String Theory Course 2005: Exercises March 14

Consider d free scalar bosons $X^\mu(z,\bar{z})$. The non-vanishing components of their energy-momentum tensor are

$$
\begin{aligned}
T(z) \equiv T_{zz}(z) &= -\frac{1}{\alpha'} : \partial X^\mu \partial X_\mu : \, , \\
\tilde{T}(\bar{z}) \equiv T_{\bar{z}\bar{z}}(\bar{z}) &= -\frac{1}{\alpha'} : \bar{\partial} X^\mu \bar{\partial} X_\mu : \, .
\end{aligned} \tag{1}
$$

The free boson OPE is

$$
X^\mu(z,\bar{z}) X^\nu(w,\bar{w}) \sim -\frac{\alpha'}{2} \delta^{\mu\nu} [\log(z-w) + \log(\bar{z}-\bar{w})] \, . \tag{2}
$$

Consider the operator

$$
: e^{ik_\mu X^\mu(z,\bar{z})} : \equiv : e^{ik\cdot X(z,\bar{z})} : \, . \tag{3}
$$

1. Use the free boson OPE to show that

$$
T(z) \; : e^{ik\cdot X(z,\bar{z})} : \sim \frac{\alpha' k^2/4}{(z-w)^2} \; : e^{ik\cdot X(z,\bar{z})} : \; + \frac{ik\cdot \partial X}{z-w} \; : e^{ik\cdot X(z,\bar{z})} : \, . \tag{4}
$$

2. Use this to show that $: \exp(ik\cdot X(z,\bar{z})) :$ is a conformal field and show that its conformal weights are $(h = \alpha' k^2/4, \bar{h} = \alpha' k^2/4)$.

3. Show that

$$
\langle 0| : e^{ik_1\cdot X(z_1,\bar{z}_1)} : : e^{ik_2\cdot X(z_2,\bar{z}_2)} : |0\rangle = (2\pi)^d \delta^{(d)}(k_1+k_2) |z_1 - z_2|^{\alpha' k_1\cdot k_2} \, . \tag{5}
$$

   Hints:

   - Use the commutation relations $[\alpha_m^\mu, \alpha_n^\nu] = m\delta^{\mu\nu}\delta_{m+n}$ (and similarly for the right-movers), the fact that α_m^μ annihilate the vacuum for $m > 0$ and the BCH formula

$$
\exp(A)\exp(B) = \exp(B)\exp(A)\exp([A,B]) \tag{6}
$$

     to show that the non-zero modes contribute a factor

$$
\left| 1 - \frac{z_2}{z_1} \right|^{\alpha' k_1\cdot k_2} \, . \tag{7}
$$

- Use the commutation relation $[x^\mu, p^\nu] = i\delta^{\mu\nu}$, the fact that p^μ annihilates the vacuum and $\langle : \exp(ik \cdot X) : \rangle = (2\pi)^d \delta^{(d)}(k)$ to show that the zero modes contribute a factor

$$(2\pi)^d \delta^{(d)}(k_1 + k_2) |z_1|^{\alpha' k_1 \cdot k_2} \ . \tag{8}$$

The generalization to n-point functions is

$$\begin{aligned} &\langle 0| : e^{ik_1 \cdot X(z_1, \bar{z}_1)} : \ldots : e^{ik_n \cdot X(z_n, \bar{z}_n)} : |0\rangle \\ &= (2\pi)^d \delta^{(d)}(k_1 + \ldots + k_n) \prod_{i<j} |z_i - z_j|^{\alpha' k_i \cdot k_j} \ . \end{aligned} \tag{9}$$

4. Show that the two and three point functions are consistent with the general structure obtained in sections 5.2 and 5.3 of the CFT notes.