1 Lecture 14

1.1 Exercise A

Consider the compactification of IIA and IIB superstrings on a six-torus T^6. For the purpose of this exercise we are only going to take the bosonic fields and ignore the fermionic fields. Assume that the four-dimensional massless fields are obtained by taking any ten-dimensional field, and by assuming that all indices are either four- or six-dimensional. Call the first type of index μ and the second type of index a. So $\mu = 0, 1, 2, 3$ and $a = 4, 5, 6, 7, 8, 9$. How many and which massless fields will be obtain in four-dimensions? Is the spectrum we obtain for IIA and IIB identical? As an example, a ten-dimensional gauge field A_M consists of A_μ and A_a. The first is four-dimensional gauge field, the second will give six different massless scalar fields in four dimensions.

1.2 Exercise B

In strings on group manifolds G, the world-sheet theory has a number of holomorphic world-sheet currents of conformal weight one which we will call $J^a(z)$ with $a = 1, \ldots, \dim G$. The operator product expansion of these currents is of the form

$$J^a(z)J^b(w) = \frac{kd^{ab}}{(z-w)^2} + \frac{f^{ab}_c J^c(w)}{z-w}.$$

Here, d^{ab} represents an invariant inner product on the algebra corresponding to G.

a) Expand each current in modes

$$J^a(z) = \sum_{m\in\mathbb{Z}} J^a_m z^{-m-1}$$

and show that

$$[J^a_m, J^b_n] = mkd^{ab}\delta_{m+n,0} + f^{ab}_c J^c_{m+n}.$$

b) Consider now $SU(2)$ with generators $J^+(z)$, $J^3(z)$ and $J^-(z)$. The structure constants f^{ab}_c can be read off from the commutators $[J^+, J^-] = 2J^3$ and $[J^3, J^\pm] = \pm J^\pm$. The Jacobi identity for J^a_{-1}, J^b_0 and J^c_{+1} fixes d^{ab} up to an overall numerical factor. Find this expression for d^{ab}.

c) Verify with the above result for d^{ab} that J^+_{+1}, J^-_{-1} and $J^3_0 + \eta$ also form an $SU(2)$ algebra for a suitable value of η and determine this value of η.

d) Now use the fact that in any unitary representation of $SU(2)$ the generator J^3 takes only half-integral values to show that η is quantized in suitable units. From this deduce that d^{ab} can only take discrete values for a unitary theory. This is the algebraic derivation of the quantization of the level of the WZW model describing string theory on a group manifold. As described in class, there is also a geometric interpretation.