice property of the theory, because this means the Dirac equation is a real equation, and hence we can take real (Majorana) spinors instead of complex ones. Since we also chose to have real X-fields, this is in fact quite a natural thing to do.

8.2 The GSO projection

The superstring theory we created in the previous lecture still has some drawbacks. To be specific, there are three problems which we would like to solve:

1. The theory contains a tachyon: a particle of negative mass squared.

2. The operators ψ^μ act in a very counterintuitive way. Because of the vector index μ, if we take a space-time boson $|A\rangle$ and we operate on it with ψ^μ, then $\psi^\mu|A\rangle$ still transforms as a boson under space-time Lorentz transformations. On the other hand, we have seen that ψ^μ consists of anticommuting variables, and from experience we would expect such variables to map bosons into fermions and vice versa.

   Note that this conceptual problem does not exist for products of *even* numbers of ψ-operators, since these products consist of ordinary commuting variables.

3. We would like our theory to be space-time supersymmetric, as well as world sheet supersymmetric. However, it is quite clear that at this stage that it is not. If we would have space-time supersymmetry, then at each mass level we would expect as many bosons as fermions. However, we have seen that the NS-sector of the theory contains half-integer masses, whereas the R-sector does not! Moreover, equality does not even hold at the massless level! On the bosonic side, we have a massless vector particle, which in 10 space-time dimensions has 8 degrees of freedom[8]. On the fermionic side, we have a 32-component real (Majorana) spinor. It can be shown that the Dirac equation relates half of its components to the other half, so we have a total number of 16 degrees of freedom, which is too large by a factor of two.

Fortunately, we can solve all of these three problems at once. This is not as miraculous as it may sound: notice that the three problems are closely related, in the sense that they all say we want to get rid of some of the states in our theory which we could describe as 'unphysical'. (On the other hand, describing non-supersymmetric systems as 'unphysical' would probably go

[7] In general, it can be proven that such a construction is possible in $D = 2, 3$ or 4 (mod 8).

[8] This can be explained in many ways. For example, we can do a light-cone quantization, just like in the bosonic case, and show that gauge fixing can be used to express α_n^+ and α_n^- in terms of the other oscillator coefficients, so they are not physical degrees of freedom. We can also use our knowledge from electrodynamics and observe that a massless photon can not have a longitudinal or a timelike polarization.

a bit too far for many physicists. However, we already chose to make our world-sheet theory supersymmetric, so the answer to this objection may be: 'If you do it, go all the way!'.)

So if we want string theory to describe the real world, it should at least be able to give us some prescription which gets rid of all the 'unphysical' states at once. This prescription turns out to exist, and it is known under the name of *GSO-projection*.

The massless level

Before giving the general form of the GSO-projection, let us take a closer look at the massless level of the open superstring theory. We have seen that at this level, we have a boson with 8 degrees of freedom from the NS-sector, and a fermion with 16 degrees of freedom from the R-sector. We would therefore like to project out half of the fermionic states. To this end, introduce the operator

$$\Gamma_{11} = \Gamma^0\Gamma^1 \cdots \Gamma^9. \tag{8.4}$$

Exercise 8.41 *Show that*

1. $\{\Gamma_{11}, \Gamma^\mu\} = 0$;
2. $(\Gamma_{11})^2 = 1$.

What are the eigenvalues of Γ^{11}*? Show that the operators* $\frac{1}{2}(1 \pm \Gamma_{11})$ *are projection operators on the subspaces of the Hilbert space consisting of states with these eigenvalues, and that the Hilbert space is the direct sum of these two spaces.*

These observations give us the solution to our problem: We can simply restrict our theory to contain only the so-called *Weyl-spinors*, i. e. the states of eigenvalue $\Gamma_{11} = +1$. This eigenvalue is called the *chirality* of a state. The projection gets rid of half of the spinors, and at least for the massless level the countings will match.

Remark: There is one subtle point in our discussion which we did not mention yet: since we started out with Majorana (real) spinors, we want our projection operators $(1 \pm \Gamma_{11})$ to be real operators, so they act within the subspace of Majorana spinors. It turns out that in $D = 10$ dimensions, it is indeed possible to make a representation of the Γ-matrices where these matrices themselves and Γ_{11} are all real. In general, this is only possible in $D = 2 \pmod 8$ dimensions.

The massive levels

Next, we want to find the general form of the GSO-projection operator. To this end, we look at our second problem above: the fact that the anticommuting operator ψ^μ maps space-time bosons into space-time bosons. We already gave a hint towards the solution to this problem: if we only keep the *even* products of these operators, we are back with our original intuition. So we would like a 'fermion counting operator' $(-1)^F$ such that

$$\begin{aligned} (-1)^F X^\mu &= X^\mu \\ (-1)^F \psi^\mu &= -\psi^\mu. \end{aligned} \tag{8.5}$$

Observe that, like Γ_{11}, this operator satisfies $((-1)^F)^2 = 1$, so we can also use it to create projection operators. Of course, we would like to find that on the massless level of the NS-sector, the operator is exactly Γ_{11}, and in fact this will turn out to be the case.

A second important observation is that X^μ and ψ^μ are a complete basis for the space of all operators in our theory. This means that once we assign a certain eigenvalue to a certain state, (8.5) will in principle tell us how $(-1)^F$ operates on *any* state. Of course, we will use this one choice we have to make the tachyon from the NS-sector a state of eigenvalue -1, so it will be projected out when we project onto the space of even fermion number.

In the Neveu-Schwarz sector, the explicit form of the operator $(-1)^F$ is not hard to find. Since the ψ^μ are linear functions of the oscillator coefficients b_n^μ, we want our operator to give a different sign when it acts on even or odd products of the operators b_n^μ. The operator that does this is

$$(-1)^F_{NS} = -(-1)^{\sum_{r=1/2}^{\infty} b_{-r}\cdot b_r}. \tag{8.6}$$

Note the overall minus sign: this makes sure that the tachyonic ground state has eigenvalue -1. The operator that is in the exponent is the familiar counting operator, so on any even product of b's (and hence ψ's), we get a minus sign; on any odd product we get a plus sign.

In the R-sector, we have a degenerate ground state, so besides the oscillator counting operator $(-1)^{\sum d_{-n}\cdot d_n}$ we can introduce an operator that changes the sign on some of the states on the massless level. Of course, the operator we pick will be Γ_{11}, so for the R-sector we propose the fermion number operator

$$(-1)^F_R = \Gamma_{11}(-1)^{\sum_{n=1}^{\infty} d_{-n}\cdot d_n}. \tag{8.7}$$

Not that, of course, we did not include the operators d_0^μ in the counting operator!

This concludes the construction of the GSO operator. From now on, we will work with superstring theories were this projection is carries out, i. e. we will only consider states $|\phi\rangle$ where

$$(-1)^F|\phi\rangle = |\phi\rangle \tag{8.8}$$

It can be shown that in these theories, the problems mentioned at the beginning of this section are solved. This is quite clear in the case of the tachyon and the anticommuting mappings from bosons to bosons; showing that the theories we obtain are space-time supersymmetric is much harder. Actually, we will not give a full proof of this fact in these lectures, but in the next section we will give a strong piece of evidence for space-time supersymmetry.

8.3 The one-loop open superstring partition function

In this section, we want to calculate the one-loop partition function for the open superstring. From now on, when we speak of 'the open superstring', the reader should realize that a GSO-projection is tacitly assumed.

As we saw in the fifth lecture, the quantity that we want to calculate is

$$Z(\tau) = \mathrm{Tr}\left(e^{2\pi i\tau H}\right), \tag{8.9}$$

where the trace is again a sum over the Hilbert space basis states and $H = L_0$ (or $L_0 - \frac{1}{2}$ in the NS-sector) is the open superstring Hamiltonian. As usual, L_0 consists of a part that goes like p^2 and a number of weighed counting operators. For now, we will ignore the p^2-part since – compare the fifth lecture – it will only contribute an overall factor to the final result.

First of all, we will calculate the partition funtion in the NS-sector. When we make the usual substitution

$$q = e^{2\pi i\tau} \tag{8.10}$$

we see that the quantity we want to calculate is

$$Z_{NS}(\tau) = \sum_{\{N_m,N_r\}} q^{\sum_{m=1}^{\infty} mN_m + \sum_{r=1/2}^{\infty} rN_r - \frac{1}{2}}, \tag{8.11}$$

where for the moment we left out the μ index for the sake of clarity. N_m and N_r are the occupation numbers of the bosonic and fermionic excitations respectively. Note that the sums over the occupation numbers run only over the occupation numbers that are allowed by the GSO-projection! We can also accomplish this by inserting the projection operator $\frac{1}{2}(1+(-1)^F)$ in the trace:

$$Z_{NS}(\tau) = \frac{1}{2}\mathrm{Tr}\left[q^H(1+(-1)^F)\right], \tag{8.12}$$

where the trace now runs over all of the basis states we had before the GSO-projection.

Exercise 8.42 *Using the same techniques as in lecture 5, show that the partition function is*

$$Z_{NS}(\tau) = \frac{1}{2\sqrt{q}}\left[\prod_{m=1}^{\infty}\left(\frac{1+q^{m-1/2}}{1-q^m}\right)^8 - \prod_{m=1}^{\infty}\left(\frac{1-q^{m-1/2}}{1-q^m}\right)^8\right]. \tag{8.13}$$

Hint: calculate $Tr(q^H)$ and $Tr\left(q^H(-1)^F\right)$ separately, and use the fact that the occupation numbers for fermions can only be 0 or 1.

Next, we want to calculate the partition function in the Ramond sector of the theory. Here, the calculation is simpler, since we do not exclude any mass levels with the GSO-projection; we simply change the degeneracy at each mass level by a factor of $\frac{1}{2}$. In particular, the ground state, after the GSO-projection, has an eightfold degeneracy. We can therefore leave out the GSO-projection and all the countings that come from the d_0-operators, and simply multiply our final result by 8. Moreover, there is no normal ordering constant in this sector, so we do not obtain a factor such as the $1/\sqrt{q}$ in the NS-case. Therefore, our final answer will be

$$Z_R(\tau) = 8\prod_{m=1}^{\infty}\left(\frac{1+q^m}{1-q^m}\right)^8. \tag{8.14}$$

We can use these two partition functions to give some strong evidence for space-time supersymmetry. Note that, basically, we are calculating

$$\mathrm{Tr}q^M \tag{8.15}$$

with M the mass operator. This means that if we Taylor expand our partition functions, the coefficient of q^m should be the number of states of mass m. If our theory is supersymmetric, there should be as many bosons as fermions at each mass level, and hence the two functions (8.13) and (8.14) should be equal. That this is indeed the case was already proved in 1829 by Jacobi! We will not present the proof here, but it is easy to check the first few terms in a Taylor expansion.

Exercise 8.43 *Calculate the number of states at the lightest three mass levels in both the Ramond an the Neveu-Schwarz sector using a Taylor expansion of (8.13) and (8.14).*

8.4 The massless closed string states

In the previous subsections, we have been discussing the open superstring – also called the type I superstring. Now, we want to discuss the mass spectrum of the closed superstring, and in particular the states that occur at the massless level.

In the previous lecture, we have seen several examples of the fact that we can obtain a closed string mass spectrum by taking two copies of the open string mass spectrum: one for the left moving modes, and one for the right moving ones. The only restriction that we have to impose is that the total mass coming from the left moving modes equals that of the right-movers.

In superstring theory, we can use the same prescription. However, there is an additional choice we have to make. Recall that in constructing the GSO-projection in the R-sector of the open string theory, we used the helicity operator Γ_{11}. However, we could have just as well used $-\Gamma_{11}$, or in other words: we could have just as well projected on states of negative helicity instead of positive helicity. In the open string case, this would have given us exactly the same theory.

However, in the closed string case, there is more to this. Here, we have to construct a GSO-operator for the left and for the right moving part of the theory, and on both sides we can pick $\pm\Gamma_{11}$. The overall sign here is not important, but the relative sign is! So we will obtain two theories: a theory where left- and right-movers have opposite chirality, and a theory where they have the same chirality. The first theory is called *type IIA* string theory; the second *type IIB* string theory.

Since the open superstring has $8 + 8 = 16$ states at its massless level, the closed superstring has $16 \times 16 = 256$ massless states; $8 \times 8 = 64$ coming from each of the four sectors. Since the d_0-operators construct space-time spinors, the $R - NS$ and $NS - R$ states are fermions. That the R-R sector consists of bosons is also clear. What are the states in the NS-NS sector? If we have one set of d_0-operators, we find a fermion – i. e. a particle whose wave function obtains a minus sign after a full rotation. Hence if we have two of these sets, we find a particle whose wave function is invariant under a full rotation: a boson. We could have also expected this from supersymmetry: we want as many bosons as fermions, so two sectors should give bosons and the other two fermions.

In the NS-NS, R-NS and NS-R sectors, the type IIA and IIB theories turn out to have the same massless spectrum. This should not be very surprising, since in these cases the relative chirality on the left and right moving fermions should not matter. The massless states are:

- **NS-NS**: A scalar called the dilaton (1 state); an antisymmetric rank 2 tensor (28 states) and a symmetric rank two tensor: the graviton (35 states).
- **NS-R**: A spin 1/2 fermion (8 states) and a spin 3/2 fermion which is interpreted as the gravitino (56 states).
- **R-NS**: Naturally, the spectrum in this sector is the same as that in the NS-R sector, so we have two more fermions of spin 1/2 and 3/2 respectively.

In the R-R sector, we find the main differences between the type IIA and type IIB theories. The massless fields are:

- **Type IIA R-R**: A vector (8 states) and an antisymmetric rank three tensor (58 states).
- **Type IIB R-R**: A scalar (1 state); an antisymmetric rank two tensor (28 states) and a selfdual antisymmetric rank four tensor (35 states).

The important thing to notice here is that type IIA string theory contains R-R fields of even rank, whereas type IIB string theory conatins R-R fields of odd rank. This observation will play an important role later.

8.5 D-branes in closed string theories

In the study of T-duality properties of open string theories, we encountered D-branes – hyperplanes on which open strings can end. We saw that open string theory could be viewed as a theory of space-filling D-branes. Compactification on circles, followed by T-duality, produced D-branes of lower dimensions.

This procedure therefore shows that D-branes of arbitrary dimensions exist, i.e. that open string theories with combinations of Neumann and Dirichlet boundary conditions are consistent. In this lecture we will make a few generalisations. First of all, although the T-duality procedure only gives D-branes whose transverse space is a compact torus, we will also want to consider D-branes that are embedded in the full non-compact target space, which for superstrings is ten-dimensional. These therefore are $p+1$-dimensional hyperplanes in $\mathbf{R}^{10}$ on which open strings end. In fact these could be viewed as a limiting case of the previous case: if we consider 10-d open string theory on a d-dimensional torus whose radii go to zero, this is T-dual to D$(9-d)$-branes whose transverse space is a torus with radii going to infinity. In the *decompactification*-limit, we have Dp-branes embedded in $\mathbf{R}^{10}$.

Now that we know D-branes exist, we can also simultaneously put D-branes of different dimensions in the 10-d target space. This corresponds to considering open string theory with various sectors. For instance, suppose we look at a theory containing a D9-brane and a D5-brane. One then has a sector of open strings having both endpoints on the D9-brane; since we are in 10 dimensions, these strings have Neumann boundary conditions in all directions, and are just the conventional open strings. Similarly one has a sector of strings connecting only to the five-brane; these $5-5$-strings have 4 D-boundary condition, and six N-boundary conditions. Finally one has strings going from 5 to 9-branes, or vice versa. Their boundary conditions are more peculiar: $5-9$ strings have 4 D and 6 N boundary conditions for the endpoint at $\sigma=0$, and 10 N-conditions at the other end, $\sigma=\pi$. One can view this system as ten-dimensional open string theory (the nine-brane theory), with $5+1$-dimensional objects in them, the D5-branes. In the bulk one just has propagating $9-9$ open strings; on the hypersurface, there are extra states, the $5-5$ strings, and the two pieces interact via the mixed $5-9$ and $9-5$ strings.

¿From this picture of a conventional string theory in the bulk, with lower-dimensional objects in it that interact with the bulk theory via the coupling of open strings, it is a small step to the final generalisation that we will make. Let us take a closed string theory, type IIA or IIB, living in ten dimensions. Add to this a number of D-branes; we then have a theory consisting of closed strings in the bulk plus open strings confined to the D-branes. The coupling of closed strings to open strings is quite natural (figure 8.1): open string diagrams are diagrams with a boundary living on the D-brane; the interior of the diagram is "made of the same stuff" as a closed string diagram, and hence there we can have an insertion of a vertex operator representing the emission or absorption of a closed string.

We will now explore the interpretation and properties of such a closed string system with D-branes.

8.5.1 Symmetries

Poincaré symmetry Closed string theories have a ten-dimensional target spacetime. The world sheet theory was shown to be invariant under global Poincaré symmetries: ten-dimensional translations and Lorentz transformations. Accordingly, the fields in the spectrum of the theory come in representations of this symmetry. In the NS-NS sector we identified a ten-dimensional graviton, a ten by ten antisymmetric matrix field[9] $B_{\mu\nu}$, and scalar field. Also in the other sectors, and at higher mass levels, the fields were ordered in ten-dimensional multiplets. The

[9] Note that because of the gauge symmetry, this statement does not mean that we have a field with $10*9/2=45$ degrees of freedom! We can go to the light cone gauge, where we find an 8×8 antisymmetric matrix having $8*7/2=28$ degrees of freedom.

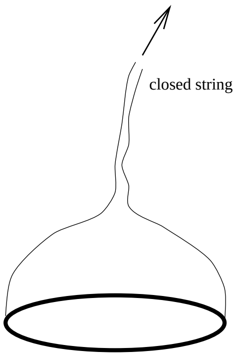

Figure 8.1: A D-brane interacting with closed strings via emission of a closed string state from an open string disk diagram

interactions between strings also respect this symmetry, so that the string theory can effectively be described by a true ten-dimensional theory with Poincaré invariant fields and interactions.

Now turn to a D-brane. As remarked before, open strings on a generic D-brane break the ten-dimensional symmetry: the ten world sheet bosons (and fermions) are not symmetric, since $p+1$ of them have Neumann boundary conditions, and the other $9-p$ Dirichlet. Instead of the ten-dimensional Lorentz group $SO(1,9)$, only a $SO(1,p) \times SO(9-p)$ subgroup respects this split-up. Translation invariance is incomplete as well: the boundary condition for the Dirichlet directions is $X^i = const$, incompatible with translational invariance in these directions.

Exercise 8.44 *Determine the massless bosonic spectrum of an open string ending on a Dp-brane, and find out in what representations of the unbroken group $SO(1,p) \times SO(9-p)$ the fields transform.*

We therefore have the situation that we have a ten-dimensional Poincaré invariant closed string theory, to which we propose to add an open string sector which does not possess this symmetry. At first sight this may seem not so sensible a thing to do. However, as alluded to earlier, this is in fact completely natural if we interpret the D-brane not as just another vacuum with less symmetry, but as the original closed string theory with an extra object. In the case that $p = 0$, this object is just a pointlike particle. We do not expect a theory describing a particle in a background to be translationally invariant, since the particle has to be located at some specific point. Likewise, for higher p, we have extended objects, that interact with the strings surrounding them. For $p = 1$, we have a so-called D-string, for $p = 2$ a membrane, etc.

The $9-p$ translational symmetries broken by the D-brane pop up elsewhere in the theory: they are in one to one correspondence with the massless scalars of the open string sector. These fields have no momentum transverse to the brane, and hence have only a dependence on the $p+1$ coordinates along the brane. The open string states can therefore be effectively described by a $p+1$ dimensional field theory. The $9-p$ scalars appear in this theory, to lowest order, as

$$\int d^{p+1}\sigma \partial_\sigma X^i \partial^\sigma X^i. \tag{8.16}$$

This is precisely of the form already encountered in the descriptions of the point particle and the string (in a gauge fixed version): the fields X^i describe the embedding of the D-brane world volume in the target space.

Supersymmetry Superstring theories were shown to have a symmetry relating the world sheet bosons X^μ to the fermions ψ^μ, called world sheet supersymmetry. Apart from that, we saw that a similar symmetry was realised on the spacetime theory: the open and closed string spectra contain, at every mass level, an equal number of bosonic and fermionic degrees of freedom. Also the interactions can be shown to be symmetric under spacetime supersymmetry transformations.

There is a difference between the open string supersymmetry on a D-brane, and spacetime supersymmetry in closed string theory, however. The massless spectrum of an open string (ending on a D9-brane) consists of a vector field A_μ, and one fermion λ. The supersymmetry variation of the gauge boson is proportional to λ, the associated variation parameter is itself a spinor ϵ. Let us now turn to the closed strings; here we have, among other fields, a graviton $g_{\mu\nu}$. Apart from that there are *two* fermionic gravitinos; the supersymmetry variation of the graviton depends on both of them, and consequently we have two spinorial parameters, ϵ_1 and ϵ_2. There are therefore two independent supersymmetry transformations in the closed string spacetime, as opposed to one for open strings; the theories are referred to as having $N = 2$ resp. $N = 1$ supersymmetry. (This is the reason for the name type II).

Again we face a similar problem as in the case of Poincaré symmetry. The closed strings have $N = 2$ symmetry, while the D-brane we add only supports $N = 1$. We conclude that the extended object living in the environment of closed strings does not only break translational invariance, but also half of the supersymmetry! Closer examination reveals that this is only true in the case of IIA theory for D-branes of even (spatial) extension: D0-branes, D2-branes, D4-branes, ...; for IIB theory for odd-dimensional branes: D1, D3, D5, ... Again, the broken supersymmetry has an interpretation in the field theory on the brane: it corresponds precisely to the fermions on the world volume. Just as a broken translation has the effect of a shift in one of the scalar fields, a broken supersymmetry transformation appears as a shift in the fermion field.

8.5.2 Coupling to the closed strings

Adding a D-brane to the closed string theory corresponds to looking at an extended object living in 10 dimensional spacetime. The theory on the brane is described by the dynamics of open strings. The open and closed string theories interact among each other as well (see figure (8.1), giving rise to interaction of the D-brane with the spacetime fields.

First of all it is of course expected that a D-brane couples to the graviton, as a massive object. Additionally, it also turns out that D-branes are charged. Recall that in type II theory the massless RR sector consists of a number of antisymmetric tensor fields. In the IIA theory for instance, we had a massless vector field A_μ, and a three-form field $A_{\mu\nu\rho}$. Both fields are gauge fields: their field strengths, $F = dA$, are invariant under the gauge transformations $A \to A + d\Lambda$ (with Λ a scalar resp. an antisymmetric matrix valued function). States in the closed string spectrum are not charged with respect to these gauge fields, they only couple to the field strengths F, not to the potentials A.

The states that do coupe to the A are the D-branes. Let us first consider the vector A_μ. Such a vector potential couples naturally to a particle's worldline, by a term in the action

$$S_A = \int d\tau A_\mu \frac{dX^\mu}{d\tau}. \tag{8.17}$$

The generlisation for the three-index potential $A_{\mu\nu\rho}$ is that it is natural to couple it to an object

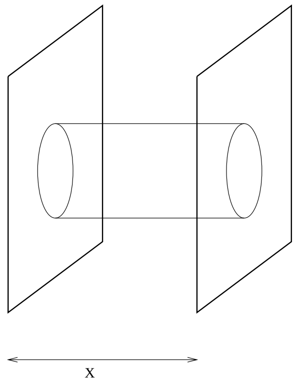

Figure 8.2: Two D-branes exchanging a closed string state, or an open string one loop diagram

with a three-dimensional world volume: a two-brane or membrane. The action is

$$S_{A_3} = \int d^3\sigma \epsilon^{\alpha\beta\gamma} \partial_\alpha X^\mu \partial_\beta X^\nu \partial_\gamma X^\rho A_{\mu\nu\rho}. \tag{8.18}$$

Comparing with the allowed one-half supersymmetric D-branes in the IIA theory, we see that this matches with the D0-brane and the D2-brane. The D4- and D6-branes are the magnetic counterparts of these branes.

Exercise 8.45 *The magnetic gauge field $\tilde{A}$ is defined by $d\tilde{A} = *dA$, where $*$ (up to a numerical factor) means contraction with the ten-dimensional anti-symmetric ϵ-tensor. Show that the magnetic fields dual to the fields A_μ and $A_{\mu\nu\rho}$ couple naturally to the D6- resp. D4-branes.*

In the IIB theory we had odd-dimensional branes, which exactly match the RR gauge potentials in this theory.

We can now try to calculate the coupling of the branes to respectively the graviton and the RR gauge fields, thereby computing the mass and charge. One way to do this would be by scattering a graviton or a RR-field off the brane, giving a diagram like figure (8.1). In this case there is an easier option, however: we will compute the force that two parallel D-branes exert upon each other by exchanging closed string states. The diagram that we have to compute is given in figure (8.2): a closed string couples to the lefthand D-brane, propagates and is absorbed by the righthand D-brane. If the closed string is a graviton, the amplitude will be proportional to the mass-squared of both branes, for an RR-state it is proportional to the charge-squared.

The remarkable fact is that the same calculation can be given a completely different interpretation. Instead of world sheet Euclidean time running along the propagation of the closed string, with the spatial coordinate being the periodic coordinate around the string, we may consider the periodic coordinate to be time, and the other one the spatial coordinate along an open string. The same diagram is then interpreted as an open string one loop annulus diagram, namely the open string partition function.

Exercise 8.46 *We are going to evaluate the partition function $\mathrm{Tr} e^{-2\pi L_0 t}$ of an open string with $9-p$ Dirichlet boundary conditions, and $p+1$ Neumann ones.*

i) Show that the contribution of the string zero-modes to L_0 is

$$M_0^2 = k^2 + \frac{X^2}{4\pi^2\alpha'^2}, \tag{8.19}$$

where k is a $p+1$-dimensional momentum vector, and X the transverse separation between the two ends of the open string.

ii) The oscillator part is not affected by the boundary conditions; it is therefore given by three terms, the NS sector with 1 and $(-1)^F$, and the R sector with 1 (in the R-sector, the trace with $(-1)^F$ gives zero). Write the total partition function as a sum over these sectors (the part Z_R should be subtracted, *since it is a fermion loop (cf. field theory)). Do not forget the trace, i.e. integration, over momentum along the brane. The transverse separation is fixed, so should not be integrated over.*

iii) Carry out the momentum integration to find

$$Z(t) = (2t)^{-(p+1)/2} e^{-\frac{X^2 t}{2\pi\alpha'^2}} \times \text{ oscillators.} \tag{8.20}$$

To find the amplitude, we have to integrate this over the modulus t of the annulus, as follows:

$$\mathcal{A} \sim \int \frac{dt}{2t} Z(t). \tag{8.21}$$

The answer will be a sum of contributions from RR and NSNS states, from the closed string calculation. To identify which is which, note that NSNS states have antiperiodic fermions along the circle, the RR ones periodic. Since in partition functions, due to their anticommutativity, fermions are naturally antiperiodic in Euclidean time, the NSNS sector coincides with the terms with 1 in the open string trace, while the periodic fermions of the RR-sector are located in the term with $(-1)^F$.

First of all we can conclude that the total answer equals zero, the contributions cancel against each other! This follows from the identity

$$Z_{NS} - Z_R = 0, \tag{8.22}$$

the manifestation of spacetime supersymmetry. The attractive force from gravity is therefore precisely cancelled by the repulsive force resulting from the RR-charge. To extract only the repulsive RR force we therefore have to look at only the term with $(-1)^F$ in the trace. This is a sum of not only the massless RR-exchange, but also all the massive modes. To extract the massless contribution, we let the transverse distance X become big; the massive exchanges are then exponentially suppressed. Big X is equivalent to small α'; we need the limit where $q = \exp -2\pi\alpha' t$ goes to one, of the expression

$$\frac{1}{2\sqrt{q}} \Pi \left(\frac{1 - q^{m-1/2}}{1 - q^m} \right). \tag{8.23}$$

This is not a straightforward limit; it is much easier to compute the asymptotic behaviour for $q \to 0$ than for $q \to 1$. Fortunately, there is a relation between these limits. Note that the terms of the partition functions are given by ratios of the functions

$$f_1(q) = q^{1/24} \Pi(1 - q^m), \tag{8.24}$$

$$f_2(q) = \sqrt{2} q^{1/24} \Pi(1 + q^m), \tag{8.25}$$

$$f_3(q) = q^{-1/48} \Pi(1 + q^{m-1/2}), \tag{8.26}$$

$$f_4(q) = q^{-1/48} \Pi(1 - q^{m-1/2}). \tag{8.27}$$

For instance, the term we are interested in is precisely $f_4^8/2f_1^8$. These functions have very nice transformation properties; they satisfy:

$$f_1(e^{-2\pi/s}) = \sqrt{s} f_1(e^{-2\pi s}), \quad f_3(e^{-2\pi/s}) = f_3(e^{-2\pi s}), \quad f_2(e^{-2\pi/s}) = f_4(e^{-2\pi s}). \tag{8.28}$$

¿From this, we can relate the asymptotics for $q \to 1$ to those for $q \to 0$.

Exercise 8.47 *Use these transformations to write the amplitude involving only the lowest order contribution. Verify that there is no tachyonic exchange (this would exponentially blow up). Show that the result goes like* $1/X^{7-p}$*, as expected from a massless exchange in* $9-p$ *transverse dimensions.*

Caputcollege string theory 1998-99

Ninth lecture

9.1 Introduction

In this lecture, we will take a closer look at the space-time interpretation of string theory. In particular, we will look at the so-called *low-energy effective actions* for all five superstring theories. These are space-time actions that tell us what happens in the low-energy regions of string theory.

Before we can present these actions, we have to discuss two field theories that we will encounter. These theories, called *supergravity* and *super Yang-Mills theory* will be introduced in section 9.2.

In section 9.3, we will discuss the low energy effective actions, and in section 4 we will use these actions to point out some of the dualities between the five superstring theories.

To keep the formulae short and clear, we will only discuss bosonic fields in this lecture. Of course, the reader should bear in mind that to every bosonic field corresponds a superpartner that is a fermionic field.

9.2 Two important field theories

9.2.1 Supergravity

We have already mentioned several times that it is very hard – if not impossible – to obtain physically interesting results from an ordinary quantum field theory that describes gravity. However, this does not mean that it is impossible to formulate actions for those theories; it only says that we will not be able to obtain finite perturbative results if we use them. In fact, we want to mention a very important example of a theory of gravity in this section.

This theory is called *supergravity*, and it arises in a very natural way. In the previous lectures, we have seen some examples of supersymmetries. An important fact about supersymmetries is that the commutator of two supersymmetry transformations is always a translation. This means that when we have supersymmetry in our theory, we automatically incorporate translations, and hence a Poincaré symmetry.

Now suppose that we want to introduce a *local* supersymmetry – i. e. a supersymmetry where the parameters depend on space and time. We will then automatically introduce a local Poincaré-invariance, and hence general relativity – i. e. gravity! Therefore, locally supersymmetric theories are automatically theories of *supergravity*.

It turns out that theories of supergravity are more or less unique. That is, we can specify the dimension and the number of supersymmetries, but unless we add other fields this completely fixes our theory. Even the choice of dimensions and degree of supersymmetry is not arbitrary: it turns out that all possible theories of supergravity can be derived from an eleven-dimensional theory of supergravity with only one supersymmetry.

The bosonic field content of this theory is very simple: it only contains the graviton field G_{MN} and an antisymmetric rank three tensor A_{MNP}. The action however is quite complicated, and we will not present it here.

Other supergravity theories are derived from this one by a procedure we have already encountered: the procedure of dimensional reduction. We simply compactify some of the dimensions of the theory on a compact manifold, and let the size of this manifold go to zero.

In this way, we can get three different supergravity theories in ten dimensions. The first is the so-called type IIA supergravity. It is obtained by compactifying eleven-dimensional supergravity on a circle. As we have seen in the previous lecture (compare exercise 8.44), this means that the eleven-dimensional fields (which form representations of $SO(1,10)$) split up in several ten-dimensional fields (which form representations of $SO(1,9)$). In particular, the metric G_{MN} splits up in the new metric $G_{\mu\nu}$ (where $\mu, \nu = 0 \ldots 9$), a vector A_μ (corresponding to $G_{\mu,10}$) and a scalar ϕ (corresponding to $G_{10,10}$). Similarly, the antisymmetric tensor A_{MNP} splits up into a rank three antisymmetric tensor $A_{\mu\nu\rho}$ and a rank two antisymmetric tensor $B_{\mu\nu}$ (corresponding to $A_{\mu\nu,10}$).

If we also include the fermionic side of the theory, we find that there are 2 supersymmetries in the reduced theory; hence the name type II. Moreover, since the theory we started out with was non-chiral (i. e. there was no preference for left or right), our resulting theory will be non-chiral as well. This means that the two fermions in the theory will be of opposite chirality.

One may wonder whether it is possible to construct a ten-dimensional theory of supergravity with two supersymmetries and two fermions of the same chirality. This turns out to be possible, and this theory is known as type IIB supergravity. It is very hard to write down an action for this theory, but we are able to write down its equations of motion.

Finally, it is possible to create a theory that is ten-dimensional and has only one supersymmetry. This theory is therefore known as type I supergravity. The bosonic field content of this theory is a graviton $G_{\mu\nu}$, an antisymmetric tensor $B_{\mu\nu}$ and a scalar ϕ.

9.2.2 Super Yang-Mills theories

The super Yang-Mills theory is basically a generzalization of the theory of electromagnetism. In electromagnetism, we start out from a complex field ϕ, and an action that is globally $U(1)$-invariant:

$$S = \int \partial_\mu \phi \partial^\mu \phi^*. \tag{9.1}$$

We see that sending $\phi \to e^{i\alpha}\phi$ does not change this action. Here, α can take any value, but it must be a *constant*. If we make α space-time dependent, the action is no longer invariant under the transformation. However, it is not very hard to introduce an action that is also invariant under these *local* $U(1)$-transformations. To this end, we introduce an extra vector field A_μ and the 'covariant derivative'

$$D_\mu \phi = \partial_\mu \phi + i A_\mu \phi \tag{9.2}$$

Moreover, we define the *field strength* corresponding to A_μ by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{9.3}$$

We use this field strength to construct an action that is invariant under both Lorentz transformations and local $U(1)$-transformations:

$$S = \int \left\{ -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + D_\mu \phi D^\mu \phi \right\}. \tag{9.4}$$

Next, we may wonder if we can also create an action that is invariant under a local group of transformations that is different from $U(1)$, and in particular under a non-abelian group. Yang and Mills showed that this is indeed possible, and therefore these non-abelian gauge theories are also called *Yang-Mills theories.*

Every Lie group can be constructed by exponentiating a set of generators T_a. For example, $U(1)$ is a very simple group that has only one generator. In general we have more generators, and they do not have to commute. However, by definition the generators always form a closed algebra, i. e. we can always express the commutator of two generators in terms of another generator:

$$[T_b, T_c] = f^a_{bc} T_a \tag{9.5}$$

Yang and Mills showed that we can make a theory that is invariant under these transformations if we take as many gauge fields A^a_μ as we have generators, and construct an action using those gauge fields. In particular, we have to define a field strength

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^a_{bc} A^b_\mu A^c_\nu. \tag{9.6}$$

This field strength enters into the action as

$$S = \int \mathrm{Tr}(F_{\mu\nu} F^{\mu\nu}), \tag{9.7}$$

where the trace runs over the group index a.

It turns out to be possible to add fermionic fields to this action and obtain a supersymmetric theory. These theories are called *super Yang-Mills theories* or SYM-theories for short. In string theory, we will encounter these theories all of the time.

9.3 Low energy effective actions

All through these lectures, we have seen that string theory differs in one very important aspect from the 'normal' quantum field theories we know: it is not a field theory with our space-time as a parameter space. Instead, we have the string world sheet as our parameter space, and space-time is the target space, i. e. the space in which (classically) the fields take their values. This difference is intimately related to the fact that we are describing extended objects instead of point particles: we can no longer – not even classically – think of the string as being at one particular point, so we cannot use normal wave functions in space-time, and hence we cannot directly use quantum field theory.

However, we know from experience that quantum field theories such as quantum electrodynamics (QED) give remarkably good results when we compare them to experiments. So the concept of a quantum field theory with space-time parameters cannot be so wrong after all. In fact, if we take the string theory point of view, this is not very hard to explain. After all, the typical length of a string is of the order of the Planck length, so in our daily life, and even in the most sophisticated experiments, strings will behave like point particles and the difference will be very hard to notice.

It is therefore natural that in a very good approximation, we can describe the 'everyday life range' of string theory using an ordinary quantum field theory in space-time. What would the action of such a theory look like? First of all, since the massive particles in string theories have masses of the order of the Planck mass, we do not expect to see many of these particles in everyday life[10]. So in principle, we could replace all the massless states by corresponding space-time fields, study how these states interact, and obtain an action. Now in general, such an action will contain an infinite number of terms. However, we can make a further approximation by realizing that taking a space- or time-derivative of our space-time fields corresponds to looking at its momentum or energy. Expressed in the natural Plank units (i. e. compared to the Planck mass), these energies will of course be very small. So instead of writing down all the terms that involve the massless fields, we can also write down only those with the lowest number of derivatives, and still obtain a very good low-energy approximation. An action that is constructed in this way is called a *low-energy effective action.*

Note that using this construction, we throw away many of the advantages of string theory, and in a sense are back at our starting point: we are using quantum field theory, including all of its divergences and its problems with respect to quantum gravity. However, we are now using this as an *approximation*, and the main advantage is that in this approximation we can compare string theory to the things we already know from quantum field theory.

If we write down the massless states in a certain string theory and use all of the symmetries of our theory, it turns out that the form of the low-energy effective action is completely determined in each case. In fact, many terms are present in all the different theories. It is not hard to see why: since all of our theories contain closed strings[11], all of the theories will contain a graviton $G_{\mu\nu}$, a dilaton Φ and an antisymmetric tensor $B_{\mu\nu}$. All of our low energy effective actions will therefore have a part in common. To be precise, we can show from symmetry principles that this part has to be of the form

$$S_{\text{eff}} = \int d^{10}x \left\{ \sqrt{G} e^{-2\Phi} (R + 4(\nabla\Phi)^2 - \frac{1}{12} H^2) \right\}, \tag{9.8}$$

where R is the curvature scalar that can be derived from the metric, and $H_{\mu\nu\lambda}$ is the field strength that can be derived from $B_{\mu\nu}$.

In the next subsections, we will present the (bosonic parts of the) effective actions for all of the five known string theories. Of course, numerical constants are a matter of taste here: we can always redefine our space-time field by multiplying it with an arbitrary factor. Besides, we can absorb some fields into other fields – for example, it is common to work with the Einstein metric $g_{\mu\nu} = e^{-\Phi/2} G_{\mu\nu}$. We will use the conventions used by A. Sen in *An Introduction to Non-perturbative String theory*; hep-th/9802051. This will have the result that (9.8) will appear more or less disguised in the effective theories; however, the reader will still be able to clearly recognize it.

9.3.1 The type II theories

In the previous lecture, we already summed up the massless bosonic field content of the type II superstrings. In the NS-NS sector, we find exactly the states that are mentioned above. The

[10]One may wonder where the masses of most particles in nature come from if we do not include the massive string particles in our discussion. The answer is that the process of symmetry breaking can give some of the massless particles a mass. We will however not discuss these matters in these lectures.

[11]The reader may object that type I theory is a theory of *open* strings only. However, as soon as we start looking at interactions, we cannot prevent the endpoints of such a string to interact with eachother and form a closed string.

fields in the R-R sector are different in the type IIA and IIB theories.

In the type IIA theory, the R-R sector gives us a vector field A_μ and a rank three antisymmetric tensor $C_{\mu\nu\lambda}$. The vector field is related in the usual (electromagnetic) way to a field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Similarly, the three-form $C_{\mu\nu\lambda}$ can be related to a four-form field strength $G_{\mu\nu\lambda\kappa}$. Using symmetries, we can show that the low energy effective action of the IIA theory is

$$\begin{aligned} S_{\text{eff}}^{II} &= \frac{1}{(2\pi)^7} \int d^{10}x \sqrt{g} \left\{ R - \frac{1}{2}(\nabla\Phi)^2 - \frac{1}{12} e^{-\Phi} H^2 - \frac{1}{4} e^{3\Phi/2} F^2 - \frac{1}{48} e^{\Phi/2} G^2 \right. \\ & \left. - \frac{1}{2304} \frac{1}{\sqrt{g}} \epsilon^{\mu_0 \cdots \mu_9} B_{\mu_0\mu_1} G_{\mu 2 \cdots \mu_5} G_{\mu 6 \cdots \mu_9} \right\}. \end{aligned} \tag{9.9}$$

It turns out that the full low energy effective action is – as the naming might have suggested – exactly the type IIA supergravity action!

In the type IIB theory, things are a bit more complicated. Remember that there, one of the space-time fields is a rank four tensor with self-dual field strength. It turns out to be very hard to insert this concept of self-duality into the effective action, so there is in fact no simple expression for the low energy effective action of type IIB string theory. On the other hand, it is possible to write down the equations of motion for the IIB-fields, and doing this shows that the low-energy effective theory corresponding to type IIB string theory is type IIB supergravity – consistent with the fact that we could not write down an action for that theory either,

9.3.2 The type I theory

As we saw in the previous lecture, the bosonic part of type I string theory contains a vector field A_μ coming from the massless open string states. In fact, by adding Chan Paton degrees of freedom, we can add a group index to this vector, and it turns out that the only consistent group is $SO(32)$. The one vector field A_μ is now replaced by 496 vector fields A_μ^a that transform into eachother under the action of this group.

However, this is not the full massless bosonic field content of the type I theory. Since we want to describe interacting strings, it is quite natural that the two endpoints of an open string will be able to join, and form closed strings. Hence, we also expect some bosonic closed string states in the theory. Actually, the closed string spectrum in type I string theory is equal to that of the type IIB theory, but restricted by a certain parity projection. It turns out that this projection only keeps the dilaton Φ and the graviton $G_{\mu\nu}$ from the NS-NS sector, and the antisymmetric rank two tensor $B_{\mu\nu}$ from the R-R sector.

Using all of these fields and their symmetries, the following effective action is found:

$$S_{\text{eff}}^{I} = \frac{1}{(2\pi)^7} \int d^{10}x \sqrt{g} \left\{ R - \frac{1}{2}(\nabla\Phi)^2 - \frac{1}{4} e^{\Phi/2} \text{Tr}(F_{\mu\nu}F^{\mu\nu}) - \frac{1}{12} e^{\Phi} H^2 \right\}, \tag{9.10}$$

where $F_{\mu\nu}^a$ is the Yang-Mills field strength corresponding to the gauge field A_μ^a, and the trace is over the group index a.

This action has some familiar features in it, and the reader may have guessed what it is: it is the action corresponding to type I supergravity coupled to $SO(32)$ Super Yang Mills theory.

9.3.3 The heterotic theories

Since we did not discuss the heterotic string in very much detail, we will not derive its massless field content explicitly. We will however mention some facts before giving the field content.

On the right moving side, the heterotic string theories are just like the other string theories. In particular, we can distinguish an NS- and an R-sector giving rise to vector and spinor representations. But even though the heterotic string is a closed string, we cannot distinguish these sectors on the left moving side. The reason for this is that the left moving fermions do not have space-time indices, and hence are scalars under space-time Lorentz transformations. So we conclude that the bosons are the states coming from the NS-sector of the theory, where NS refers to the boundary conditions on the right moving fermions only.

We already mentioned that there are only two consistent groups that can act on the 32 fermionic world-sheet scalars: either we have $SO(32)$ or we have $E_8 \times E_8$. This gives us two different theories, but the massless field contents are practically the same.

First of all, we obtain the standard fields $G_{\mu\nu}$, Φ and $B_{\mu\nu}$. Moreover, we obtain 496 vector fields A^a_μ that transform under either $SO(32)$ or $E_8 \times E_8$. This field content looks a lot like the type I field content, and so does the effective action: it is (note the two minus signs with respect to the type I effective action!):

$$S^H_{\text{eff}} = \frac{1}{(2\pi)^7} \int d^{10}x \sqrt{g} \left\{ R - \frac{1}{2}(\nabla\Phi)^2 - \frac{1}{4} e^{-\Phi/2} \text{Tr}(F_{\mu\nu}F^{\mu\nu}) - \frac{1}{12} e^{-\Phi} H^2 \right\}, \tag{9.11}$$

These are the actions of type I supergravity coupled to $SO(32)$ or $E_8 \times E_8$ SYM theory.

9.4 Dualities

In lecture 6, we already discussed the concept of a duality. In this section, we will encounter two examples of dualities between superstring theories.

9.4.1 Type I - Heterotic duality

If we compare the two effective actions (9.10) and (9.11) coming from the type I and heterotic string theories respectively, we see a remarkable similarity. In fact, if we send

$$\Phi \to -\Phi, \tag{9.12}$$

we interchange the two actions. If on the heterotic side, we take the $SO(32)$ theory, the symmetries of the theories are also exactly the same, and we are actually describing the same theory twice. This led to the conjecture that the full type I and $SO(32)$ heterotic string theories would also be two descriptions of the same theory, or in other words: they would be dual to eachother.

How can two theories that – apart from their low energy effective actions – look so different, actually be the same? The answer is in the transformation (9.12). Note that in all the effective actions that were presented so far, we left out constants such as α' and the string coupling constant g_s. Of course, they do appear when we try to formulate the effective actions, but by a rescaling we can absorb them into the fields. In particular, it turns out that all the effective actions are invariant under the simultaneous translation and rescaling

$$\begin{aligned} \Phi &\to \Phi - c \\ g_s &\to e^c g_s, \end{aligned} \tag{9.13}$$

where c is an arbitrary constant. This implies that if we want to remove the string coupling constant from our effective actions, we should set $e^c = g_s^{-1}$, and hence

$$e^\Phi \to g_s e^\Phi. \tag{9.14}$$

From this, we see that in an effective action where g_s is absorbed in Φ, sending $\Phi \to -\Phi$ corresponds to inverting the string coupling constant.

This explains why the type I and $SO(32)$ heterotic string theories look so different: the weak coupling limit of one (which is what we describe using perturbation theory) corresponds to the strong coupling limit of the other.

In fact, this makes proving (or even testing) the conjectured duality very hard, since practically all quantities we can calculate in quantum field theory are based on perturbative calculations, and hence we do not know much about the strong coupling limits of string theories. However, there are some non-perturbative techniques that we can use to test this duality, and no discrepancies have been found. On the other hand, an exact proof has not been found either.

9.4.2 Type II T-duality

A duality conjecture that can be proved – even though we will not do so in these lectures – is the conjectured duality between the two type II theories. In fact, the relationship between the two turns out to be a simple T-duality like the ones we discussed in the bosonic case in lecture 6.

We have seen that T-duality can be regarded as a one-sided parity transformation. In other words: a T-duality takes one side (say the right moving one) of a closed string theory, and adds a minus sign to all of the oscillators on this side. We can view this as holding a 'mirror' in front of the right moving side of the theory, and hence it is natural that all the Lorentz generators on right moving states obtain a minus sign. In particular, Γ_{11} on this part of the theory will obtain a minus sign, and hence a state where left and right have opposite chiralities turns into a state where left and right have the same chiralities, and vice versa.

This strongly suggests that a type IIA state on a space-time compactified on a circle of radius R is T-dual to a type IIB-state on a space-time compactified on a circle of radius l^2/R. Since this transformation sends perturbative states to perturbative states, it is actually possible to prove this T-duality conjecture.

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Eleventh lecture

In this lecture, we will continue the story of string duality. So far, we have seen that five apparently different string theories exist, and we have also seen some relations between the theories.

One of the five string theories, type I string theory, has only been mentioned briefly so far. In section 11.1 we will describe it more precisely.

In section 11.2, we will introduce the concept of a charge lattice. Using this concept, we will see several interesting things: first of all it will indicate why the heterotic string can only have the gauge groups $SO(32)$ and $E_8 \times E_8$; secondly the concept can be used to show that these two theories are also T-dual to eachother.

In section 11.3, we will show that D-branes are not completely rigid objects – even though we did introduce them in this way – but are in fact dynamical. This is the basis for another duality conjecture: we can in fact see the fundamental string and the D-string as two different manifestations of the same type of object. We will say some things about this duality conjecture in sections 11.4 and 11.5.

11.1 Type I string theory

In the previous lectures, we have introduced type I string theory as the only string theory that contained open strings. There is however a more elegant way to arrive at type I strings, which is by considering *unoriented* closed strings.

So far, all of our strings have been oriented. By 'oriented', we mean that we can unambiguously define a 'left to right' direction on the string. Since we have a parameter σ that runs in a certain direction, it is clear that in general we can do this.

What would an unoriented string look like? Consider an operator Ω that changes the direction of the string:

$$\Omega : \sigma \to 2\pi - \sigma. \tag{11.1}$$

A string state which has no intrinsic orientation is a state that is invariant under this operation:

$$\Omega|\Phi\rangle = |\Phi\rangle \tag{11.2}$$

If we could restrict our Hilbert space to contain only these invariant states, we would have a theory of unoriented strings. It is not very difficult to do so: since $\Omega^2 = 1$, we can construct the projection operator $\frac{1}{2}(1 + \Omega)$ which projects the Hilbert space on the subspace consisting of unoriented states.

Let us carry out this prescription for type IIB string theory[12]. The bosonic fields in this theory can be expandeded in left- and rightmoving modes:

$$X(\tau,\sigma) = X_L(\tau+\sigma) + X_R(\tau-\sigma). \tag{11.3}$$

We see that the operator Ω exchanges X_L and X_R, which is of course what we expect from a parity-operator. On the level of the oscillators, this means that Ω exchanges α_n^μ and $\tilde{\alpha}_n^\mu$:

$$\Omega\alpha_n^\mu|\Phi\rangle = \tilde{\alpha}_n^\mu\Omega|\Phi\rangle \tag{11.4}$$

or

$$\Omega\alpha_n^\mu\Omega^{-1} = \tilde{\alpha}_n^\mu. \tag{11.5}$$

What does the projection on $\Omega = +1$ do to the massless spectrum of the theory? In the NS-NS sector, this question is easily answered. Remember that the bosonic massless states in this sector are of the form

$$M_{\mu\nu}\alpha_{-1}^\mu\tilde{\alpha}_{-1}^\nu|0,k\rangle. \tag{11.6}$$

This set of states consists of three irreducible representations of the Lorentz group: the trace Φ of the matrix M (the dilaton), the antisymmetric part $B_{\mu\nu}$ and the traceless symmetric part $G_{\mu\nu}$ (the graviton). Only the symmetric part of $M_{\mu\nu}$ is invariant under Ω, so in the oriented theory we only keep the fields $G_{\mu\nu}$ and Φ.

There is another, more intuitive explanation of this fact. Since $B_{\mu\nu}$ is antisymmetric, it can be regarded as a 2-form, and as we have seen these forms couple in a very natural way to the string world sheet. In other words: a string can have a magnetic charge coming from the field $B_{\mu\nu}$. However, the action of Ω is to change the sign on $B_{\mu\nu}$, and hence the sign of the charge. The states that are invariant under this operation are the states that are not charged with respect to B, and hence we do not expect this field to be present in an unoriented theory.

Next, consider the NS-R and R-NS sectors of the type IIB string theory. In this sector, we found two gravitini $\Psi_\pm^{a\mu}$. Since we only have one graviton, and the gravitino is its supersymmetric partner, we concluded that type IIB theory should have two supersymmetry generators, or in short: $N=2$ supersymmetry.

The two gravitini in type IIB theory are of opposite chirality, so a parity operation will exchange them:

$$\Omega : \Psi_+ \leftrightarrow \Psi_-. \tag{11.7}$$

This means that the separate gravitini are not conserved under Ω, but only a linear combination of the two is. In other words: we are only left with one gravitino, and hence we expect to obtain an $N=1$ supersymmetric theory.

Finally, look at the R-R sector Here, we found three antisymmetric tensor fields: $C, C_{\mu\nu}$ and $C_{\mu\nu\rho\sigma}$. In an exercise, we will show that under the parity projection, only the two-tensor $C_{\mu\nu}$ survives. Using a suggestive notation, we will call it $B_{\mu\nu}^R$.

To summarize: we see that the parity projection leaves us with three bosonic fields: a metric and a dilaton coming from the NS-NS sector, and a antisymmetric two-form coming from the R-R sector. This field content may sound familiar by now: it is almost exactly the bosonic field content of the heterotic string – although there the B-field came from the NS-sector. The only difference is that the heterotic string also contains 496 gauge fields A_μ^a. Can we also find these states in the unoriented string theory?

[12]The reason for taking type IIB is that it is non-chiral; type IIA theory is chiral and hence has an intrinsic orientation.

11.1.1 Adding open strings

Surprisingly, the answer to this question is positive. It turns out that the unoriented version of type IIB theory is in itself not consistent. This has to do with so-called *anomalies*: inconsistencies we encounter when we quantize a classically consistent theory. Unfortunately, discussing these anomalies in detail is beyond the scope of these lectures. It turns out to be very difficult to cancel the anomalies in unoriented IIB string theory, but there is a way: by including open strings to the spectrum.

Of course, the open strings in this theory will also have to be unoriented. Since we chose the open string parameter σ to run from 0 to π, the action of Ω on these strings is

$$\Omega : \sigma \to \pi - \sigma. \tag{11.8}$$

The oscillator part of the bosonic open string fields is $X^\mu = \sum \alpha_n^\mu \cos n\sigma$, so we see that the action of Ω on the oscillators can be expressed as

$$\Omega \alpha_n^\mu \Omega^{-1} = (-1)^n \alpha_n. \tag{11.9}$$

At first sight, this does not seem to get us any further. At the massive level, the open superstring has a state $\alpha_{-1}^\mu |0, k\rangle$, but this state obtains a minus sign under the parity operation, and hence is not present in the unoriented string! It looks like we can only add massive states to our unoriented theory in this way.

However, there is a way out of this problem: we can introduce Chan Paton degrees of freedom at the ends of the open string. If we give the string endpoints a label running from 1 to N, the massless string states become a matrix $A_\mu^{ij} = \alpha_{-1}^\mu |0, k\rangle \times |ij\rangle$. Ω still gives a minus sign on the oscillator, but moreover it exchanges the two endpoints of the string, and hence the indices i and j:

$$\Omega(\alpha_{-1}^\mu |0, k\rangle \times |ij\rangle) = -\alpha_{-1}^\mu |0, k\rangle \times |ji\rangle \tag{11.10}$$

This means that on the massless level, we only keep the antisymmetric part of A_μ^{ij}. More generally, at even mass levels we will obtain antisymmetric states; at odd mass levels symmetric ones. The number of these vector fields of course depends on the number N of index values.

Coming back to our anomaly story: it turns out that the anomalies in the unoriented theory are only cancelled if we add an open string sector with 32 Chan Paton charges to our theory. In this case, we have $\frac{1}{2} \times 32 \times 31 = 496$ vector fields – exactly the same number as the one we encountered for the heterotic string! Moreover, since the vector fields are antisymmetric and real, they can be viewed as the vector fields in an $SO(32)$ Yang-Mills theory.

Collecting all the massles fields, we see that the type I string theory contains exactly the same massless fields as the $SO(32)$ heterotic string theory – a fact that we already briefly mentioned in the previous lectures. This makes the statement that these two theories are completely dual to each other quite plausible.

11.2 Lattices, gauge groups and T-duality

In the sixth lecture, we introduced the concept of T-duality by compactifying a theory of closed strings on a circle. We can of course go further, and compactify such a theory on a d-dimensional torus instead. Just like we created a circle by taking a real line and identifying the points $X^1 + 2\pi R$, we can create a torus by taking the Euclidean space $\mathbb{R}^d$ and identifying points that are periodic with respect to a certain d-dimensional lattice $2\pi\Lambda$:

$$T^d = \mathbb{R}^d / 2\pi\Lambda. \tag{11.11}$$

This will imply certain periodicity restrictions on the fields $X^i, i = 1 \ldots d$, and it will quantize the momenta p^i_L and p^i_R, just like in the $d = 1$ case. To be precise, the total momentum $p^i_L + p^i_R = k^i$ will be in a lattice that is proportional to the dual lattice[13] Λ^*. Moreover, the winding vector, which like in the 1-dimensional case is given by $w^i = p^i_L - p^i_R$, is of course a vector in the original lattice. Combining these facts, and including the right factor of proportionality for k^i, we find that

$$p_{L,R} = \frac{1}{\sqrt{2}}(k \pm w). \tag{11.12}$$

¿From this, we see two important things. First of all, the inner product of p_L and p_R is an integer, and hence these vectors are in dual lattices. This means that the total momentum vector $p = (p^i_L, p^j_R)$ is in a lattice that has the following structure:

$$p \in \Lambda' \oplus \Lambda'^*. \tag{11.13}$$

¿From this construction, it is clear that this lattice is equivalent (i. e. the same, up to a rotation) to its own dual: it is *self-dual*. Moreover, if we define the inner product on this lattice to have signature (d, d), we see that the lattice vectors have even length:

$$p^2 = p_L^2 - p_R^2 = 2k \cdot w \in 2\mathbb{Z}. \tag{11.14}$$

We can summarize these results as follows: the (momentum) charge lattice $\Gamma^{d,d}$ of a closed string theory compactified on a d-dimensional torus is *even* and *self-dual*.

We can also derive these facts from a more general argument, involving the partition function for the one-loop amplitudes in closed string theory. Since these amplitudes correspond to Feynman diagrams that are tori, we expect the partition function to be invaraint under the usual $SL(2, \mathbb{Z})$-transformations of the modulus τ of the torus:

$$\tau \to \frac{a\tau + b}{c\tau + d}, \tag{11.15}$$

where a, b, c, d are integers such that $ad - bc = 1$. This group is generated by the two transformations $\tau \to \tau + 1$ and $\tau \to -1/\tau$, so it is enough to check invariance of the partition function under these two transformations. It turns out that this exactly implies that (p_L, p_R) is in an even, self-dual lattice.

Now, we will state some mathematical facts about lattices. Let $\Gamma^{(r,s)}$ be an even, self-dual lattice of signature[14] (r, s). Then

1. $r - s = 0 \bmod 8$;

2. if $r, s \neq 0$ then Γ is unique. In particular, if $s = r + 8n$, then Γ is of the form

$$\Gamma^{(1,1)} \oplus \ldots \oplus \Gamma^{(1,1)} \oplus E_8 \oplus \ldots \oplus E_8, \tag{11.16}$$

   where the first sum contains r terms, and the second one n terms. $\Gamma^{(1,1)}$ is the – essentially unique – lattice of signature $(1, 1)$; E_8 is a lattice of signature $(0, 8)$ that is intimately connected to the group E_8; it is called the *root lattice* of this group.

[13]The *dual lattice* corresponding to a certain lattice Λ is the set of all vectors that have integer inner product with all vectors in Λ. The fact that momentum-vectors are in the dual lattice is a well-known fact from – for example – solid state physics.

[14]The signature of a lattice is the signature of its inner product.

3. If $r = 0$ or $s = 0$, we have a complicated classification of possible lattices. The first two levels only have a low number of possibilities: $\Gamma^{(0,8)} = E_8$ is the only possibility, and $\Gamma^{(0,16)}$ is either $E_8 \oplus E_8$ or D_{16}, where the latter is the root lattice of $SO(32)$. After these levels, the number of possibilities grows enormously.

Now let us look at the heterotic string. In lecture 7, we saw that it was obtained by adding 32 right moving fermions to a ten-dimensional closed string theory. Without proof, we state that we can translate these fermionic degrees of freedom into the degrees of freedom of 16 bosons – the so-called process of *bosonization*.

Since these are only right moving bosons, their momenta have to be put on a lattice of signature $(0, 16)$. The reader will not be surprised to hear that the fact that there are only two possible lattices for this, immediately implies that there are only two possible gauge groups. These are the gauge groups corresponding to the lattices $E_8 \oplus E_8$ and D_{16}: the possible groups are $E_8 \times E_8$ and $SO(32)$.

Moreover, we can see a hint for another duality conjecture here. If we compactify a heterotic string theory on a circle, we add a normal boson to the momentum latice. So now we want the bosons to be on a lattice of signature $(1, 17)$, and we have seen that this is unique! In other words: when we compactify on a circle, it does not matter with which heterotic theory we start: they are equivalent. This is analogous to the T-duality we saw in the type II case.

In fact, we can now extend our concept of T-duality to tori of higher dimension than one. Of course, every lattice has a corresponding symmetry group. For Euclidean lattices – i. e. for lattices of definite signature – these groups are usualy small: they consist of reflections, and sometimes some rotations over angles like $\pi/2$. However, due to the minus signs in the inner product, the group is much larger in the non-Euclidean cases where the signature is (r, s). In fact, the full symmetry group of the lattice is $SO(r, s, \mathbb{Z})$. This is called the *T-duality group*.

To see that this group is really larger than what we would expect in a point particle case, look at the T-duality group for a type II string theory compactified on a two-torus. For an ordinary field theory, we would expect a duality group that simply transforms the modulus of the torus into equivalent moduli: the by now well-known $SL(2, \mathbb{Z})$. However, from the above discussion, we see that the full T-duality group is $SO(2, 2, \mathbb{Z})$, which is isomorphic to $SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z})$. We see that part of this group is the group we would naively expect; the extension is a real string-effect, that comes from the interchanging of momentum- and winding-numbers.

11.3 Dynamical D-branes

So far, we have viewed D-branes as completely rigid objects: fixed p-dimensional hyperplanes in space-time. However, as we will now show, this is only a low-energy approximation: in fact, D-branes are just as dynamical as strings are!

Let us take a closer look of the massless modes of an open string with its endpoint on a p-brane. We will use a notation where greek indices take values along the brane (i. e. $\mu = 0, \ldots, p$), lower case latin indices take values in the transverse directions ($a = p + 1, \ldots, D - 1$), and upper case indices are full space-time indices ($M = 0, \ldots, D - 1$).

The massless string modes are the states $\alpha^{\mu}_{-1}|0, k\rangle$ and $\alpha^{a}_{-1}|0, k\rangle$. In order for these string states to be massless, they have to be very short, and hence we can translate them into space-time fields that only live on the surface of the brane. Using a suggestive name, we will call these space-time fields $A^{\mu}(x^{\mu})$ and $X^{a}(x^{\mu})$.

It turns out that the field A^{μ} is a gauge field on the surface of the D-brane. What is the interpretation of the fields X^{a}? The answer may be surprising: these fields correspond to the transverse oscillations of the brane: they are the transverse coordinates of the brane!

To see why this is the case, recall from the previous lecture that space-time fields can couple directly to the string world sheet. For example, the graviton $g_{\mu\nu}\alpha_{-1}^{\mu}\alpha_{-1}^{\nu}|0,k\rangle$ can also be written in the string action as a term $\int d^2\sigma\, G_{\mu\nu}(x)\partial x^{\mu}\bar{\partial}x^{\nu}$. To avoid confusion, we used lower case letters for the string world-sheet fields x here.

The same thing is true for the fields X^a. In fact, since these are fields that live on the D-brane only, they only couple to the endpoints of the open strings, which live on the D-brane. The coupling term in the action is

$$\int d\tau \partial_\sigma x^a X_a(x), \tag{11.17}$$

where the integral is over the boundary of the open string world sheet. Clearly, this addition to the action does not change the bulk properties of the string, but it does change its boundary conditions. In an exercise, we will show that the new boundary conditions for the open string are

$$x_i(\sigma = 0, \pi) = X_i(x^\mu), \tag{11.18}$$

proving that the endpoints of the string – and hence the D-brane – are at position X_i.

11.4 Coupling constants

We have already mentioned several times that T-duality is a very special kind of duality, since it is perturbative: perturbative expansions in one theory correspond to perturbative expansions in the other. To make this more qualitative, let us look at the string coupling constants for type IIA and type IIB theories; call them g_s and $\tilde{g}_s$. The low energy effective action for these theories has the form

$$\frac{1}{g_s^2}\int d^{10}x\ \{\ldots\}, \tag{11.19}$$

for type IIA, and the same expression with a tilde for type IIB. If we compactify these theories on a circle, we obtain a nine-dimensional action, and the volume of the circle will be all that remains from the tenth dimension. The actions wil therefore be of the form

$$\begin{aligned} \text{IIA:}\qquad & \frac{2\pi R}{g_s^2}\int d^9x\ \{\ldots\} \\ \text{IIB:}\qquad & \frac{2\pi \tilde{R}}{\tilde{g}_s^2}\int d^9x\ \{\ldots\}. \end{aligned} \tag{11.20}$$

However, we know from T-duality that if we take the two radii to be dual, $\tilde{R} = \alpha'/R$, the two theories will be the same, and in particular they will have the same coupling constant. Inserting this, we see that for compactified IIA theory, the coupling constant for the corresponding IIB theory is

$$\tilde{g}_s = g_s\frac{\sqrt{\alpha'}}{R}. \tag{11.21}$$

Since α' is a very small length, in general the factor between the two coupling constants will be very large. However, notice that it is still a proportionality, so in principle, any perturbative expansion in one of the constants directly corresponds to a perturbative expansion in the other.

Now, let us turn to the coupling constants of the D-branes. Just like strings have a world-sheet action, D-branes have a $(p+1)$-dimensional world volume action. It can be shown that this action is of the form

$$\frac{T_p}{g_s}\int d^{p+1}x\,\{\ldots\}, \tag{11.22}$$

where T_p is the D-brane tension. Since T-dualizing in one of the brane directions turns a p-brane into a $(p-1)$-brane, we find the relation

$$\frac{2\pi R_p T_p}{g_s} = \frac{T_{p-1}}{\tilde{g}_s}, \tag{11.23}$$

and hence the D-brane tensions obey the relation

$$T_p = \frac{1}{2\pi\sqrt{\alpha'}} T_{p-1}. \tag{11.24}$$

Checking dimensions, we see that the tension of a D p-brane

$$T_p \sim \frac{1}{g_s(\alpha')^{p/2}}. \tag{11.25}$$

This shows us two important things. First of all, it shows that the D-brane tension is infinite if we perturb around $g_s = 0$. In other words, we only see dynamical D-branes if we use non-perturbative arguments. Moreover, the D-string is much lighter than all the higher dimensional D-branes. In other words: if we would go to strong coupling, we would expect the D-strings to become the fundamental degrees of freedom. We will elaborate on this point of view in the next section.

11.5 $SL(2,\mathbb{Z})$ S-duality of type IIB string theory

From the results of the previous section, we expect a duality in type IIB string theory if we exchange the fundamental string and the D-string, and invert the string coupling constant.

Can we find more evidence for this conjecture? Yes, we can: there are many similarities between the fundamental string and the D-string. For example, the fundamental string can be charged under the NS-NS B-field, but not under the R-R B-field. For the D-string, it is just the other way around. So both strings carry one kind of magnetic charge, and we expect the duality to incorporate the change $B^R_{\mu\nu} \leftrightarrow B^{NS}_{\mu\nu}$.

Moreover, the world sheet actions of both objects show many similarities. If we look at the bosonic part of the theory only, we know that the fundamental string action is

$$S_F = \frac{1}{\alpha'} \int d^2\sigma \, (\partial_\alpha X^i)^2, \tag{11.26}$$

where in the light cone gauge, $i = 1, \ldots, 8$.

We already saw that the D-string has a gauge field A^α and eight scalar fields X^i living on its world sheet. The world sheet action, which can be obtained by dimensionally reducing ten-dimensional SYM-theory, is

$$S_D = \frac{1}{g_s\alpha'} \int d^2\sigma \left\{ F^2_{\alpha\beta} + (\partial_\alpha X^i)^2 \right\}, \tag{11.27}$$

where $F_{\alpha\beta}$ is the field strength corresponding to A_α. However, notice that a two-dimensional gauge field has no degrees of freedom $(2-2=0)$ in the light cone gauge! Therefore, the effective degrees of freedom on the D-string world sheet are exactly the same as those on the F-string!

This type of duality is called an *S-duality*. In fact, it can be shown that we cannot only relate the D-string and the F-string in this way, but that we can take any linear combination of m D-strings and n F-strings as our fundamental object, as long as m and n are relatively prime[15]. Therefore, the S-duality of type IIB string theory is not a single duality, but in fact an entire $SL(2,\mathbb{Z})$ group of these dualities!

[15]This requirement is basically to guarantee the existence of a bound state: if m and n have a common divisor d, the system can split up in to d smaller subsystems.

Caputcollege string theory 1998-99

Twelfth lecture

In this final lecture, we will pick up some loose ends that we left in the previous lectures on string duality. So far, we have seen that string theory is much more than a collection of five perturbative theories: many of the theories are connected by (usually non-perturbative) dualities, suggesting that we are in fact looking at several aspects of some larger, and hopefully even more beautiful theory.

However, there are still some missing links in our story. First of all, we related the type I theory and the heterotic theories, and we related the two type II theories, but we have not found a relation between these groups of theories yet. This is not very surprising, since as we have seen the two groups of theories are very different – even in their low-energy behaviour. However, there is still a quite simple way to relate them.

To find this relation, we will first study the concept of an orbifold in section 12.1. In section 12.2, we will then use the generalized concept of an orientifold to show that type I and type IIA theory (ore more accurately: a theory that in the bulk is equivalent to type IIA) are related by a duality.

Another still open question is related to S-duality. We have seen that taking the coupling constant to infinity exchanges type I and heterotic $SO(32)$ string theory, and maps the IIB theory onto itself. The natural question arises: what happens if we take the coupling constant to infinity in type IIA or heterotic $E_8 \times E_8$ theory? In section 12.3 we will answer this question for type IIA theory. This leads to the concept of M-theory, which has a conjectured description in terms of matrices. This so-called matrix theory is briefly discussed in section 12.4.

In section 12.5, we describe the strong coupling limit of the $E_8 \times E_8$ heterotic string theory, which concludes our search for dualities. Section 12.6 gives an overview of the relations we found between the perturbative string theories.

12.1 Orbifolds

Until now, we have compactified our string theories on the simplest possible compact manifolds: circles S^1, or products of circles (tori) T^n. Of course, there are many other possibilities. One of these possibilities is to compactify string theories on a so-called *orbifold*.

An orbifold is constructed out of a space X (for which we will usually take spaces such as $\mathbb{R}^n$ or T^n) and a symmetry group G of this space. The orbifold is then the space X/G. For example, take $X = \mathbb{R}^2$, the Euclidean plane. This plane has a mirror symmetry that sends the point x to $-x$. Since the square of this opration is the identity, this generates a symmetry group that is isomorphic to $\mathbb{Z}_2$.

We can now construct the orbifold $\mathbb{R}^2/\mathbb{Z}_2$ by identifying the points that are in the same orbit under the group action, i. e. identifying x and $-x$. This is equivalent (see figure 12.1) to taking only the upper half plane, and indentifying the positive and negative x-axes. If we would do this using a sheet of paper, we would obtain a cone. It is clear that the manifold is smooth (and flat) everywhere, except at the special point $x = (0,0)$. The reason for this is that $(0,0)$ is a fixed point under the group action. This statement is general: points that are invariant under some non-trivial group element ($gx = x$ for some $g \in G \setminus \{1\}$) become singular points on the orbifold.

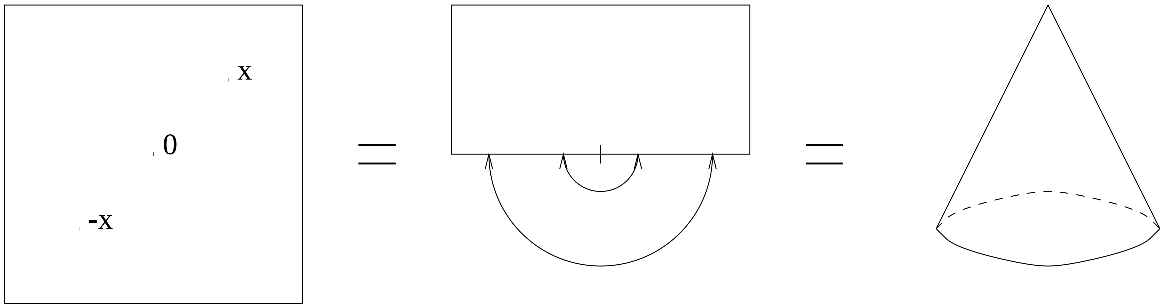

Figure 12.1: Construction of the orbifold $\mathbb{R}^2/\mathbb{Z}_2$.

Another important example of an orbifold is the space $S^1/\mathbb{Z}_2$. The circle is obtained from the real line by identifying the points x and $x + 2\pi R$. After doing this, we can also mod out a $\mathbb{Z}_2$ symmetry: $x \sim -x$. Pictorially, this looks like figure 12.2. We see that we obtain a line segment with two singular endpoints: one at $x = 0$ and one at $x = \pi R$. Again, these are the points that are invariant under the $\mathbb{Z}_2$-symmetry.

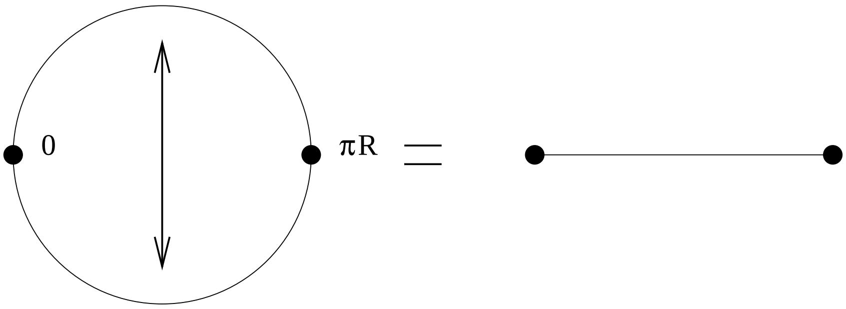

Figure 12.2: Construction of the orbifold $S^1/\mathbb{Z}_2$.

Now, we may wonder how we can do quantum mechanics on a space like this. States that live on the orbifold are the same thing as states that live on the original manifold and are invariant under the group transformations. In other words, we can project our original Hilbert space into the Hilbert space H^G of states that are G-invariant:

$$g|\psi\rangle = |\psi\rangle \qquad \forall g \in G. \tag{12.1}$$

Now if we want to calculate one-loop partiton functions, we take the usual trace over this new Hilbert space. It is easily seen that this is equivalent to taking a 'weighed trace' over the original space:

$$\mathrm{Tr}_{H^G}(q^{L_0}) = \frac{1}{\#G}\sum_{g\in G}\mathrm{Tr}_H(gq^{L_0}). \tag{12.2}$$

In a picture (see figure 12.3), this corresponds to the following. Instead of creating the torus by sewing together the opposite sides of a rectangle, we 'twist' the timelike direction by a factor of g, and then sew the sides together. We then average over all $g \in G$.

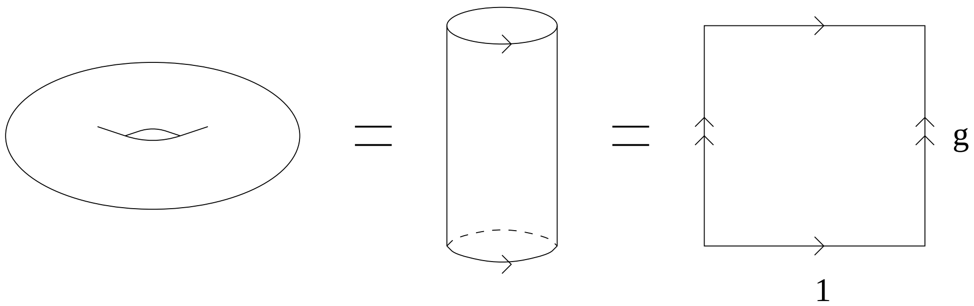

Figure 12.3: The orbifold partition fuction in quantum mechanics is obtained by 'twisting' the time-like direction using the group elements g.

In ordinary quantum mechanics, this would be the entire story, but since we are talking about strings here, a second thing can happen. Look at the closed string state on the orbifold $\mathbb{R}^2/\mathbb{Z}_2$ in figure 12.4a. In the original space $\mathbb{R}^2$, this corresponds to two copies of the same closed string state. However, a completely different thing can happen if we consider a closed string state that winds around the singular point in the orbifold, as in figure 12.4b. This corresponds to two copies of an *open* string in the original $\mathbb{R}^2$!

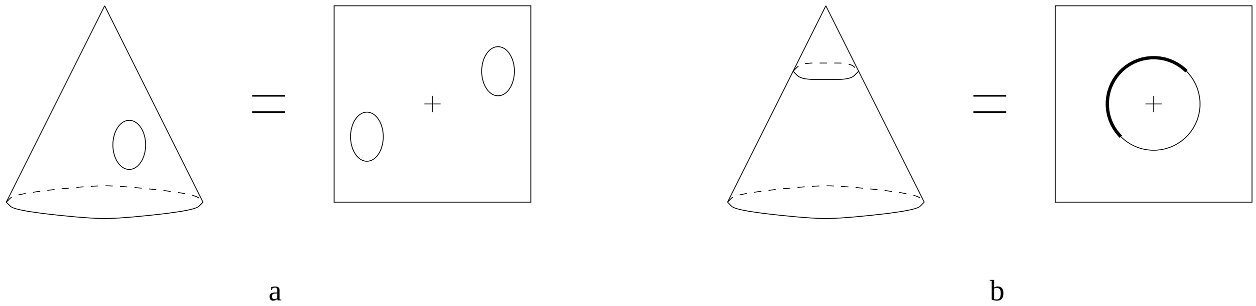

Figure 12.4: Twisted states in $\mathbb{R}^2$ lead to closed string states in $\mathbb{R}^2/\mathbb{Z}_2$.

We see that not all closed string states that live on the orbifold can be obtained by looking at closed string states on the original space! In general, we will have to take into account that the strings in the original space can start at some point, and end at *any* point that is in the same group orbit. In a formula:

$$X^\mu(\sigma + 2\pi) = gX^\mu(\sigma), \tag{12.3}$$

for some $g \in G$. These states are called *twisted states*.

The existence of twisted states in the space direction is also quite natural from another point of view: that of modular invariance. We want our theory to be invariant under the $SL(2,\mathbb{Z})$ group of modular transformations of the torus. A particular invariance is the one under $\tau \to -1/\tau$. This transformation exchanges the two sides of a torus, so if we have twists in the time-like direction, we expect to have twists in the space-like direction as well.

There is one condition we have to impose on the space-like and time-like twists. Consider the fundamental region of a torus, depicted in figure 12.5. In general, we can have any twist g in the space-like direction, and any twist h in the time-like direction. However, if we look at the curve γ that we drew on the surface, we see that not any combination of g and h will work. If we start at the left bottom corner of the curve, and follow it anti-clockwise, we see that we obtain a total twist $ghg^{-1}h^{-1}$. However, note that we can continuously deform the curve γ into a single point. Since we are dealing with a discrete group G, the total twist cannot change under this deformation. Therefore, it must be equal to the identity element of G, and we arrive at the important condition that the space-like and time-like twists must *commute*.

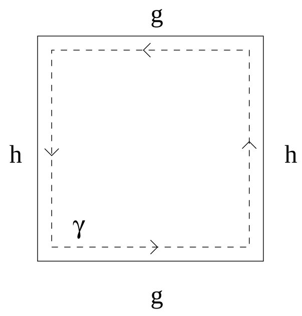

Figure 12.5: The space-like and time-like twists of a torus must commute.

Using all of the information obtained above, we see that the final formula for the one-loop partition function is

$$Z = \frac{1}{\#G} \sum_{gh=hg} \mathrm{Tr}_{H_g}(hq^{L_0}), \tag{12.4}$$

where Tr_{H_g} stands for the trace over all twisted states that have a space-like twist g.

Finally, it is interesting to look at the twisted states at low energy (i. e. $\alpha' \to 0$). In tis limit, all string lengths go to zero, and the string sits at a certain point in space-time. However, the twisted states are connected to their image states, that live at another point of space-time, related to the first one by a group transformation. Therefore, the string length can only go to zero if the second point is the same as the first point! In other words: in the low energy limit, twisted states can only live at fixed points under the group action, i. e. at the orbifold singularities! Away from these singularities, we only have the ordinary, non-twisted states.

12.2 Orientifolds and type I - type I' duality

With the dualities that we have seen so far, we have been able to relate the type I and heterotic string theories to eachother, and to relate the two type II string theories. To unite all the string theories, we therefore need one more duality. In this section, we will consider such a duality.

To be precise, we will consider what happens if we apply a T-duality to type I string theory compactified on a circle. In the previous lecture, we saw that type I string theory can be constructed by applying a parity opration Ω to type IIB string theory. This gives us a theory of unoriented closed strings, to which we have to add open strings for consistency. In this section, we will only consider the open string sector of the theory.

Note that the parity operator Ω does *not* create an orbifold, since it acts as a parity operator on the world sheet – not on space-time. However, this picture changes if we apply a T-duality to the type I string theory. To see this, recall that Ω interchanges the left- and right-moving parts of the fields X:

$$X = f(z) + g(\bar{z}) \xrightarrow{\Omega} X' = g(z) + f(\bar{z}). \tag{12.5}$$

What does this operation look like in the T-dual theory? We have seen that T-duality amounts to a parity operation (i. e. a minus sign) on the right-moving part of the theory, so in the dual theory, the action of Ω is given by

$$\tilde{X} = f(z) - g(\bar{z}) \xrightarrow{\Omega} \tilde{X}' = g(z) - f(\bar{z}). \tag{12.6}$$

If we express this in terms of operations on the dual theory, we see that we have a combination of a parity operation $\tilde{\Omega}$ on the world sheet and a space time parity operation g.

If we only projected on $\tilde{\Omega}$-invariant states, we would obtain another theory of unoriented closed strings. If we only projected on g-invariant states, we would obtain an orbifold. However, the above prescription tells us to do both, so we obtain a new type of construction which is called the *orientifold* construction.

Under this projection, two types of states survive (see figure 12.6). First of all, we have the bulk theory. Here, we still have oriented strings. The reason for this is that in the dual theory, Ω does not project out the oriented states, but it relates an oriented state to its mirror image in space-time, which has opposite orientation. Since we started from a type IIB theory, the strings in the bulk of the T-dual theory wil be type IIA strings.

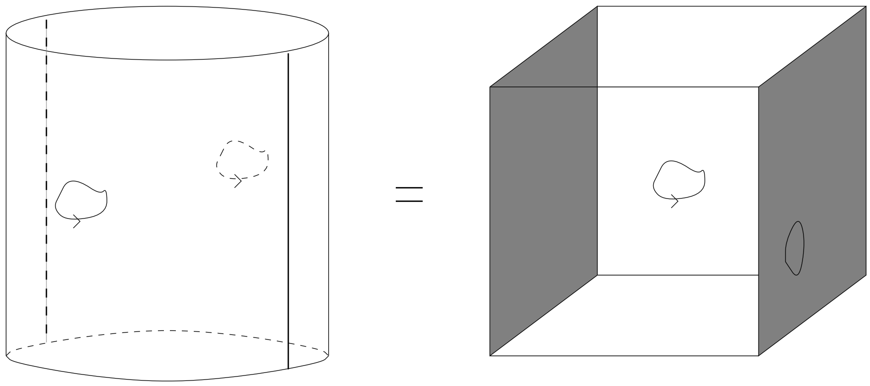

Figure 12.6: The orientifold construction.

Secondly, there are the two orientifold planes. These are the (8+1)-dimensional objects that are invariant under the space-time parity operation. String states that live on these branes are invariant under the g-operation, and hence must be invariant under the $\tilde{\Omega}$-operation as well. This means that on the orientifold planes, we do obtain a theory of unoriented strings.

To sum up: the T-dual of type I theory is a theory that lives on an $\mathbb{R}^9$ with two orientifolds at the boundary. In the bulk, it is equivalent to type IIA string theory, and on the orientifold it is equivalent to unoriented closed string theory. The full theory is also called type I' theory.

If we let the compactification radius of the original type I theory tend to 0, the radius of the dual theory will go to infinity, and hence the two orientifold planes will move off to infinity as well. In the end, we are only left with the bulk theory, which is a type IIA theory. This establishes the final duality we were looking for.

12.3 The strong coupling limit of type IIA string theory

In the previous lectures, we have seen two examples of *S-dualities*: dualities that relate the strong coupling limit of one theory to the weak coupling limit of another. First of all, we have seen that type I and $SO(32)$ heterotic string theory are related in this way. Secondly, it turned out that type IIB string theory was S-dual to itself. The natural question arises: what are the strong coupling limits of the other two string theories? In this section, we will look at the strong coupling limit of type IIA string theory.

Let us first look at the low energy effective action. We know that the low energy effective action for type IIA string theory is ten-dimensional type IIA supergravity. Moreover, this theory can be derived by dimensionally reducing the (essentially unique) eleven-dimensional supergravity to ten dimensions. There are several ways to describe this dimensional reduction: one is to simply require that the fileds do not depend on the eleventh direction, but another way which leads to the same results is to compactify the theory on a circle and make the radius of the circle go to zero.

We now claim that this correspondence does not only hold for the low-energy effective action, but also for the full type IIA string theory: it is the dimensional reduction of some eleven-dimensional theory. This claim has not been proved yet, but there are several strong arguments that support it.

A very important role in these arguments is played by the D0-branes. As we have seen, these are the lightest solitonic states in type IIA theory, having a mass of order $1/l_s g_s$. In fact 'lightest' may be a strange word, since in perturbative string theory (where $g_s = 0$) these states have infinite mass, but since all other D-branes have even higher negative powers of g_s in their mass, the D0-branes are much lighter in the limit $g_s \to 0$.

Another claim that has been made (and partially proven) is that there exist bound states of N D0-branes, which have a mass of $N/l_s g_s$. Moreover, these states are BPS-states, which means that (up to a factor), their charge under the type IIA one-form field is equal to their mass. Hence we have a tower of evenly spread massive states, which is usually the sign of a compactification on a circle.

To see this, suppose that we have some eleven-dimensional theory containing gravity, and that we compactify it on a circle. The massless graviton in eleven dimensions satisfies $m_{11}^2 = -p_M^2 = 0$, where $M = 0, \ldots, 9, 11$. [16] However, in ten dimensions this object has a mass:

$$m_{10}^2 = -p_\mu^2 = p_{11}^2, \tag{12.7}$$

where $\mu = 0, \ldots, 9$. Furthermore, we know that since we are in a periodic dimension, $p_{11} = N/R_{11}$, and hence we obtain

$$m = \frac{N}{R_{11}}, \tag{12.8}$$

which is the tower of states we mentioned before.

More support for our claim comes from the observation that in the low-energy theory, the eleven-dimensional metric G_{MN} reduces to the ten-dimensional metric $g_{\mu\nu}$, the one-form $A_\mu =$

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[16] To adapt to the literature, we label the extra dimension 11 instead of 10.

$g_{\mu,11}$ and the dilaton $\Phi = g_{11,11}$. Therefore, the charge corresponding to A_μ should be related to the charge corresponding to $g_{\mu,11}$, which is exactly the momentum p_{11}. This explains the BPS-property: the charge of the D0-brane is equal to its mass.

Now, compare the two coefficients that give the distance between two consecutive masses in the tower of states. In the case of D0-branes, we have $m = N/l_s g_s$, whereas from dimensional reduction, we have $m = N/R_{11}$. If our claim is correct, we should have the relation

$$R_{11} = l_s g_s, \tag{12.9}$$

which says that the radius of compactification (up to a constant) is equal to the string coupling constant! This fits in perfectly with our discussion, since this means that we recover perturbative type IIA string theory if we let $g_s \to 0$, and hence $R_{11} \to 0$. In other words: even though it comes from an eleven-dimensional theory, perturbative type IIA string theory is ten-dimensional!

We can of course turn around the argument: if we start from perturbative type IIA string theory, and increase the coupling constant, the theory will 'grow another dimension', and become eleven-dimensional. If we let the coupling constant go to infinity, the radius of compactification will go to infinity, and we obtain an uncompactified eleven-dimensional theory. This theory is called M-theory. Note that we do not have any perturbative description of this theory; all we have is a strong claim that it exists, and the notion that its low energy effective action will be eleven-dimensional supergravity.

In the above discussion, we derived the relation $R_{11} = l_s g_s$, which is a relation between an eleven-dimensional quantity on the one side, and two ten-dimensional quantities on the other. For completeness and for later use, we will also derive another relation between the eleven-dimensional parameters and the ten-dimensional ones.

Eleven-dimensional supergraity has an action that behaves like $\int d^{11}x \sqrt{g}R + \dots$. Since the Lagrangian has dimension $[length]^{-2}$, and integrating adds eleven dimensions of length, we need a coupling constant in front of this action that has dimension $[length]^{-9}$. Since the only natural unit of length in supergravity is the eleven-dimensional Planck length, the action should up to numerical factors be

$$S_{11} = \frac{1}{l_p^9} \int d^{11}x \, \{\sqrt{g}R + \dots\}. \tag{12.10}$$

After dimensional reduction to ten dimensions, all that remains from the eleventh dirction is a volume factor $2\pi R_{11}$. The action now looks like

$$S_{10} = \frac{R_{11}}{l_p^9} \int d^{10}x \, \{e^{-2\Phi}\sqrt{g}R + \dots\} \tag{12.11}$$

Let us compare this to the ten-dimensional low energy effective action for type IIA string theory. We know that the Lagrangian is the same as the one above. Moreover, the coupling constant in front of the Lagrangian must be g_s^{-2}. Since this is a dimenionless constant, we need eight more powers of length to make the action dimensionless. In string theory, the natural length scale is the string length, so we obtain

$$S'_{10} = \frac{1}{l_s^8 g_s^2} \int d^{10}x \, \{e^{-2\Phi}\sqrt{g}R + \dots\}. \tag{12.12}$$

But since we claimed that $S_{10} = S'_{10}$, the coupling constants for both theories should be the same, so we obtain

$$l_p = g_s^{1/3} l_s. \tag{12.13}$$

Together with equation (12.9), this expresses the eleven-dimensional parameters in terms of the ten-dimensional ones, and vice versa.

Finally, let us look at what happens if we compactify M-theory on a torus of radii R_9 and R_{11}. This is equivalent to type IIA string theory with coupling constant g_s coming from R_{11}, compactified on a circle of radius R_9. By varying R_9 and applying T-duality, we can go from uncompactified type IIA theory ($R_9 \to \infty$) to uncompactified IIB theory ($R_9 \to 0$).

However, the choice of an eleventh dimension is of course completely arbitrary. We might just as well have taken R_9 to be the coupling constant g_s. This means that varying R_9 interpolates between compactified type IIA theory ($R_9 = g_s \to 0$) and M-theory on a circle ($R_9 = g_s \to \infty$). In other words: S-dualities and T-dualities can be the same thing!

12.4 Matrix theory

In the previous section, we mentioned that even though we dan define M-theory, we do not have a perturbative description of it yet. However, there is a conjecture that suggests a certain perturbative description: the matrix theory conjecture that was put forward by Banks, Fischler, Shenker and Susskind in 1996. In this section, we will say a few words about this conjecture.

Since D0-branes are so important in deriving M-theory from type IIA string theory, we will take a closer look at a collection of N D0-branes. As we have seen, the low-energy modes of the open strings stretching from brane i to brane j become fields X^μ_{ij} in the low-energy effective action. Since we are dealing with D0-branes here, the low energy effective action is the dimensional reduction to (0+1) dimensions of (9+1)-dimensional SYM-theory, in which the matrices X_{ij} are the fields. The bosonic part of the Lagrangian is

$$L = \mathrm{Tr}\left(\dot{X}_\mu^2 + [X_\mu, X_\nu]^2\right). \tag{12.14}$$

The second term in this Lagrangian is a potential for the matrices. Since it is positive definite, the states of lowest energy are the states that make this potential vanish. This means that we have a set of matrices that do all commute. There is a mathematical theorem that ensures us that this means that we can simultaneously diagonalize these matrices.

So instead of N^2 degrees of freedom, the lowest energy states only have N degrees of freedom per coordinate. It is tempting to interpret these degrees of freedom as the coordinates of the D0-brane, and in fact we saw in the previous lecture that we are allowed to do so.

We may wonder when the energies of the excited states are so high that we can ignore them, and indeed talk about N separate D-branes. To find the answer, perturb around the diagonal states and write

$$X_{ij} = Y_i\delta_{ij} + \delta X_{ij}. \tag{12.15}$$

As a definite example, take the case $N = 2$. If we calculate the Lagrangian in terms of these new fields, we find

$$L = (\delta\dot{X}_{ij})^2 + |Y_1 - Y_2|^2 \delta X_{12}\delta X_{21}. \tag{12.16}$$

We see that the off-diagonal modes are harmonic oscillators of mass (frequency) $|Y_1 - Y_2|$. Therefore, as long as the differences between the Y_i are large (in terms of the string length), we can indeed ignore the massive modes[17] and consider the degrees of freedom as those coming form N separate D0-branes.

[17]There is a subtlety here: the zero-modes of the harmonic oscillators still contribute to the energy, so we might expect that the D0-branes can not move away from eachother. However, if we introduce the fermions as well, the zero-mode energies are seen to vanish, and there is no energetic barrier that prevents the D0-branes from moving freely.

However, as soon as the Y_i come close together, we have to take the off-diagonal fields into account, and we can no longer speak of N separate D0-branes.

The Lagrangian (12.14) has many more interesting properties: it can describe all sorts of bound states, and even higher-dimensional branes. In fact, it is so rich, that Banks, Fischler, Shenker and Susskind conjectured that it completely describes M-theory in a specific limit. In the past two years, a lot of evidence has been found to support this conjecture.

12.5 The strong coupling limit of $E_8 \times E_8$ heterotic string theory

Finally, we will discuss the strong coupling limit of the $E_8 \times E_8$ heterotic string theory – the only strong coupling limit that we have not discussed yet. To do so, we will use all of our knowledge, and combine a series of T- and S-dualities.

First of all, we compactify the theory on a circle of radius R_9, and apply a T-duality. Using the results of previous sections, we know that we will obtain an $SO(32)$ heterotic theory with parameters

$$\begin{aligned} R'_9 &= \frac{l_s^2}{R_9} \\ g'_s &= \frac{l_s}{R_9} g_s \end{aligned} \tag{12.17}$$

Next, we apply an S-duality. This leads to a type I theory, with parameters

$$\begin{aligned} R''_9 &= \frac{R'_9}{\sqrt{g'_s}} &&= \frac{l_s^{3/2}}{\sqrt{R_9 g_s}} \\ g''_s &= \frac{1}{g'_s} &&= \frac{R_9}{l_s g_s}. \end{aligned} \tag{12.18}$$

Finally, we apply another T-duality. In this lecture, we have seen that we obtain a type I' theory: a type IIA theory with two orientifold 8-planes at the boundary. The parameters are

$$\begin{aligned} R'''_9 &= \frac{l_s^2}{R''_9} &&= \sqrt{R_9 g_s l_s} \\ g'''_s &= \frac{l_s}{R''_9} g''_s &&= R_9^{3/2} l_s^{-3/2} g_s^{-1/2} \end{aligned} \tag{12.19}$$

Now, we want to take the appropriate limits. First of all, we want to take R_9 to infinity, to obtain an uncompactified version of our original theory. After that, we want to take the coupling constant to infinity and see what happens.

It seems like we are hopelessly stuck, since the first limit already makes both the radius and the coupling constant of the type I' theory infinite. However, we now know that this theory (since in the bulk it is IIA) can be seen as coming from an eleven-dimensional M-theory with two nine-dimensional orientifold planes. In this theory, we now have three parameters: the Planck length l_p, the distance between the nine-branes R'''_9, and the compactification radius R_{11}. The last two should be expressed in terms of the Planck length:

$$\begin{aligned} l_p &= {g'''}_s^{1/3} l_s \\ \frac{R'''_9}{l_p} &= g_s^{2/3} \\ \frac{R_{11}}{l_p} &= \frac{R_9}{g_s^{1/3} l_s}. \end{aligned} \tag{12.20}$$

Using these units, we can let $R_9 \to \infty$. We see that the compactification radius R_{11} tends to infinity, whereas the distance between the 9-branes is unchanged. In other words: uncompactified heterotic $E_8 \times E_8$ string theory with coupling constant g_s is dual to M-theory compactified on an orientifold S_1/Z_2 with radius $g_s^{2/3}$. If we send the coupling constant to infinity, we obtain the uncompactified M-theory.

12.6 An overview of string dualities

We end this final lecture by giving an overview of the different string dualities we have encountered. They are summarized in figure 12.7

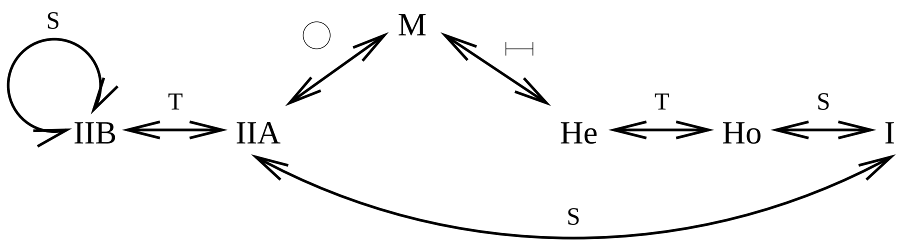

Figure 12.7: An overview of string dualities.

There are T-dualities relating the two type II theories and relating the two heterotic theories. Moreover, there are S-dualities relating IIB to itself, I to IIA and Ho to I. Finally, there is M-theory, which has a special position in the diagram since it is not a string theory and since it is in a way an 'overarching' theory. Compactifying M-theory on a circle gives IIA theory; compactifying it on an S^1/Z_2 orientifold gives heterotic $E_8 \times E_8$ theory. Uncompactified M-theory is S-dual to both perturbative IIA and He string theories.