GRAPHS THAT ARE NOT COMPLETE PLURIPOLAR

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Abstract. Let $D_1 \subset D_2$ be domains in $\mathbb{C}$. Under very mild conditions on D_2 we show that there exist holomorphic functions f, defined on D_1 with the property that f is nowhere extendible across ∂D_1, while the graph of f over D_1 is **not** complete pluripolar in $D_2 \times \mathbb{C}$. This refutes a conjecture of Levenberg, Martin and Poletsky [6].

1. Introduction

Levenberg, Martin and Poletsky [6] have conjectured that if f is a holomorphic function, which is defined on its maximal set of existence $D \subset \mathbb{C}$, then the graph

$$\Gamma_f = \{(z, f(z)) : z \in D\}$$

of f over D is a complete pluripolar subset of $\mathbb{C}^2$. I.e. there exists a plurisubharmonic function on $\mathbb{C}^2$ such that it equals $-\infty$ precisely on Γ_f (see e.g. [4]). They gave support for this conjecture in the sense that they could prove it for some lacunary series. More support was provided by Levenberg and Poletsky [7] and by the second author [9, 10, 11]. Nevertheless, in this paper we show that the conjecture is false.

In fact we have

Theorem 1.1. *Let $D_1 \subset D_2$ be domains in $\mathbb{C}$. Assume that $D_2 \setminus D_1$ has a density point in D_2. Then there exists a holomorphic function f with domains of existence D_1 such that the graph Γ_f of f over D_1 is not complete pluripolar in $D_2 \times \mathbb{C}$.*

In case $D_2 \setminus D_1$ has no density point in D_2, it is known that Γ_f is complete pluripolar in $D_2 \times \mathbb{C}$ (see [10]). If we take in Theorem 1.1 for D_1 the unit disc $\mathbb{D}$ and for D_2 the whole plane $\mathbb{C}$, we obtain the following corollary.

Corollary 1.2. *There exists a holomorphic function f defined on $\mathbb{D}$, which does not extend holomorphically across $\partial\mathbb{D}$, such that Γ_f is not complete pluripolar in $\mathbb{C}^2$.*

Theorem 3.2 then states that such a function can even be smooth up to the boundary of $\mathbb{D}$.

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2. Graphs with non-trivial pluripolar hull

The *pluripolar hull* of a pluripolar set $K \subset \Omega$ is the set

$$K^*_\Omega = \{z \in \Omega : u|_K = -\infty, u \in \mathrm{PSH}(\Omega) \Longrightarrow u(z) = -\infty\}.$$

By $\omega(z, E, D)$ we denote as usual the harmonic measure of a subset E of the boundary of a domain D in $\mathbb{C}$ at the point z in D (see e.g. [8]).

Let $A = \{a_n\}_{n=1}^\infty$ be a countable dense subset of ∂D_1. Under the assumptions of Theorem 1.1, there exists an $a \in (\partial D_1) \cap D_2$ such that $a \in \overline{A \setminus \{a\}}$. We may assume that $a \notin A$. Our function f will be of the form

$$f(z) = \sum_{j=1}^{\infty} \frac{c_j}{z - a_j}. \tag{2.1}$$

We will choose c_n very rapidly decreasing to 0. In particular,

$$\sum_{j=1}^{\infty} |c_j| < +\infty, \tag{2.2}$$

so the limit in (2.1) exists and is a holomorphic function on D_1.

Moreover, we will choose c_n such that $\sum_{n=1}^\infty \frac{|c_j|}{|a-a_j|} < +\infty$. Hence, the series (2.1) will converge at $z = a$. We will denote its limit by $f(a)$. We will prove a version of Theorem 1.1 that elaborates on (2.1). However, no statement about extendibility is made at this point.

Theorem 2.1. *Let $D_1 \subset D_2$ be domains in $\mathbb{C}$, such that $D_2 \setminus D_1$ has a density point in D_2. There exists a sequence $\{R_n\}_{n=1}^\infty$ of positive numbers such that for any sequence of complex numbers $\{c_n\}_{n=1}^\infty$ with $|c_n| \le R_n$ we have $(a, f(a)) \in (\Gamma_f)^*_{D_2 \times \mathbb{C}}$, where f is given by* (2.1). *Here Γ_f is the graph of f over D_1.*

Proof. We may assume that $a = 0$. For $b \in \mathbb{C}$ and $r > 0$ we set $\mathbb{D}(b, r) = \{z \in \mathbb{C} : |z - b| < r\}$, $\mathbb{D}_r = \mathbb{D}(0, r)$ and $\mathbb{D} = \mathbb{D}_1$, the unit disc. Put

$$\pi_n^k(z) = e^{\frac{2\pi k i}{n}} z, \quad z \in \mathbb{C}, \quad k, n \in \mathbb{N},$$

and

$$B = \cup_{n=1}^\infty \cup_{k=1}^n \pi_n^k(A).$$

Note that $0 \notin B$ and that B is also countable (and therefore thin at 0). By Corollary 4.8.3 in [4], there exists an open set $U \supset B$ such that U is thin at 0.

Step 1. We construct a sequence of radii $\{\rho_n\}_{n=1}^\infty$ with special properties, the main one being that $\cup_n \cup_{k=1}^n \pi_n^k(\mathbb{D}(a_n, \rho_n))$ is thin at 0.

It is a corollary of Wiener's criterion (see [8], Theorem 5.4.2) that there exists a sequence $r_n \to 0$ such that

$$\partial\mathbb{D}_{r_n} \cap U = \varnothing, \quad n \in \mathbb{N}. \tag{2.3}$$

Since U is thin at 0, there exists a subharmonic function u on $\mathbb{C}$ such that

$$\limsup_{U \ni z \to 0} u(z) = -\infty < u(0)$$

(see e.g. Proposition 4.8.2 in [4]). Moreover, by scaling and adding a constant, we can assume that $u(0) = -\frac{1}{2}$ and $u < 0$ on $\mathbb{D}$. By (2.3) there exists a $\rho > 0$ such that $\overline{\mathbb{D}}_\rho \subset D_2$, $\partial\mathbb{D}_\rho \cap D_1 \neq \varnothing$, $\partial\mathbb{D}_\rho \cap U = \varnothing$, and $u \leq -1$ on $U \cap \mathbb{D}_\rho$ (take $\rho = r_n$ with sufficiently big n).

Let $J \subset \partial\mathbb{D}_\rho \cap D_1$ be a closed arc. We can assume that

$$J = \{e^{i\theta}\rho : \frac{2\pi k_0}{n_0} \leq \theta \leq \frac{2\pi(k_0+1)}{n_0}\}$$

for some $k_0, n_0 \in \mathbb{N}$.

Now we choose a sequence of positive numbers $\rho_n \in (0,1)$, $n \in \mathbb{N}$, in the following way:

(1) Let $0 < \rho_1 < 1$ be such that
   (a) $\cup_{k=1}^{n_0} \pi_{n_0}^k\big(\mathbb{D}(a_1, \rho_1)\big) \subset U$;
   (b) $\mathbb{D}_\rho \setminus \cup_{k=1}^{n_0} \pi_{n_0}^k\big(\overline{\mathbb{D}}(a_1, \frac{\rho_1}{2})\big)$ is connected.
(2) Assume that $\rho_1, \dots, \rho_{n-1}$ are chosen. Choose $0 < \rho_n < 1$ such that
   (a) $\cup_{k=1}^{n_0} \pi_{n_0}^k\big(\mathbb{D}(a_n, \rho_n)\big) \subset U$;
   (b) $\mathbb{D}_\rho \setminus \cup_{j=1}^{n} \cup_{k=1}^{n_0} \pi_{n_0}^k\big(\overline{\mathbb{D}}(a_j, \frac{\rho_j}{2})\big)$ is connected.

Put $Y_n = \bigcup_{j=1}^{n} \bigcup_{k=1}^{n_0} \pi_{n_0}^k\big(\overline{\mathbb{D}}(a_j, \frac{\rho_j}{2})\big)$. So, $Y_n \subset U$ is a closed set such that $\mathbb{D}_\rho \setminus Y_n$ is a domain and $\partial\mathbb{D}_\rho \cap Y_n = \varnothing$ for any $n \in \mathbb{N}$.

Step 2. We want to show that

$$\omega(0, \partial\mathbb{D}_\rho, \mathbb{D}_\rho \setminus Y_n) \geq \frac{1}{2}, \quad n \in \mathbb{N}. \tag{2.4}$$

Fix $n \in \mathbb{N}$. Put $v_n(z) = -\omega(z, \partial\mathbb{D}_\rho, \mathbb{D}_\rho \setminus Y_n) + u(z)$. It suffices to show that

$$v_n \leq -1 \quad \text{on } \mathbb{D}_\rho \setminus Y_n. \tag{2.5}$$

Observe that $-\omega(\cdot, \partial\mathbb{D}_\rho, \mathbb{D}_\rho \setminus Y_n) \leq 0$ and $u(\cdot) \leq 0$ on $\mathbb{D}_\rho \setminus Y_n$. Moreover, we have $\limsup_{z\to\partial\mathbb{D}_\rho} -\omega(z, \partial\mathbb{D}_\rho, \mathbb{D}_\rho \setminus Y_n) \leq -1$ and $\limsup_{z\to Y_n} u(z) \leq -1$. So, from the maximum principle for the subharmonic function v_n we get (2.5) and, therefore, (2.4).

Step 3. Here we want show that

$$\omega\big(0, J, \mathbb{D}_\rho \setminus \cup_{j=1}^{n}\overline{\mathbb{D}}(a_j, \frac{\rho_j}{2})\big) \geq \frac{1}{2n_0}, \quad n \in \mathbb{N}. \tag{2.6}$$

Put

$$w_n(z) = \omega\big(z, \partial\mathbb{D}_\rho, \mathbb{D}_\rho \setminus Y_n\big) - \sum_{k=1}^{n_0} \omega\big(z, \pi_{n_0}^k(J), \mathbb{D}_\rho \setminus Y_n\big).$$

Note that $\cup_{k=1}^{n_0} \pi_{n_0}^k(J) = \partial\mathbb{D}_\rho$. Again from the maximum principle we obtain that $w_n \leq 0$ on $\mathbb{D}_\rho \setminus Y_n$, $n \in \mathbb{N}$.

Because $\pi_{n_0}^k(\mathbb{D}_\rho \setminus Y_n) = \mathbb{D}_\rho \setminus Y_n$, for any $k, n \in \mathbb{N}$, we find

$$\omega\big(0, \pi_{n_0}^k(J), \mathbb{D}_\rho \setminus Y_n\big) = \omega\big(0, J, \mathbb{D}_\rho \setminus Y_n\big), \quad k \in \mathbb{N}.$$

Hence,

$$\omega\big(0, J, \mathbb{D}_\rho \setminus \cup_{j=1}^{n}\overline{\mathbb{D}}(a_j, \frac{\rho_j}{2})\big) \geq \omega\big(0, J, \mathbb{D}_\rho \setminus Y_n\big) \geq \frac{1}{2n_0}, \quad n \in \mathbb{N}.$$

Step 4. Let $\{R_n\}_{n=1}^{\infty}$ be a sequence of positive numbers such that $C_1 := \sum_{n=1}^{\infty} \frac{R_n}{\rho_n} < +\infty$ and, therefore, $\sum_{n=1}^{\infty} R_n < C_1$ (take e.g. $R_n = \frac{\rho_n}{n^2}$). Consider any sequence of complex numbers $\{c_n\}_{n=1}^{\infty}$ with $|c_n| \leq R_n$ and let f be defined by (2.1).

Put

$$f_n(z) = \sum_{j=1}^{n} \frac{c_j}{z - a_j} - \sum_{j=n+1}^{\infty} \frac{c_j}{a_j}, \quad n \in \mathbb{N}.$$

Then $|f_n(z)| \le 2C_1$ for every $z \in \mathbb{D}_\rho \setminus \cup_{j=1}^n \overline{\mathbb{D}}(a_j, \frac{\rho_j}{2})$ and all n.

Let $h \in \mathrm{PSH}(D_2 \times \mathbb{C})$ have the property that $h\big(z, f(z)\big) = -\infty$, $z \in D_1$. The function s_n defined on $D_2 \setminus \{a_1, \dots, a_n\}$ by $s_n(z) := h(z, f_n(z))$ is subharmonic. Let $A_n := \sup_{z\in J} s_n(z)$ and let $C_2 := \sup_{z\in\overline{\mathbb{D}}_\rho, |w|\le 2C_1} h(z, w)$. Then $A_n \to \sup_{z\in J} h(z, f(z)) = -\infty$ as $n \to \infty$.

From the two-constant theorem (see e.g. [8], Theorem 4.3.7) we infer

$$\frac{C_2 - s_n(0)}{C_2 - A_n} \ge \omega\big(0, J, \mathbb{D}_\rho \setminus \cup_{j=1}^n \overline{\mathbb{D}}(a_j, \frac{\rho_j}{2})\big) \ge \frac{1}{2n_0}, \quad n \in \mathbb{N}.$$

Letting $n \to \infty$, we conclude that $h(0, f(0)) = s_n(0) = -\infty$ and therefore $(0, f(0)) \in (\Gamma_f)^*_{D_2\times\mathbb{C}}$. ☐

For the proof of Theorem 1.1 we need to know that the function defined by (2.1) is not extendible across the boundary of D_1. This will be done in the next section.

3. Non-extendible sums

Without additional conditions on a_n and c_n a function defined by (2.1) may well extend holomorphically beyond the boundary of D_1, think of Lambert-type series $\sum_{n=1}^\infty c_n \frac{z^n}{1-z^n}$, cf. [5]. It may even yield 0 on D_1, cf. [1]. We will see that suitable choice of a_n and c_n prevents this from happening. We are grateful to Marek Jarnicki and Peter Pflug who suggested the idea of the proof of the next lemma.

Lemma 3.1. *Let D be a domain in $\mathbb{C}$. Then there exist a dense subset $A = \{a_n\}_{n=1}^\infty$ of ∂D and a sequence $\{R_n\}_{n=1}^\infty$ of positive numbers such that for any sequence of complex numbers $\{c_n\}_{n=1}^\infty$ with $0 < |c_n| \le R_n$ the holomorphic function f given by* (2.1) *is not holomorphically extendible across ∂D.*

Proof. Let $B = \{b_n\}_{n=1}^\infty$ be a dense subset of D (take e.g. $U = D \cap \mathbb{Q}^2$). For any $b_n \in B$ there exists a point $a \in \partial D$ such that $\mathrm{dist}(b_n, \partial D) = |b_n - a|$. We denote by a_n one of them. Set $A = \{a_n\}_{n=1}^\infty$. Note that A is a dense subset of ∂D. Taking subsequence of $\{a_n\}_{n=1}^\infty$ we may assume that $a_i \ne a_j$, $i \ne j$.

Fix $n \in \mathbb{N}$. Let $B_n = \{z \in D : \mathrm{dist}(z, \partial D) = |z - a_n|\} \subset D$. Note that $B_n \cup \{a_n\}$ is a closed set on the plane and $\widetilde{B} = \cup_{n=1}^\infty B_n$ is dense in D (because $\widetilde{B} \supset B$). Moreover, if $z_0 \in B_n$ then the open segment with the ends at the points z_0 and a_n is contained in B_n.

For any $j \in \mathbb{N}$ we put $\epsilon_{nj} = \mathrm{dist}(a_j, B_n)$. Since $a_j \notin B_n$ for $j \ne n$, we see that $\epsilon_{nj} > 0$ for $j \ne n$.

Put

$$R_j = \frac{\min\{\epsilon_{1j}, \dots, \epsilon_{(j-1)j}\}}{j^2}, \quad j \in \mathbb{N}.$$

For any $n \in \mathbb{N}$ and any $j > n$ we have $R_j \leq \frac{\epsilon_{nj}}{j^2}$ and therefore

$$\sum_{j \neq n} \frac{R_j}{|z - a_j|} \leq \sum_{j \neq n} \frac{R_j}{\epsilon_{nj}} < +\infty, \quad z \in B_n.$$

Take a sequence of complex numbers $\{c_n\}_{n=1}^{\infty}$ with $0 < |c_n| \leq R_n$. Then for a fixed $n \in \mathbb{N}$ we have

$$\begin{aligned} (3.1) \qquad & \liminf_{B_n \ni z \to a_n} |(z - a_n) f(z)| \\ & \geq |c_n| - \lim_{B_n \ni z \to a_n} |z - a_n| \cdot \limsup_{B_n \ni z \to a_n} \sum_{j \neq n} \frac{|c_j|}{|z - a_j|} = |c_n| > 0. \end{aligned}$$

Observe that for any $n \in \mathbb{N}$ the Taylor series at any point of z_0 B_n has a radius of convergent equal to $\operatorname{dist}(z_0, \partial D)$ (because of (3.1) and $|z_0 - a_n| = \operatorname{dist}(z_0, \partial D)$). Hence, by Lemma 1.7.5 from [3] we see that D is the domain of existence of f. $\square$

Proof of Theorem 1.1. If a set $E \subset \Omega$ is complete pluripolar in a domain Ω, then $E_{\Omega}^* = E$. By Lemma 3.1 and Theorem 2.1 there exists a holomorphic function f on D_1 for which D_1 is a domain of existence and $(\Gamma_f)_{D_2 \times \mathbb{C}}^* \neq \Gamma_f$. Hence, Γ_f is not complete pluripolar in $D_2 \times \mathbb{C}$. $\square$

Theorem 3.2. *There exists a sequence $\{a_n\}_{n=1}^{\infty} \subset \mathbb{C} \setminus \overline{\mathbb{D}}$ and a sequence $\{c_n\}_{n=1}^{\infty}$ such that the function f defined by (2.1) is C^{∞} on $\overline{\mathbb{D}}$, is nowhere extendible over the boundary of $\mathbb{D}$, while Γ_f is not complete pluripolar in $\mathbb{C}^2$.*

Proof. Let $r_j = 1 + 1/(j+1)$. The sequence a_n is formed by

$$a_{2^j + k} = r_j e^{2\pi i \frac{k}{2^j}}, \quad k = 0, \ldots 2^j - 1, \ j = 0, 1, \ldots.$$

The proof of Theorem 2.1 provides us with a sequence $\{R_n\}$ such that for every sequence $\{c_n\}$ with $|c_n| < R_n$ the series (2.1) represents a function on $\mathbb{D}$, the graph of which is not complete pluripolar. Assembling all $a_n \in C(0, r_j)$ we find that there exists a sequence $\{R_j'\}$ such that for every choice of $0 < \varepsilon_j < R_j$ the function f_ε on $\mathbb{D}$ defined by

$$f_\varepsilon(z) = \sum_{j=0}^{\infty} \frac{\varepsilon_j}{r_j^{2^j} - z^{2^j}} \tag{3.2}$$

has a graph that is not complete pluripolar.

We observe that independently of the choice of ε_j

$$\sum_{j=1}^{n} \frac{\varepsilon_j}{r_j^{2^j} - z^{2^j}} = \sum_{k=1}^{\infty} d_{n,k} z^k$$

is holomorphic on $\mathbb{D}_{r_n}$ with singularities on the boundary. (The $d_{n,k}$ are defined by the equality.) Therefore we have $\limsup_{k \to \infty} |d_{n,k}^{1/k}| = 1/r_n$. Hence there is a k_n such that

$$|d_{n,k_n}| > r_{n+1}^{-k_n}. \tag{3.3}$$

We will now make an appropriate choice for the ε_j to insure that f cannot be extended over the boundary of $\mathbb{D}$. Along the way we will determine constants C_j that are needed for smoothness at the boundary.

Choose $\varepsilon_0 = R_0'$. Then

$$f_{\varepsilon_0}(z) = \frac{1}{r_0 - z} = \sum_{k=1}^{\infty} d_{0,k} z^k$$

with $\limsup_{k\to\infty} |d_{0,k}^{1/k}| = 1/r_0$; in particular there is a C_0 such that $|d_{0,k}| < C_0$.

Suppose $\varepsilon_0, \dots, \varepsilon_{n-1}$ and $C_0, \dots, C_{n-1}$ have been chosen in such a way that we have found $k_0, \dots, k_{n-1}$ with

$$|d_{l,k_j}| > r_{j+1}^{-k_j} \quad \text{for } j = 0, \dots, n-1,\ l = j, \dots n-1 \tag{3.4}$$

and

$$|d_{j,k}| < \frac{C_l}{k^l} \quad \text{for } l = 0, \dots, n-1,\ j = 0, \dots, n-1 \quad \text{and all } k. \tag{3.5}$$

Then choose

$$C_n > \sup_k |d_{j,k}| k^n \quad j = 0, \dots, n-1. \tag{3.6}$$

This is finite because of (3.4). Next choose $\varepsilon_n < R_n'$ so small that

(1) The inequality (3.5) holds for $l = 0, \dots, n$ and $j = 0, \dots, n$. This is possible because of (3.4).

(2)
$$|d_{n,k_j}| > r_{j+1}^{-k_j} \quad \text{for } j = 0, \dots, n-1,$$
which is again possible because of (3.4).

Having chosen ε_n, we can by (3.3) choose k_n so large that $|d_{n,k_n}| > r_{n+1}^{-k_n}$.

Observe that the coefficients $d_{n,k}$ converge to the coefficients d_k of the power series expansion of f_ε as $n \to \infty$. From (3.4) we see that $|d_{k_j}| > r_{j+1}^{-k_j}$ so that the radius of convergence of the power series of f_ε is at most 1, and since f_ε is holomorphic on $\mathbb{D}$, it equals 1. So f_ε has a singular point b on $C(0,1)$. We split f_ε as

$$f_\varepsilon = f_1 + f_2 = \left(\sum_{j=0}^{n-1} + \sum_{j=n}^{\infty}\right) \frac{\varepsilon_j}{r_j^{2^j} - z^{2^j}}.$$

Then f_1 is holomorphic in a neighborhood of the closed unit disc and f_2 has at least one singular point on $C(0,1)$, but f_2 is invariant under rotation over a 2^n-th root of unity, which implies that there is a singularity in each arc of length $> 2\pi/2^n$. Therefore, f can nowhere be extended analytically over $C(0,1)$.

Next we show that f is smooth up to the boundary of $\mathbb{D}$. We have to show that there exist constants $C_l > 0$ such that for every l

$$|d_k| \le \frac{C_l}{k^l}, \quad \text{for all } k,$$

but this follows from (3.5). □

Remark 3.3. Let D be a domain in $\mathbb{C}$ and let A be a closed polar subset of D. Using methods presented in this paper and in [10], the authors give in [2] a complete characterization of the holomorphic functions f on $D \setminus A$ such that $\Gamma_f = \{(z, f(z)) : z \in D \setminus A\}$ is complete pluripolar in $D \times \mathbb{C}$.

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