THE LEMPERT FUNCTION AND THE PLURICOMPLEX GREEN FUNCTION ARE NOT EQUAL IN THE BIDISC

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ABSTRACT. We give a counterexample to Coman's conjecture stating that in convex domains the pluripolar Green function with several poles and weights equals the corresponding Lempert function.

1. Introduction

The pluricomplex Green function with several poles is introduced as follows. Let Ω be a domain in $\mathbb{C}^n$, and

$$A = \{(w_1, \nu_1), \ldots, (w_p, \nu_p)\} \subset \Omega \times \mathbb{R}^+,$$

where $\mathbb{R}^+ = [0, \infty)$. Define

$$U_{A,\Omega} = \{u \in PSH(\Omega) : u(\zeta) - \nu_j \log|\zeta - w_j| \leq C_u, \zeta \to w_j, j = 1, \ldots, p\},$$

and the *pluricomplex Green function* $g(z, A) = \sup\{u(z) : u \in U_{A,\Omega}, u \leq 0\}$.

We also introduce the Lempert function. Let D be the unit disc, Ω and A as before. For each point $z \in \Omega$, we let F_z denote the family of all mappings $f : D \to \Omega$ such that $f(0) = z$ and there are points $\zeta_1, \ldots, \zeta_p$ with $f(\zeta_j) = w_j$, $j = 1, \ldots, p$. We call these mappings as well as their images *analytic discs*. Then for $f \in F_z$ we define $d(f) = \sum_{j=1}^{p} \nu_j \log|\zeta_j|$ and the *Lempert function* $\delta(z, A) = \inf\{d(f) : f \in F_z\}$. It is easy to see that $\delta(\cdot, A) \geq g(\cdot, A)$ with equality if and only if δ is plurisubharmonic. A remarkable theorem of Lempert [Lem] says that this is indeed the case if Ω is convex and $p = 1$. Much later Coman [Com] showed that it is also the case when Ω is the unit ball, $p = 2$, and the weights are equal. He conjectured that in convex domains the inequality $\delta(\cdot, A) \geq g(\cdot, A)$ is always an equality.

The Lempert function and its relation to the Green function have recently been studied by Wikström [Wik], Paper V and VI. One result is the following. Define

$$\delta^A(z) = \min_{\emptyset \neq B \subseteq A} \delta(z, B).$$

Theorem 1.1. [Wik], Paper V, Corollary 2.3. *Let Ω be a bounded taut domain in $\mathbb{C}^n$, and let A be as above. Then for every $z \in \Omega$ there exists an analytic disc f, such that $f(0) = z$, passing through a (non-empty) subset $\{w_{j_1}, \ldots, w_{j_m}\}$ of $\{w_1, \ldots, w_p\}$, such that $d(f)$ equals the infimum in the definition of*

$$\delta^A(z) = \delta(z, \{(w_{j_1}, \nu_{j_1}), \ldots, (w_{j_m}, \nu_{j_m})\}).$$

Motivated by this theorem, he makes the following definition. If $\delta(z, A) = d(f)$ for some disc f passing through z and a non-empty subset of A, f is called an *extremal disc* for z and A. Then he characterizes the extremal discs in convex domains, thereby generalizing a theorem of Lempert. However, he points out that the weights are invisible in the characterization, which indicates that the conjecture might be problematic, or at least not possible to prove by that method.

The purpose of this note is to give a counterexample to Coman's conjecture. In fact we have the following theorem.

Theorem 1.2. *Coman's conjecture fails for the bidisc when the poles lie on a coordinate axes and the weights are unequal.*

The corresponding Green function was computed in our earlier work [Car-Wie]. The failure of Coman's conjecture in our example is largely due to the fact that it concerns a Green function that is *not* an extreme point in the cone of negative PSH functions. It is therefore tempting to modify the conjecture in the sense that one restricts to Green functions that are extreme points of this cone. However, we feel that presently there is not enough support to make any conjecture at all.

2. The example

The following theorem was proved in [Car-Wie].

Theorem 2.1. *Let $\Omega = D \times D$ be the bidisc in $\mathbb{C}^2$, and assume that $w_j = (a_j, 0)$, $j = 1, \dots, p$. Define $T_j(z_1) = \log |\frac{z_1 - a_j}{1 - \bar{a}_j z_1}|$, $j = 1, \dots, p$. Without loss of generality, we assume that $1 = \nu_1 \geq \nu_2 \geq \dots \nu_p$. Then the pluricomplex Green function is*

$$g_\nu = g_{(1,\nu_2,\dots,\nu_p)} = \nu_p h_p(z) + \sum_{j=1}^{p-1} (\nu_j - \nu_{j+1}) h_j(z),$$

where $h_j(z) = \max\{T_1(z_1) + \dots + T_j(z_1), \log |z_2|\}$ is the Green function for $D \times D$ with weight 1 at $w_1, \dots, w_j$.

We will also use the following result.

Theorem 2.2. [Wik], Paper V, Theorem 2.4. *Let Ω be a bounded convex domain in $\mathbb{C}^n$, and let A be as above. Then $\delta^A(z) = \delta(z, A)$ for all $z \in \Omega$.*

In the rest of this paper we will only consider the following special case. We choose two poles $(a, 0)$ and $(b, 0)$, $a \neq b$ and $a, b \neq 0$, and a point $z = (0, \gamma)$, such that $|ab| < |\gamma| < \min\{|a|, |b|\}$. We let $g_{\mu,\nu}$ denote the Green function with weight μ at $(a, 0)$ and ν at $(b, 0)$, evaluated at z, and similarly for $\delta_{\mu,\nu}$. Note that, according to Theorem 2.1, $g_{1,1} = \log |\gamma|$, $g_{1,0} = \log |a|$, and $g_{2,1} = g_{1,1} + g_{1,0} = \log |\gamma| + \log |a|$.

We can now prove our theorem.

Proof of Theorem 1.2. It suffices to show that $\delta_{2,1} > g_{2,1}$. We already know that $\delta_{2,1} \geq g_{2,1}$. Suppose that equality holds. Then

$$\begin{aligned} g_{2,1} =& g_{1,1} + g_{1,0} \leq \delta_{1,1} + \delta_{1,0} \\ &= \inf\{\log |\zeta_1| + \log |\zeta_2|\} + \inf\{\log |\zeta_1|\} \\ &\leq \inf\{2 \log |\zeta_1| + \log |\zeta_2|\} = \delta_{2,1} = g_{2,1}, \end{aligned} \tag{1}$$

where all the infima are over the same family F_z. Hence all the inequalities are in fact equalities. Now, by Theorems 1.1 and 2.2, the *last* infimum is attained by an extremal disc f that passes through $(a, 0)$ or $(b, 0)$ or both.

It follows that f must be extremal for $\delta_{1,1}$ and $\delta_{1,0}$ as well. However, this is impossible, as the following lemma shows. The contradiction gives the theorem. □

Lemma 2.3. *If $|ab| < |\gamma| < \min\{|a|, |b|\}$, there are no common extremal discs for $\delta_{1,1}$ and $\delta_{1,0}$.*

Proof. We first characterize all extremal discs for $\delta_{1,0}$. Let $f = (f_1, f_2) : D \to D \times D$ be such a disc. By definition there is a $\zeta_1 \in D$ such that $f_1(\zeta_1) = a$ and $f_1(0) = 0$. Now, we have $g_{1,0} = \log|a|$, and we see that this also equals $\delta_{1,0}$, since the disc

$$\zeta \mapsto (\zeta, \frac{\gamma}{a}\frac{\zeta - a}{1 - \bar{a}\zeta})$$

gives that value. Hence $|\zeta_1| = |a|$. The Schwarz lemma now gives the conclusion that a disc passing through $(a, 0)$ and z is extremal for $\delta_{1,0}$ if and only if it is a rotation in the first variable.

Fix now such a disc $\zeta \mapsto (\alpha\zeta, f_2(\zeta))$, where $|\alpha| = 1$, and assume that it is extremal for $\delta_{1,1}$. By Theorem 1.1 there are now two cases. If the disc passes through just one pole, it must be $(a, 0)$ which would be the image of $\zeta_1 = a/\alpha$. Hence we get $\delta_{1,1} = \log|a/\alpha| = \log|a|$. On the other hand, if the disc passes through both poles, they would be the image of $\zeta_1 = a/\alpha$ and $\zeta_2 = b/\alpha$ respectively. Hence $\delta_{1,1} = \log|a/\alpha| + \log|b/\alpha| = \log|ab|$ in this case.

To finish the proof we will prove that $\delta_{1,1} = g_{1,1} = \log|\gamma|$, which will rule out both cases. We have $|\gamma| < |a| < |a/b|$, hence $|\gamma b/a| < 1$. Similarly we get $|\gamma a/b| < 1$. Hence we can find $\zeta_1 \in D$ such that $\zeta_1^2 = \gamma a/b$. Define $\zeta_2 = \gamma/\zeta_1$, and $\beta = a/\zeta_1$. Then $\zeta_2 \in D$, since $|\zeta_2|^2 = |\gamma b/a|$, and $\beta \in D$ since $|\beta|^2 = |ab/\gamma|$. We now define an analytic disc by

$$f : \zeta \mapsto \left(\beta\zeta, \frac{\zeta - \zeta_1}{1 - \bar{\zeta}_1\zeta}\ frac\ \zeta - \zeta_2 1 - \bar{\zeta}_2\zeta\right).$$

It is easy to verify that it maps ζ_1 onto $(a, 0)$, ζ_2 onto $(b, 0)$, and 0 onto $(0, \gamma)$. Hence $d(f) = \log|\zeta_1| + \log|\zeta_2| = \log|\gamma|$, which completes the proof of the lemma. □

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