PLURIPOLAR SETS: HULLS AND COMPLETENESS

JAN WIEGERINCK

In honour of J. B. Poly

Abstract. This is a survey of recent work in which completeness of certain graphs of analytic functions was established. The analytic functions are either gap series or have isolated singularities.

1. Introduction

In this paper we survey some recent work in which completeness of certain pluripolar sets was established and also related work concerning the determination of pluripolar hulls. The paper is an expanded version of my lecture held at the " Journées d'analyse complexe du Sud", at Poitiers in March 1999. Rather than giving complete proofs, which may be found in the literature anyway, I will try to emphasize background and motivation.

We start with two problems posed Sadullaev in his seminal paper [15]: Consider the sets

$$E_1 = \{(x, y) \in \mathbb{C}^2 : \ y = x^\alpha, \, x \in (0, 1)\}$$

with α irrational, and

$$E_2 = \{(x, y) \in \mathbb{C}^2 : \ y = e^{-1/x}, \, x \in (0, 1)\}.$$

Are the sets E_j plurithin at the origin? That is, does there exist a plurisubharmonic (PSH) function h on a neighborhood V of $\overline{E_j}$ such that

$$h(0) > \limsup_{\substack{x \to 0 \\ x \in E_j}} h(x).$$

In [11] Levenberg and Poletsky show that E_1 is plurithin at 0. In [16] the author shows that the same goes for E_2. In fact, for both sets stronger results were obtained.

In section 2 we review the part of classical potential theory that stands as a model for what we do in the setting of pluripotential theory. In section 3 we deal with the relevant background in pluripotential theory. In section 5 we sketch a proof of completeness of graphs of certain gap series, as is in [15] and [12]. The final sections deal with hulls and completeness in the presence of isolated singular points and branch points.

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2. Polar sets in Newtonian potential theory

Recall that a function $u \not\equiv -\infty$ is subharmonic on a domain $\Omega \subset \mathbb{R}^n$, ($u \in SH(\Omega)$) if it has values in the extended reals $\mathbb{R} \cup \{-\infty\}$, is upper semicontinuous and satisfies $\Delta u \geq 0$, where the Laplacian is taken in distribution sense. A set $E \subset \Omega$ is called *polar* in Ω if there exists a $u \in SH(\Omega)$ such that $u|_E = -\infty$. A set E is called *complete polar* in Ω if $\exists u \in SH(\Omega)$ such that $E = \{x \in \Omega : u(x) = -\infty\}$.

The following properties are well known, cf., [9].

1. If E is polar in Ω then E is polar in $\mathbb{R}^n$.
2. If E is locally polar, that is, for all $x \in \mathbb{R}^n$ there is a ball B_x such that $E \cap B_x$ is polar, then E is polar in $\mathbb{R}^n$.
3. E is complete polar in Ω if and only if E is polar in Ω and E is a G_δ.
4. If E is complete polar in Ω, then E is complete polar in $\mathbb{R}^n$.

Remark 1. Note that 4. follows directly from 3. and 1., while the direct part of 3 is just a property of the $-\infty$-sets of upper semicontinuous functions. The fact that subharmonic functions may be realized as difference of a harmonic function and a Newtonian potential, is instrumental in proving the other properties.

Remark 2. Although polar sets are in general not complete, problems of the type that Sadullaev posed are easily solved for subharmonic functions. If E is polar and $x \notin E$ then there exists $h \in SH(\mathbb{R}^n)$ such that $h|_E = -\infty$, while $h(x) > -\infty$. Indeed, start with $h_0 \in SH(\mathbb{R}^n)$ such that $h_0|_E = -\infty$ and let $F = \{y : \ h_0(y) = -\infty\}$; F is a polar G_δ and so is $F \setminus \{x\}$, hence there exists $h \in SH(\mathbb{R}^n)$ as required.

3. Pluripotential theory

In complex analysis we want only those subharmonic functions that are invariant under holomorphic transformations. This leads to plurisubharmonic functions, pluripolar sets etcetera. We speak loosely of pluripotential theory, although potentials are only of limited importance. For an introduction see Klimek's book [8].

A function h on a domain $\Omega \subset \mathbb{C}^n$ is called *pluri(sub)harmonic* if it is (sub)-harmonic on Ω and the restriction to any complex one-dimensional line is (sub)-harmonic too. The plurisubharmonic functions on Ω are denoted by $PSH(\Omega)$. A subset $E \subset \Omega$ is called *pluripolar* in Ω if there exists $h \in PSH(\Omega)$ such that $h|_E = -\infty\}$. The set E is called *complete pluripolar* if there exists $h \in PSH(\Omega)$ such that

$$E = \{z \in \Omega : \ h(z) = -\infty\}.$$

Now it is reasonable to ask if the properties 1 through 4 of the previous section hold in the pluripotential setting. Properties 1 and 2 remain valid; the result is known as Josefson's theorem, [7]:

Theorem 1. *If E is locally pluripolar, then E is pluripolar in $\mathbb{C}^n$.*

The proof is *not* easy. Bedford and Taylor, [2], have found a second proof based on the fact that pluripolar sets have Monge-Ampère capacity 0. Hörmander has an illuminating version of Josefson's proof in [6].

Properties 3 and 4 are no longer valid in the new setting.

Example 1. Let $E = \{(z, w) \in \mathbb{C}^2 : \ |z| = 1, \ w = 0\}$. The function $\log |w| \in PSH(\mathbb{C}^2)$ equals $-\infty$ on E, so E is pluripolar. Moreover E is a G_δ. However since

$\{|z|=1\}$ is non-polar in $\mathbb{C}$, every $h \in PSH(\mathbb{C}^2)$ that equals $-\infty$ on E must have $h|_{\{w=0\}} = -\infty$, therefore E is not complete.

Note that remark 2 prevents such a phenomenon to occur in Newtonian potential theory. We are led to the following definition:

Definition . Let $E \subset \Omega \subset \mathbb{C}^n$ be pluripolar in the domain Ω. The *pluripolar hull* of E with respect to Ω is the set

$$E^*_\Omega = \{z \in \Omega : \forall h \in PSH(\Omega)\ h|_E = -\infty \implies h(z) = -\infty\}.$$

The *negative pluripolar hull* of E with respect to Ω is

$$E^-_\Omega = \{z \in \Omega : \forall 0 \geq h \in PSH(\Omega)\ h|_E = -\infty \implies h(z) = -\infty\}.$$

The pluripolar hull E^* was first studied by Zeriahi in [18]. The negative plurisubharmonic hull is closely related to pluriharmonic measure, and serves to determine the pluripolar hull in many cases as may be seen from the following result of Levenberg and Poletsky, [11]. Obviously $E^*_\Omega \subset E^-_\Omega$, but moreover

Theorem 2. *Let Ω be pseudoconvex in $\mathbb{C}^N$ and $E \subset \Omega$ pluripolar. Suppose $\Omega = \cup_j \Omega_j$, where Ω_j form an increasing sequence of relatively compact subdomains of Ω. Then*

$$E^*_\Omega = \bigcup_j (E \cap \Omega_j)^-_{\Omega_j}.$$

Moreover, if Ω is hyperconvex, that is, Ω admits a bounded plurisubharmonic exhaustion function, then

$$E^-_\Omega = \bigcup_j (E \cap \Omega_j)^-_{\Omega_j}.$$

If E is complete pluripolar in Ω, then $E = E^*_\Omega$ and E is a G_δ. Whether the converse is true, is not known. However, in [18] Zeriahi proved

Theorem 3. *Let E be a pluripolar subset of a pseudoconvex domain Ω and suppose that $E = E^*_\Omega$. If E is a G_δ and an F_σ, then E is complete pluripolar in Ω.*

In fact, he proved somewhat more:

Theorem 4. *Let E be pluripolar in a pseudoconvex domain Ω, F an F_σ, G a G_δ and suppose that $F \subset E \subset E^*_\Omega \subset G$. Then there exists a complete pluripolar set $\tilde{E} \subset \Omega$ such that $F \subset \tilde{E} \subset G$.*

The failure of properties 3 and 4 can be much worse than example 1 suggests.

Example 2. Let

$$E = \{(z,w) \in \mathbb{C}^2 :\ |z-1| < 1, w = p.v.z^\alpha\},$$

where α is irrational and $p.v.$ denotes principal value. The function $\log|w - p.v.\,z^\alpha|$ on $|z-1|<1$ exhibits E as a complete pluripolar subset of $\{(z,w) : |z-1|<1\}$. However, if $u \in PSH(B)$, where B is the unit ball in $\mathbb{C}^2$ and $u|_E = -\infty$, then $u = -\infty$ on the graph of the maximal analytic continuation X of $p.v.\,z^\alpha$. We shall see in section 7 that $E^*_B = X \cap B$. Note that X is a dense subset of the first Baire category of $\{(z,w) : |w| = |z|^\alpha\}$ and thus X (and $X \cap B$) are not G_δ. We infer that E fails to be complete pluripolar in B, and even E^*_B is not complete pluripolar.

4. Pluriharmonic Measure

Let E be a subset of a domain $\Omega \subset \mathbb{C}^n$. The *pluriharmonic measure* at $z \in \Omega$ of E relative to Ω is the number

$$\omega(z, E, \Omega) = -\sup\{u(z) : \ u \in PSH(\Omega), \ u \leq 0, \ u|_E \leq -1\}. \tag{4.1}$$

Observe that in general the sup in (4.1) is *not* plurisubharmonic. The upper regularization of the sup *is* plurisubharmonic, and some authors define pluriharmonic measure in this way. We are interested in the situation where E is pluripolar; then the regularization in (4.1) equals 0 and yields no information.

Nevertheless, pluriharmonic measure boils down to the usual concept of harmonic measure if $n = 1$ and E is a non-polar subset of $\Omega \subset \mathbb{C}$.

Pluriharmonic measure may be used to compute pluripolar hulls.

Proposition 5. *Let Ω be a hyperconvex domain in $\mathbb{C}^N$ and let $E \subset \Omega$ be pluripolar. Then*

$$E^-_\Omega = \{z \in \Omega : \omega(z, E, \Omega) > 0\}.$$

Proof. If $\omega(z, E, \Omega) = 0$, then for every $n > 0$ there exists a negative $h_n \in PSH(\Omega)$ with $h_n|_E = -1$ and $h_n(z) > -\frac{1}{2^n}$. Then $h(w) = \sum_{n=1}^\infty h_n(w)$ is a limit of a decreasing sequence of negative plurisubharmonic functions. Therefore $h \in PSH(\Omega)$. Moreover, $h(z)$ is finite, but $h|_E = -\infty$.

Now suppose that $\omega(z, E, \Omega) = \delta > 0$ and that h is a negative plurisubharmonic function with $h|_E = -\infty$. Then $\frac{1}{n}h(z)$ competes for the sup in 4.1, so $\frac{1}{n}h(z) < -\delta$ or $h(z) < -\delta n$. It follows that $h(z) = -\infty$. □

The following trivial observation will polish our computations.

Lemma 6. *Let Ω be a domain in $\mathbb{C}^n$, and E a pluripolar subset of Ω. Then $\omega(z, E, \Omega) = 0$ if and only if there is a $C > 0$ such that for every $\epsilon > 0$ there exists $h \in PSH(\Omega)$ such that*

1. $h \leq \epsilon$,
2. $h|_E \leq -\log 2 + \epsilon$
3. $h(z) > -\epsilon$.

While this is all we will use in sections 5 and 6, we list a few additional properties, some of which are needed in section 7. Most of them are taken from [11], except for the last lemma which comes from [16]

4.1. Regularity.

$$\omega(z, E, \Omega) = \inf_{\substack{V \text{open} \\ E \subset V \subset \Omega}} \omega(z, V, \Omega)$$

4.2. Subadditivity.

$$\omega(z, E_1 \cup E_2, \Omega) \leq \omega(z, E_1, \Omega) + \omega(z, E_2, \Omega). \tag{4.2}$$

4.3. Negligability.

Lemma 7. *Let $\Omega \subset \mathbb{C}^N$ be a domain, $E \subset \Omega$ and let $A \subset \Omega \setminus E$ be closed and pluripolar. Then for $z \in \Omega \setminus A$ we have $\omega(z, E, \Omega) = \omega(z, E, \Omega \setminus A)$.*

The above three properties follow directly from the definition 4.1.

4.4. **Duality.**

$$\omega(z,V,\Omega)=\frac{1}{2\pi}\sup_f\{m(\{t\in(0,2\pi):\ f(e^{it})\in V\})\},$$

where f runs over all analytic disks $f:\{|\zeta|\le 1\}\to\Omega$ with $f(0)=z$, while m denotes Lebesgue measure on the circle.

This duality result is due to Poletsky, [13]. It is the basis of the proof of lemma 8 and proposition 9 below. For more information, see also [10].

4.5. **Maximum principle.**

Lemma 8. *Let Ω be a relatively compact subdomain of a domain $G\subset\mathbb{C}^N$. Suppose that $E\subset\Omega$ is compact, $V\subset G\setminus E$ is a domain and $z\in V\cap\Omega$. Put $K=\overline{\partial V\cap\Omega}$. If $\omega(z,E,\Omega)=a$ then $\omega(w,E,G)\ge a$ for some $w\in K$.*

4.6. **Maps and coverings.**

Proposition 9. *Let $\Omega_1\subset\mathbb{C}^N$ and $G\subset\mathbb{C}^M$ be domains and let $h:\Omega_1\to G$ be holomorphic. Then for $z\in\Omega_1$ and $E\subset G$*

$$\omega(z,h^{-1}(E),\Omega_1)\le\omega(h(z),E,G). \tag{4.3}$$

Moreover, equality holds in (4.3) if the following additional requirements are met: The map h is a covering map; E admits a simply connected open neighborhood V such that $h^{-1}(V)$ is a disjoint union of connected open V_j; $z\in\Omega_1$ has the property

$$\lim_{j\to\infty}\omega(z,\cup_{k=j}^{\infty}V_k,\Omega_1)=0 \tag{4.4}$$

Lemma 10. *Let $\Omega_1\subset\mathbb{C}^N$ and $G\subset\mathbb{C}^M$ be domains and let $h:\Omega_1\to G$ be a finite branched holomorphic covering. Then for $z\in\Omega_1$ and $E\subset G$*

$$\omega(z,h^{-1}(E),\Omega_1)=\omega(h(z),E,G). \tag{4.5}$$

In all our applications E will be (a piece of) an analytic variety in Ω. Let E_0 be a non-polar subset of E, so that $E_\Omega^*=(E_0)_\Omega^*$. Then also

$$E_\Omega^-=\{z\in\Omega:\ \omega(z,E_0,\Omega)>0\}. \tag{4.6}$$

5. Graphs of gap series.

In this section we will consider gap series on the unit disk of the form

$$f(z)=\sum_{k=1}^{\infty}a_{n(k)}z^{n(k)}, \tag{5.1}$$

where the radius of convergence of the series equals 1 and $a_{n(k)}\ne 0$. For the integers $n(k)$ we demand

$$\frac{n(k)}{n(k+1)}\to 0 \tag{5.2}$$

It is well-known that such functions do not admit analytic continuation outside the unit disk, cf., [4] or [14]. Sadullaev [15] obtained a stronger result.

Theorem 11. *Let f be an gap series on $|z|<1$ satisfying (5.1) and (5.2). Then*

$$A_f=\{(z,f(z):|z|<1\}$$

is plurithin at each point of $\overline{A_f}\setminus A_f$.

Levenberg, Martin and Poletsky [12] improved on this by showing

Theorem 12. *Let f be as above. Then A_f is a complete pluripolar subset of $\mathbb{C}^2$.*

Proof. First observe that A_f is a G_δ as well as an F_σ. Therefore by Zeriahi's theorem 3 and theorem 2 it suffices to prove that for $A_f = (A_f)_\Omega^-$ for all polydisks Ω with radius greater than 1. Let

$$E = \{(z, w) \in A_f : \ |z| < 1/3\}.$$

In view of (4.6) it suffices to compute $\omega((z_0, w_0), E, \Omega)$; we have to see that for each $(z_0, w_0) \in \Omega \setminus A_f$ there exists an h as in lemma 6.

Let $M = \max_{|z|\leq 2/3} |f(z)|$ and

$$P_m = \sum_{k=1}^{m} a_{n(k)} z^{n(k)}$$

be a partial sum of f. The function h will be of the form

$$v_m(z, w) = \frac{1}{n(m+1)} \log |w - P_m(z)|.$$

We find by Cauchy's theorem $|a_j| \leq M(3/2)^j$. Hence, if $(z, w) \in E$, then

$$\begin{aligned} v_m(z, w) &= \frac{1}{n(m+1)} \log |\sum_{k\geq m+1} a_{n(k)} z^{n(k)}| \leq \frac{1}{n(m+1)} \log M(1/2)^{n(k+1)-1} \\ &\leq -\log 2 + \frac{1}{n(m+1)} \log 2M < -\log 2 + \epsilon \end{aligned} \tag{5.3}$$

if m is large enough.

Next, on $\Omega = \{|z|, |w| < R\}$ we use that $(n(m) + 1)/n(m + 1) \to 0$ as $m \to \infty$ and find that

$$v_m(z, w) \leq \frac{1}{n(m+1)} \log(M(3R/2)^{n(m)+1} + R) < \epsilon,$$

if m is sufficiently large. Thus 1. and 2. of lemma 6 are satisfied by v_m if m is large enough.

Now for 3. we may have to choose m depending on z_0: If $|z_0| < 1$ and $(z_0, w_0) \notin E$, we put $\delta = |w_0 - f(z_0)|$ and compute

$$|w_0 - P_m(z_0)| \geq |w_0 - f(z_0)| - |f(z_0) - P_m(z_0)| \geq 1/2\delta,$$

if m is sufficiently large. Hence

$$v_m(z_0, w_0) \geq \frac{1}{n(m+1)} \log \delta/2 \geq -\epsilon,$$

if m is sufficiently large.

Finally, if $|z_0| \geq 1$, we proceed as follows. Suppose that for all m the inequality $v_m(z_0, w_0) < -\epsilon$ would hold. Then

$$|a_{n(m)}||z_0|^{n(m)} = |w_0 - P_m(z_0) - (w_0 - P_{m-1}(z_0))| < e^{-\epsilon n(m+1)} + e^{-\epsilon n(m)} < 2e^{-\epsilon n(m)}.$$

This leads to

$$|a_{n(m)}|^{1/n(m)} < 2^{1/n(m)} e^{-\epsilon} |z_0|^{-1},$$

which contradicts the assumption that the radius of convergence of f is 1. Thus for some m we find that $v_m(z_0, w_0) > -\epsilon$ and we are done.

□

Remark 3. Turning to arbitrary functions in $A(D)$, Levenberg, Martin and Poletsky continue to observe that if f extends continuously to $T = \{|z| = 1\}$, then the pluripolar hull of $\Gamma = \{(z, f(z) : z \in T\}$ contains E. Moreover, they give examples of gap series, where $E \cup \Gamma$ is complete pluripolar. Finally, they construct $f \in A(D)$ such that Γ is non pluripolar (This is based on a construction of Diederich and Fornæss of a non pluripolar smooth arc in $\mathbb{C}^2$, [3]).

The above result led them to conjecture that if f is a holomorphic function on its natural domain of existence $G \subset \mathbb{C}$, then the graph of f is over G is a complete pluripolar set. Although this seems quite out of reach yet and it seems difficult to proceed with the gap series approach, there is more support of this conjecture. If $f = g + h$ where g is a gap series as before, and h is holomorphic on a larger disk with radius $R > 1$, then the graph of f over the unit disk is complete pluripolar:

After a change of coordinates, the same method as before gives that the graph is complete pluripolar in $\Omega = \{|z| < R\} \times \mathbb{C}$. Thus we have a negative plurisubharmonic function h on the polydisks $\{|z| < R-\delta\} \times \{|w| < M\}$ that equals $-\infty$ on the graph of f. Then it is easy to find a plurisubharmonic function on $\mathbb{C} \times \{|w| < M\}$ which coincides with h on $\{|z| < R - 2\delta\}$, and equals $-\infty$ only on the graph of f. Now apply theorem 2.

6. Graphs with isolated singularities

In this section we will discuss the main result of [17].

Theorem 13. *Suppose that D is a domain in $\mathbb{C}$ and S a sequence of points in D without density point in D. Let f be holomorphic on $D \setminus S$. Let A_f denote the graph of f in $(D \setminus S) \times \mathbb{C}$. Then A_f is complete pluripolar in $D \times \mathbb{C}$.*

We give an outline of the proof and refer to [17] for details. Let D_0 be a compactly contained subdomain of D and let $\Omega = D_0 \times \{|w| < M\}$, with $M > 0$ We proceed as in the previous section by observing that $A_f \cap \Omega$ is a G_δ as well as an F_σ. We will show that

$$E^-_\Omega(A_f \cap \Omega) = A_f \cap \Omega. \tag{6.1}$$

The proof is then finished by invoking 2 and 3.

In proving 6.1 we may assume that $= A_f \cap \Omega \neq \emptyset$. Then we can find a small disk in $D_0 \setminus S$ such that E, the graph of f over this disk is contained in Ω.

After this preliminary setting of the stage, we need

Claim 1:

$$E^-_\Omega(A_f \cap \Omega) \subset A_f \cap \{z = a_i, a_i \in S \cap D_0\} \tag{6.2}$$

Zeriahi's theorem 4 implies that there exists a complete pluripolar set F such that

$$A_f \cap \Omega \subset F \subset (A_f \cap \Omega) \cup (S \cap D_0) \cup \mathbb{C}.$$

It follows that $(A_f \cap \Omega) \cup \{(z, w) : z = a_i, a_i \in S \cap D_0\}$ is a complete pluripolar subset of Ω. This allows one to prove

Claim 2:

$$\omega((a, w), E, \Omega) = 0, \quad (a \in S \cap D_0). \tag{6.3}$$

As 6.2 and 6.3 imply 6.1, the proof is reduced to proving Claim 1 and 2.

For (6.2) one starts off with the Mittag-Leffler decomposition of f on $D_0 \setminus S = D_0 \setminus \{a_1, \dots, a_m\}$:

$$f = f_0 + f_1 + \cdots + f_m,$$

where f_0 is holomorphic on D_0 and $f_i = \sum_{j=-\infty}^{-1} c_{ij}(z-a_i)^j$ converges outside a_i. A change of coordinates allows us to assume that $f_0 \equiv 0$ on D_0. Now we consider the functions

$$P_N(z,w) = (w - \sum_{i=1}^{n}\sum_{m=1}^{N} c_{im}(z-a_i)^{-m})\prod_{i=1}^{n}(z-a_i)^N$$

and

$$h_N(z,w) = \frac{1}{N}\log|P_N(z,w)|.$$

Using the fact that $|c_{jm}|^{1/m} \to 0$ as $m \to \infty$, one can show that h_N tend to $-\infty$ on E with N, while $|h_N|$ remain bounded on compacts in $\Omega \setminus (A_f \cup \{(z,w) : z = a,\ a \in S\})$. Rescaling provides for each $(z_0, w_0) \in \Omega \setminus (A_f \cup \{z = a, a \in S\})$ and each $\epsilon > 0$ a function h satisfying 1–3 of lemma 6.

For (6.3) let G be the projection of $A_f \cap \Omega$ on the first coordinate. Let $\epsilon > 0$. There exists a negative subharmonic function h on a neighborhood of $\bar{G} \cup (S \cap D_0)$ such that $h|_K = -1$, $h(a) > -\epsilon$. The construction of h in [17] is a variation of the classical construction of barriers as may be found in [5]. Now we view h as a plurisubharmonic function on a neighborhood of $\bar{G} \cup (S \cap D_0) \times \mathbb{C}$. As this is also a neighborhood of the complete pluripolar set $X = (A_f \cap \Omega) \cup (S \cap D_0 \times \mathbb{C})$, the function h can be extended from X to a negative plurisubharmonic function on a slightly smaller polydisk. This essentially proves the claim.

7. The graph of $w = z^\alpha$

In this final section we sketch the surprizing answer of Levenberg and Poletsky to Sadullaev's question. The details can be found in [11].

Theorem 14. *Let $E = \{(z,w) \in \mathbb{C}^2 : w = z^\alpha, z \neq 0\}$ with α irrational. Then $E^*_{\mathbb{C}^2} = E$.*

This theorem came as a surprize (at least to me) because one might expect that the origin would belong to $E^*_{\mathbb{C}^2}$, in view of the rational case and because there seems to be so much of E close to the origin.

Let

$$F = \{(z,w) : |z - 1/2| < 1/4,\ w = p.v.z^\alpha\}$$

and let Ω be any large polydisk about the origin. As before the proof is reduced to showing that for $(z,w) \in \Omega$ one has

$$\omega((z,w), F, \Omega) \neq 0 \quad \Longleftrightarrow \quad (z,w) \in E.$$

Moreover we may take $\Omega = \Delta^2$, where Δ denotes the unit disk. Computing the pluriharmonic measure is now more complicated and will involve most of the results mentioned in section 4.

Let $\Pi : \mathbb{C}^2 \to \mathbb{C}^2 \setminus \{z = 0\}$ be the covering map

$$\Pi : (\zeta, w) \mapsto (e^\zeta, w).$$

Let $H = \{\zeta : \Re\zeta < 0\}$, so that $\Pi(H \times \Delta) = \Delta \setminus \{0\} \times \Delta$. The following observation is easily checked, but crucial to understand the different behaviour for rational and irrational α.

Lemma 15. *If α is irrational, then $\Pi^{-1}(E) = \cup_{j\in\mathbb{Z}} E_j$, where $E_j = \{\zeta, w) : w = e^{i\alpha 2\pi j} e^{\alpha\zeta}\}$ and $\Pi|_{E_j}$ is biholomorphic.*

If $\alpha = p/q$ is rational, then $\Pi^{-1}(E) = \cup_{j=1}^{q} E_j$, with $E_j = \{\zeta, w) : w = e^{i2\pi pj/q} e^{\alpha\zeta}\}$ and $\Pi|_{E_j}$ is an infinite holomorphic covering map.

We set $F_j = E_j \cap \Pi^{-1}(F)$. From now on α will be irrational. Let $V = \{(z, w) : |z - 1/2| < 1/3, |w| < 1\}$, a neighborhood of F. Then $\Pi^{-1}(V) = \cup_{j\in\mathbb{Z}} V_j$, with $V_j = \{(\zeta, w) : |e^\zeta - 1/2| < 1/3, -\pi/2 + j < \Im\zeta < \pi/2 + j\}$. Let $p : H \times \mathbb{C} \to H$ be the projection on the first coordinate.

Notice that for each $\zeta \in H$

$$\omega(\zeta, p(\cup_{|j|>k} V_j), H) \to 0 \quad \text{as } k \to \infty.$$

Hence from proposition 9 applied to p, we find that

$$\omega((\zeta, w), \cup_{|j|>k} V_j, H \times \Delta) \to 0 \quad \text{as } k \to \infty.$$

and the technical condition for applying propostion 9 to Π is met.

For $(\zeta, w) \in H \times \Delta \setminus E_j$, the function $\log|w - e^{i\alpha 2\pi j} e^{\alpha\zeta}|$ shows that

$$\omega((\zeta, w), F_j, H \times \Delta) = 0.$$

If $(\zeta, w) \in E_j$ this is of course not true, but again by propostion 9 we have that

$$\omega((\zeta, w), F_j, H \times \Delta) \leq \omega(\zeta, p(F_j), H) \to 0 \quad \text{as } \Re\zeta \to -\infty. \tag{7.1}$$

This is where the lemma comes in, $p(F_j)$ is a bounded set in H; if α were rational, then $p(F_j)$ would contain a set of the form $\cup_j (B + 2\pi iqj)$ and (7.1) would not hold.

Now we see by theorem 9 that for $(e^\zeta, w) \in (\Delta \setminus \{0\} \times \Delta) \setminus E$

$$\omega((e^\zeta, w), F, \Delta \setminus \{0\} \times \Delta) = \omega((\zeta, w), \Pi^{-1} F, H \times \Delta) \leq \sum_j \omega((\zeta, w), F_j, H \times \Delta) = 0.$$

If $\Re\zeta \to -\infty$, we find similarly that

$$\lim_{\Re\zeta \to -\infty} \omega((e^\zeta, w), F, \Delta \setminus \{0\} \times \Delta) = 0.$$

Using Lemma 7 we obtain

$$\omega((z, w), F, \Delta^2) = 0 \quad \text{if } (z, w) \in \Delta^2 \setminus E,\ z \neq 0 \tag{7.2}$$

$$\omega((z, w), F, \Delta^2) \to 0 \quad \text{if } (z, w) \to (0, w_0). \tag{7.3}$$

In 7.3 the convergence is uniform on compact sets in Δ^2 and independent of w_0. Finally, (7.3) and Lemma 8 ensure that $\omega(0, w_0), F, \Delta^2) = 0$, and the sketch is complete.

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E-mail address, Jan Wiegerinck: `janwieg@wins.uva.nl`