Fourier Analysis

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Fourier Analysis can indicate the study of Fourier transformations, Fourier series, and their extensions. One studies e.g. the convergence properties of Fourier series of certain classes of functions. It may also indicate the use of the Fourier mechanism in other subjects, e.g. in differential equations or in signal analysis. This course contains something of both these worlds. The course is intended for the master level mathematics at Dutch universities. Thus we assume some knowledge of elementary Fourier Analysis, Functional Analysis and Integration theory. At the UvA the courses Integration theory and Linear analysis of the bachelor program are more than sufficient. For convenience of the reader the essentials of all this are mentioned in the notes, but we do not dwell on the proofs.

The notes are based on a course that I first gave in 1996 at the UvA. They have been modified and slightly been edited, but there are bound to be many typo's and other errors. I would certainly appreciate it if a reader pointed out some to me.

Everything in these notes can be found in the literature, but one may have to look for it a while.

Jan Wiegerinck

CHAPTER 1

Classical Fourier series

1.1. Introduction and Reminder

In this section we recall a few facts from the linear analysis course of the third year. See the notes by Prof. Koornwinder [**13**] or the books [**20, 12**]. As usual we identify functions on the unit circle T with 2π-periodic functions on $\mathbb{R}$; if f is defined on T, then $g(t) = f(e^{it})$ is the associated 2π-periodic function on $\mathbb{R}$. We denote either by $L^p_{2\pi}$ or by $L^p(T)$, $(1 \le p < \infty)$ the set of 2π-periodic measurable functions that satisfy

$$\|f\|_p \stackrel{\text{def}}{=} \left(\int_0^{2\pi} |f(t)|^p \frac{dt}{2\pi}\right)^{1/p} < \infty \tag{1.1.1}$$

Notice that we normalized $L^p(T)$ spaces utilizing the measure $\frac{dt}{2\pi}$. The pleasant effect is that the norm of the function 1 equals 1. We know that $\| \, \|_p$ is a norm, which turns L^p into a Banach space, while $L^2(T)$ is even a Hilbert space with inner product

$$\langle f, g\rangle \stackrel{\text{def}}{=} \int_0^{2\pi} f(t)\overline{g(t)} \frac{dt}{2\pi}.$$

Other function spaces on T are $L^\infty(T)$ the space of 2π periodic, essentially bounded measurable functions, and $C(T) \subset L^\infty(T)$ the space of continuous 2π-periodic functions. Both spaces are Banach spaces when equipped with the sup-norm.

We will also have use for sequence spaces: $l^p(\mathbb{Z})$ $(1 \le p < \infty)$ is the space of sequences

$$\left\{ a = \{a_j\}_{j=-\infty}^{\infty} : |a|_p \stackrel{\text{def}}{=} \left(\sum_{j\in\mathbb{Z}} |a_j|^p\right)^{1/p} < \infty \right\} \tag{1.1.2}$$

Again $l^p(\mathbb{Z})$ is a Banach space with norm $| \, |_p$, and $l^2(\mathbb{Z})$ is a Hilbert space, the inner product being $\sum_{j\in\mathbb{Z}} a_j \bar{b}_j$, $(a, b \in l^2(\mathbb{Z}))$. Other sequence spaces that we will meet are the space of bounded sequences $c(\mathbb{Z})$, and its subspace $c_0(\mathbb{Z})$, which consists of sequences $\{a_j\}$ tending to 0 if $|j| \to \infty$. Both are Banach spaces when equiped with the sup-norm.

The *Fourier series* of $f \in L^1(T)$ is

$$\sum_{-\infty}^{\infty} a_n e^{int}, \tag{1.1.3}$$

where $a_n = \hat{f}(n) = \int_0^{2\pi} f(s)e^{-ins}\frac{ds}{2\pi}$ are the *Fourier coefficients* of f. We know by the *Riemann-Lebesgue Lemma* that $\hat{f}(n) \to 0$ if $|n| \to \infty$, that is, $\hat{f} \in c_0(\mathbb{Z})$. A formal sum of the form (1.1.3) with arbitrary a_n is called a *trigonometric series.* If we start with a Borel measure supported on $[-\pi, \pi)$ we can also form the Fourier coefficients of μ

$$a_n = \hat{\mu}(n) = \int_{-\pi}^{\pi} e^{-int} d\mu(t).$$

The series (1.1.3) is then called a *Fourier-Stieltjes series.* Of course $|\hat{\mu}(n)| \le \|\mu\|$, but it is in general not true that $\hat{\mu}(n) \to 0$, if $|n| \to \infty$. Taking $\mu = \delta$, point mass at 0, we find $\hat{\delta}(j) = 1/2\pi$ for all j and the series (1.1.3) does not converge in the usual sense

We see that taking a Fourier series can be seen as a map ˆ from a space of functions, or measures, or more general, to a space of sequences. Natural questions are: For what kind of things can one define a Fourier series? Can you say something about the target space if you start in $L^p(T)$? Is ˆ surjective to some $l^p(\mathbb{Z})$? Is it maybe even an isometry? Is (1.1.3) convergent in L^p if $f \in L^p$? Is it perhaps convergent in any other sense?

Some of these questions will be answered in the course.

Partial sums of the series (1.1.3) are expressed by means of the *Dirichlet kernels* D_N. These are defined as follows

$$D_N(t) = \sum_{-N}^{N} e^{int} = \begin{cases} \frac{\sin((N+1/2)t)}{\sin(t/2)} & \text{if } t \notin 2\pi\mathbb{Z} \\ 2N+1 & \text{if } t \in 2\pi\mathbb{Z}. \end{cases} \tag{1.1.4}$$

For the N-th partial sum $S_N[f](t) = \sum_{n=-N}^{N} \hat{f}(n)e^{int}$ of the Fourier series of f we find

$$S_N[f](t) = \sum_{n=-N}^{N} \left(\int_{-\pi}^{\pi} f(s)e^{-ins} ds \right) e^{int} = \int_0^{2\pi} f(s)D_N(t-s)\frac{ds}{2\pi} =: f * D_N(t). \tag{1.1.5}$$

Similarly, for the N-th *Césaro sum* σ_N, i.e. the average of the partial sums S_0 upto S_N, there is an expression by means of the N-th *Fejér kernel* K_N. The latter is defined by

$$\begin{aligned} K_N(t) &= \frac{1}{N+1}\sum_{n=0}^{N} D_n(t) = \sum_{n=-N}^{N} \frac{N+1-|n|}{N+1} e^{int} \\ &= \begin{cases} \frac{1}{N+1}\left(\frac{\sin((N+1)t/2)}{\sin(t/2)}\right)^2 & \text{if } t \notin 2\pi\mathbb{Z} \\ N+1 & \text{if } t \in 2\pi\mathbb{Z}. \end{cases} \end{aligned} \tag{1.1.6}$$

The Césaro sum of f is given by

$$\sigma_N[f](t) = \frac{1}{N+1}\sum_{n=0}^{N} S_n[f](t) = f * K_N(t).$$

The Fejér kernels K_N are *good kernels*, they have the three characteristic properties of an *approximate identity*:

- $K_N \geq 0$.
- $\int_0^{2\pi} K_N(t)\, \frac{dt}{2\pi} = 1$.
- For every $0 < \delta < \pi$, $K_N(t) \to 0$ as $N \to \infty$ uniformly on $[\delta, 2\pi - \delta]$.

Let $f \in L^1(T)$. If a family of integral kernels L_N on $[0, 2\pi]$ has these three properties, then at a point of continuity a of f one has that $L_N * f(a) \to f(a)$ and, moreover, for $f \in C(T)$ the convergence of $L_N * f$ to f is uniform on T. We indicate the proof.

(1.1.7)

$$|f(a) - L_N * f(a)| = \left| \int_0^{2\pi} \Big(f(a) - f(a-t)\,)L_N(t)\,dt \right| \leq \left| \int_{[\delta,2\pi-\delta]} \cdots \right| + \left| \int_{[0,\delta]\cup[2\pi-\delta,2\pi]} \cdots \right|$$

The first term is small for small δ by continuity of f at a and property i and ii. Fixing such a small δ, the second term is bounded by $\max_{\delta \leq t \leq 2\pi-\delta} L_N(t)\|f\|_1 + |f(a)| \int_\delta^{2\pi-\delta} L_n(t)dt$. This tends to 0 when $N \to \infty$. Now if f is continuous on T, then it is uniformly continuous on T and δ can be chosen independently of a. Moreover the second term can be estimated uniformly, leading to uniform convergence of $L_N * f$ on T. With a bit more effort, if $f \in L^p(T)$ $(1 \leq p < \infty)$, then $L_N * f$ tends to f in L^p sense as $N \to \infty$. See [**10**] for a clever proof with a slightly weaker condition iii.

In particular these things hold for the Fejér kernel, giving the well-known fact that the Césaro sums of $f \in C(T)$ converge to f uniformly on T. In particular every $f \in C(T)$ can be approximated uniformly by goniometric polynomials, namely by its Césaro sums.

$$f(t) = \lim_{N\to\infty} \sigma_N[f](t). \tag{1.1.8}$$

The exponentials e^{int}, $n \in \mathbb{Z}$, form clearly an orthonormal system in $L^2(T)$. Because $C(T)$ is dense in $L^2(T)$ and the goniometric polynomials are dense in $C(T)$, this orthonormal system is complete, hence an orthonormal *basis* for $L^2(T)$. Observe that for an $f \in L^2(T)$ its Fourier series is the expansion of f on the basis $\{e^{int}\}$. As a consequence for $f, g \in L^2(T)$ *Parseval's formula* holds:

$$\int_{-\pi}^{\pi} f\bar{g}\frac{dt}{2\pi} = \sum_{\mathbb{Z}} \hat{f}(n)\bar{\hat{g}}(n), \tag{1.1.9}$$

Observe that the lefthand side of 1.1.9 is the inner product in $L^2(T)$, while the righthand side is the inner product in $l^2(\mathbb{Z})$. So 1.1.9 expresses that $\hat{}$ is an isometry from $L^2(T) \to l^2(\mathbb{Z})$. In fact it is also surjective. If $\{a_j\}_j \in l^2(\mathbb{Z})$ then the partial sums $\sum_{j=-N}^{N} a_j e^{ijt}$ form a Cauchy sequence in $L^2(T)$ that converges to some $f \in L^2(T)$ with $\hat{f}(j) = a_j$.

We will need a few additional estimates on K_N:

$$K_N(t) \le \min\{N+1, \frac{\pi^2}{(N+1)t^2}\}, \quad t \in [-\pi, \pi] \tag{1.1.10}$$

and, using Parseval's formula,

$$\|K_N\|_2^2 = \sum_{n=-N}^{N} \left(\frac{N+1-|n|}{N+1}\right)^2 \ge N/2. \tag{1.1.11}$$

The sum in the middle can of course be computed, but the last estimate follows easily by comparison with $\int_0^{N+1} (1 - x/(N+1))^2\, dx$.

Pointwise Convergence of the Fourier series is not nearly as good as L^2-convergence. The classical result is as follows.

THEOREM 1.1.1. *Suppose that $f \in C(T)$ is* Hölder continuous*, i.e. there exist $\alpha, C > 0$ such that*

$$|f(s) - f(t)| < C|s-t|^\alpha.$$

Then

$$S_N[f](t) \to f(t), \quad \textit{uniformly, as } N \to \infty.$$

However, there exist continuous functions on T, the Fourier series of which does not converge uniformly on T. Indeed, for every $x \in T$ the map $\Lambda_n^x : f \mapsto S_n[f](x)$ is a bounded linear functional on $C(T)$. One can show that $\|\Lambda_n^x\| \ge C \log n$ and in particular tends to ∞. The Banach Steinhaus Theorem then gives that for a dense set of functions $f \in C(T)$ one has

$$\sup_n |\Lambda_n^x f| = |S_n[f](x)| = \infty.$$

See [**16**] for details. A more or less constructive proof can be found in [**20**].

Concerning point wise convergence of the Fourier series of L^p functions we state two classical results. Andrey Kolmogorov constructed $L^1(T)$-functions whose Fourier series does not converge in any point of T, [**11**]. Lennart Carleson, on the other hand, showed that the Fourier series of an f in $L^2(T)$ converges almost everywhere on T, [**1**]. His result was extended by Richard Hunt to L^p for $p > 1$, cf. [**8**].

1.2. Sine versus Cosine Series

If we start with an *even function* f, $(f(t) = f(-t))$, in (1.1.3), we will find that $a_n = a_{-n}$ (the sequence is *even*). Then taking together the n-th and $-n$-th term, we obtain

$$a_0 + \sum_{n=1}^{\infty} a_n(e^{int} + e^{-int}) = a_0 + \sum_{n=1}^{\infty} 2a_n \cos(nt),$$

a *cosine series.* Similarly if f is an *odd function* $(f(t) = -f(-t))$, we find $a_n = -a_{-n}$, (the sequence is *odd*), and we obtain the *sine series*

$$\sum_{n=1}^{\infty} a_n(e^{int} - e^{-int}) = \sum_{n=1}^{\infty} 2ia_n \sin(nt).$$

Notice that every function on $(0, \pi)$ can be extended as an even, but also to an odd 2π-periodic function. In that light it is remarkable that sine and cosine series have different convergence behavior even if the coefficients are the same. (Of course the series belonging to even and odd continuation do not have the same coefficients) In the present section we prove two theorems which enlighten this behavior of sine and cosine series.

THEOREM 1.2.1. *Suppose that $(a_n)_{n=-\infty}^{\infty}$ is an even sequence of positive numbers which tend to 0 if $|n| \to \infty$. If (a_n) satisfies the convexity condition*

$$a_{n-1} + a_{n+1} - 2a_n \geq 0, \quad (n \geq 1),$$

then there exists $f \in L^1_{2\pi}$ such that $\hat{f}(n) = a_n$.

PROOF. The convexity condition implies that $a_n - a_{n+1}$ is monotonically decreasing to 0. From this we have

$$\begin{aligned} n(a_n - a_{n+1}) &\leq (a_k - a_{k+1}) + (a_{k+1} - a_{k+2}) \dots (a_n - a_{n+1}) + (k-1)(a_n - a_{n+1}) \\ &= a_k - a_{n+1} + (k-1)(a_n - a_{n+1}) \to 0, \end{aligned} \tag{1.2.1}$$

by choosing k fixed and large, so that a_k is small, and then letting $n \to \infty$. By cleverly rearranging of the series, so-called *summation by parts*, we also find

$$\begin{aligned} &\sum_{n=1}^{N} n(a_{n-1} + a_{n+1} - 2a_n) = \sum_{n=1}^{N} n((a_{n+1} - a_n) - (a_n - a_{n-1})) \\ &= \sum_{n=1}^{N} n(a_{n+1} - a_n) - \sum_{n=0}^{N-1} (n+1)(a_{n+1} - a_n) = a_0 - a_N - N(a_N - a_{N+1}) \longrightarrow a_0, \end{aligned} \tag{1.2.2}$$

for $N \to \infty$. Put

$$f_N(t) = \sum_{n=1}^{N} n(a_{n-1} + a_{n+1} - 2a_n) K_{n-1}(t).$$

This series has non-negative terms and is Cauchy in L^1 sense. In view of (1.2.2), for $N > M$

$$\begin{aligned} \int_0^{2\pi} |f_N - f_M| \, dt &= \sum_{n=M+1}^{N} n(a_{n-1} + a_{n+1} - 2a_n) \int_0^{2\pi} K_{n-1}(t) dt \\ &= \sum_{n=M+1}^{N} n(a_{n-1} + a_{n+1} - 2a_n) < \varepsilon \quad \text{if } M \text{ is sufficiently large.} \end{aligned} \tag{1.2.3}$$

Therefore $\lim_N \to \infty f_N = f$ exists in $L^1(T)$. Using that by (1.1.6) $\hat{K}_{n-1}(p) = \frac{n-|p|}{n}$ if $n > |p|$ and a dilated version of (1.2.2), we compute $\hat{f}$.

$$\hat{f}(p) = \sum_{n>|p|} (a_{n-1} + a_{n+1} - 2a_n)(n - |p|) = \sum_{j=1}^{\infty} j(a_{|p|+j-1} + a_{|p|+j+1} - 2a_{|p|+j}) = a_{|p|}.$$

□

THEOREM 1.2.2. *Suppose that f is in $L^1(T)$ and that $\hat{f}(n) = -\hat{f}(-n) \geq 0$ for $n \geq 0$. Then $\sum_{n\neq 0} \frac{\hat{f}(n)}{n}$ converges.*

PROOF. Let $F(t) = \int_0^t f(s)\,ds$, $t \in [-\pi, \pi]$. Then F is continuous and $F(-\pi) = F(\pi)$, because $\hat{f}(0) = 0$, i.e. $F \in C(T)$. The Fourier coefficients of F are

$$\hat{F}(n) = \frac{\hat{f}(n)}{in}, \quad (n \neq 0).$$

The Césaro sums of F will converge uniformly to F, therefore, subtracting $\hat{F}(0)$ from F and evaluating at 0,

$$\lim_{N\to\infty} \sum_{1\leq|n|\leq N} \frac{N+1-|n|}{N+1} \frac{\hat{f}(n)}{n} = i(F(0) - \hat{F}(0)) = -i\hat{F}(0).$$

All terms in the sum are positive, so this sum converges absolutely. Next

$$\sum_{1\leq|n|\leq N} \frac{\hat{f}(n)}{n} \leq 2 \sum_{1\leq|n|\leq N} (1 - \frac{|n|}{2N+1}) \frac{\hat{f}(n)}{n} < i\hat{F}(0),$$

which proves the theorem. □

COROLLARY 1.2.3. *Let $b_n = \frac{1}{\log(n+2)}$, then $\sum b_n \cos(nt)$ is the Fourier series of an L^1 function, but $\sum b_n \sin(nt)$ is not.*

1.3. Weak Topologies

Occasionally we will use weak-* convergence of measures. In this section we recall this notion for readers who are not familiar with it.

1.3.1. Weak-* convergence. A sequence of Borel measures $(\mu_j)_j$ on a compact Hausdorff space X converges weak-* to μ if for every $f \in C(X)$

$$\lim_{j\to\infty} \int f\,d\mu_j = \int f\,d\mu. \tag{1.3.1}$$

Similarly, in a Hilbert space H with inner product $\langle\cdot,\cdot\rangle$, a sequence f_j converges weakly to f if for every $g \in H$

$$\lim_{j\to\infty} \langle f_j, g\rangle = \langle f, g\rangle. \tag{1.3.2}$$

This and the Banach-Alaoglu Theorem below is basically all we need to know. Nevertheless some background may be useful.

1.3.2. The weak topology. Recall that a topology τ_1 on a set X is called *weaker* than τ_2 on X if every τ_1 open set in X is also τ_2 open; then τ_2 is called *stronger* than τ_1. Also recall that the product topology is defined by requiring that it is the weakest topology on the set theoretical product such that all projections are continuous mappings.

We can do something similar in topological vector spaces. Thus let X be a topological vector space such that its dual X^* separates points of X, i.e. for every $x \in X$ there exists a continuous linear functional $L \in X^*$ with $Lx \neq 0$. This is certainly the case if X is a Banach or a Hilbert space. The *weak topology* on X is the weakest topology that makes all $L \in X^*$ continuous. Since they are already continuous in the original topology of X, the weak topology is weaker than the original one.

A local subbasis for the weak topology on X consists of the sets

$$V_L^\varepsilon = \{x \in X : |Lx| < \varepsilon\},$$

where $\varepsilon > 0$ and $L \in X^*$. This means that $U \subset X$ is a neighborhood of 0 if there exist $\varepsilon_i > 0$, $L_i \in X^*$ such that $(\cap_{i=1}^p V_{L_i}^{\varepsilon_i}) \subset U$. How does this relate to convergence? Well, $x_j \to x$ if and only if $x_j - x \to 0$, that is, every neighborhood U of 0 must, for sufficiently large j, contain $x_j - x$. Therefore, for every choice of finitely many $V_{L_i}^{\varepsilon_i}$, it holds that $x_j - x \in (\cap_{i=1}^p V_{L_i}^{\varepsilon_i})$ if j is sufficiently large. This happens if and only if $Lx_j \to Lx$ for every L. Compare this to (1.3.2).

1.3.3. The weak-* topology. Recall that X can be seen as a subset of X^{**} via $xL := Lx$, $(x \in X,\ L \in X^*)$ and that the subset $X \subset X^{**}$ already separates points on X^*. The *weak-* topology* on X^* is defined as the weakest topology that makes all $x \in X$ continuous functionals on X^*. We do not require continuity of functionals in $X^{**} \setminus X$ (In many important cases, however, this set is empty, compare [**22**]). Similarly to weak convergence, a sequence $L_j \in X^*$ converges weak-* to $L \in X^*$ if and only if for every $x \in X$ we have for every $x \in X$ that $L_j x \to Lx$.

Finally we quote

THEOREM 1.3.1 (Banach-Alaoglu). *If V is a neighborhood of 0 in a topological vector space X and*

$$K_V = \{L \in X^* : \ |Lx| \le 1 \textit{ for every } x \in V\}.$$

Then K_V is weak- compact.*

A proof can be found in [**17**].

EXAMPLE 1.3.2. Let $X = C(T)$, $V = \{f \in C(T) : \ \|f\|_\infty < 1\}$, then $K_V = \{\mu \in M(T) : \|\mu\| \le 1\}$ is compact. Theorem 1.3.1 tells us that every sequence of Borel measures $(\mu_\alpha)_\alpha$ on T with uniformly bounded mass has a weak-* convergent subsequence. In other words, there exists a subsequence $(\mu_j)_j$ and a measure $\mu \in M(T)$ such that (1.3.1) holds.

1.4. Lacunary Series

DEFINITION 1.4.1. A sequence $\{\lambda_j\}$, $j = 1, 2, \ldots$, of positive integers is called (Hadamard) *lacunary* with constant $q > 1$ if $\lambda_{j+1} > q\lambda_j$ for all $j \ge 1$. A power series is called lacunary if it is of the form $\sum c_j z^{\lambda_j}$, while a trigonometric series is called lacunary if it is of the form $\sum c_j e^{i\lambda_j t} + \sum d_j e^{-i\lambda_j t}$ with $\{\lambda_j\}$ lacunary.

LEMMA 1.4.2. *Let $n_0 \in \mathbb{Z}$. Suppose that $f \in L^1_{2\pi}$ and $f(t) = O(t)$ as $t \to 0$. If*

$$\hat{f}(j) = 0, \quad \textit{for all } 1 \le |n_0 - j| \le 2N, \tag{1.4.1}$$

then

$$|\hat{f}(n_0)| \le 2\pi^4 (N^{-1} \sup_{|t| \le N^{-1/4}} |f(t)/t| + N^{-2} \|f\|_1).$$

PROOF. If g_N is any trigonometric polynomial of degree $2N$ with $\hat{g}(0) = 1$, then

$$\hat{f}(n_0) = \int_{-\pi}^{\pi} e^{-in_0 t} f(t) g_N(t) \frac{dt}{2\pi},$$

because (1.4.1) expresses that $S_N[e^{-in_0 t} f(t)] = \hat{f}(n_0)$. We take $g_N = K_N^2 / \|K_N\|_2^2$. Then in view of (1.1.10) and (1.1.11) $\int_{-\pi}^{\pi} g_N \frac{dt}{2\pi} = 1$, $g_N \ge 0$, $g_N(t) \le \frac{\pi^4 2}{N(N+1)^2 t^4}$. We use this to estimate

$$|\hat{f}(n_0)| \le \int_{-\pi}^{\pi} |f(t)| g_N(t) \frac{dt}{2\pi} = \int_{|t| \le N^{-1}} + \int_{N^{-1} \le |t| \le N^{-1/4}} + \int_{N^{-1/4} \le |t| \le \pi}.$$

Now these three integrals are estimated as follows.

$$\int_{|t| \le N^{-1}} |f(t)| g_N(t) \frac{dt}{2\pi} \le \frac{1}{N} \sup_{|t| \le N^{-1}} \frac{|f(t)|}{|t|} \int_{-\pi}^{\pi} g_N(t) \frac{dt}{2\pi} = \frac{1}{N} \sup_{|t| \le N^{-1}} \frac{|f(t)|}{|t|}.$$

$$
(1.4.2) \qquad \begin{aligned}
\int_{\frac{1}{N}\le|t|\le\frac{1}{N^{1/4}}} |f(t)|g_N(t)\frac{dt}{2\pi} &\le \sup_{|t|\le\frac{1}{N^{1/4}}} \frac{|f(t)|}{|t|}\int_{\frac{1}{N}\le|t|\le\frac{1}{N^{1/4}}} \frac{|t|\pi^4 2}{N(N+1)^2t^4}\frac{dt}{2\pi} \\
&\le \sup_{|t|\le\frac{1}{N^{1/4}}} \frac{|f(t)|}{|t|}\frac{\pi^4 2N}{2\pi(N+1)^2}.
\end{aligned}
$$

$$
\int_{N^{-1/4}\le|t|\le\pi} |f(t)|g_N(t)\frac{dt}{2\pi} \le \frac{\pi^4 2}{N(N+1)^2(N^{-1/4})^4}\int_{-\pi}^{\pi}|f(t)|\frac{dt}{2\pi}.
$$

These three estimates prove the lemma. $\square$

COROLLARY 1.4.3. *Suppose that* $f = \sum_{n=1}^{\infty} a_n\cos(\lambda_n t) \in L^1_{2\pi}$, *with* $\lambda_{n+1} \ge q\lambda_n$ *and* $q > 1$. *If* f *is differentiable at a point* p *then* $a_n = o(\lambda_n^{-1})$.

PROOF. Considering $f_p(t) = f(t+p)$ which has $\hat{f}_p(n) = e^{inp}\hat{f}(n)$ we may assume that $p = 0$. Replace f by $f - f(0) - f'(0)\sin t$. This has no effect on the tail of the series and now $f(t) = o(|t|)$ at 0. We have $\hat{f}(j) = 0$ for $0 < |j - \lambda_n| < (1 - 1/q)\,\lambda_n$. We apply the Lemma and obtain

$$
|\hat{f}(\lambda_n)| \le \frac{o(1)}{\lambda_n} + \frac{C}{\lambda_n^2} = \frac{o(1)}{\lambda_n}).
$$

$\square$

COROLLARY 1.4.4 (Weierstrass' nowhere differentiable function).

$$
f(t) = \sum_{n=0}^{\infty}\frac{\cos(2^n t)}{2^n}
$$

is continuous, but nowhere differentiable.

PROOF. The series is lacunary and uniformly convergent, so f is continuous and the previous corollary gives that f is nowhere differentiable. $\square$

1.5. Riesz products

Let $\{\lambda_n\}$ be lacunary with $q \ge 3$. A trigonometric polynomial of the form

$$
P_N(t) = \prod_{n=1}^{N}(1 + a_n\cos(\lambda_n t + \varphi_n))
$$

is called a (finite) *Riesz product*. Observe that, since $q \ge 3$, an integer M can at most in **one** way be written as

$$
M = \sum_{1}^{\infty} c_n\lambda_n, \quad c_n \in \{-1, 0, 1\}.
$$

In fact M will be a finite sum and unless $q = 3$, not all M can be expressed as such a sum. We use this when expanding P_N. A typical factor of P_N is $1 + (a_ne^{i\varphi_n}/2)e^{i\lambda_n t} + (a_ne^{-i\varphi_n}/2)e^{-i\lambda_n t}$. In the expansion of P_N we will thus find exponentials of the form $e^{ikt} = e^{i(\sum c_n\lambda_n)t}$ and by the preceding observation such an exponential can be obtained in at most one way. It follows that

$$
(1.5.1) \qquad \hat{P}(k) = \begin{cases} \prod\left(\frac{a_ne^{i\varphi_nc_n}}{2}\right) & \text{if } k = \sum c_n\lambda_n, \text{ with } c_n \ne 0, \\ 0 & \text{elsewhere.} \end{cases}
$$

Also, from $P_{N+1} = P_N + \frac{a_{N+1}e^{i\varphi_{N+1}}}{2}P_Ne^{i\lambda_{N+1}t} + \frac{a_{N+1}e^{i\varphi_{N+1}}}{2}P_Ne^{-i\lambda_{N+1}t}$ we see that the Fourier series of P_{N+1} is obtained from the Fourier series of P_N by adding two copies of P_N multiplied by a constant, one shifted λ_{N+1} to the right, the other shifted λ_{N+1} to the left. As $q \ge 3$ there is no overlap. In particular, whatever the sequence $\{a_n\}$, if $N \to \infty$, then $\hat{P}_N$ becomes stationary on every finite subset of $\mathbb{Z}$. We find that $\lim_{N\to\infty}\hat{P}_N$ is a well-defined trigonometric series. If $P = \lim P_N$ in some sense then $\hat{P} = \lim\hat{P}_N$.

For us there are two cases of interest:

(1) Suppose $-1 \le a_n \le 1$. Then all P_N are nonnegative and $\int_{-\pi}^{\pi} P_N = 1$. Thus the P_N are (densities of) probability measures on $(-\pi, \pi)$. There exists at least one weak-* limit point. If μ_1, μ_2 are two weak-* limit points of $P_N\,dt$, we have $\hat\mu_1 = \hat\mu_2$, so $\mu_1 = \mu_2$ and P_N converges weak-* to a probability measure with Fourier-Stieltjes series $\lim P_N$.

(2) Suppose that $a_n = ib_n$, $b_n \in \mathbb{R}$ and $\sum_n b_n^2 < \infty$. We have $1 \le |(1 + ib_n \cos(\lambda_n t)| \le (1 + b_n^2)^{1/2}$, therefore, with suitable constant C.

$$1 \le |P_N| \le e^{(\sum_n \log(1+b_n^2))/2} \le e^{\frac{1}{2}\sum_n b_n^2} < C. \tag{1.5.2}$$

Thus $|P_N|$ is uniformly bounded and P_N converges weak-* to (a measure given by) an $L^\infty_{2\pi}$ function. (If O is open in T with Lebesgue measure $|O|$, and f is continuous $0 \le f \le 1$ with support in O, then $|\int f P_N\,dt| \le C|O|$; the same goes for the weak-* limit, giving that the weak-* limit is absolutely continuous with respect to Lebesgue measure and the density is in L^∞.)

LEMMA 1.5.1. *Let $\{\lambda_j\}$ be lacunary with constant q. Put $\lambda_{-j} = -\lambda_j$, $\lambda_0 = 0$. There exist constants A_q, B_q such that if $f(t) = \sum_{-N}^{N} c_j e^{i\lambda_j t}$, then*

$$\begin{aligned} \sum |c_j| &\le A_q \|f\|_\infty, \\ \|f\|_2 &\le B_q \|f\|_1. \end{aligned} \tag{1.5.3}$$

PROOF. Notice that if we prove the Lemma for real valued f, then it follows for complex valued f with the constants A_q and B_q doubled. We first deal with the case $q \ge 3$ and assume that f is real, which means that $c_j = \bar c_{-j}$. To prove the first inequality, we set

$$P_N(t) = \prod_{j=1}^{N} (1 + \cos(\lambda_j t + \varphi_j)).$$

We choose $\varphi_j = \arg c_j$, $j \ge 1$. We have

$$\int_{-\pi}^{\pi} P_N \bar f \frac{dt}{2\pi} = \sum_{j=-N}^{N} \hat P_N(\lambda_j) \bar{\hat f}(\lambda_j) = \frac{1}{2} \sum_{j=-N}^{N} e^{i\mathrm{Sign}\,(j)\varphi_{|j|}} \bar c_j = \frac{1}{2} \sum_{-N}^{N} |c_j|,$$

Also

$$\left|\int_{-\pi}^{\pi} P_N \bar f \frac{dt}{2\pi}\right| \le \|f\|_\infty \int_{-\pi}^{\pi} P_N \frac{dt}{2\pi} = \|f\|_\infty.$$

Thus we have proved the first equality for $q \ge 3$ with $A_q = 4$.

For the second inequality we set

$$P_N(t) = \prod_{j=1}^{N} \left(1 + i\left(\frac{|c_j|}{\|f\|_2}\right) \cos(\lambda_j t + \varphi_j)\right).$$

We proceed as above and find with the same choice of φ_j

$$\begin{aligned} \|f\|_2 &= \sum_{j=-N}^{N} \frac{|c_j|^2}{\|f\|_2} = -2i \sum_{j=-N}^{N} i \frac{|c_j|}{\|f\|_2} \frac{e^{i\mathrm{Sign}\,(j)\varphi_{|j|}}}{2} \bar c_j \\ &= -2i \sum_{j=-N}^{N} \hat P_N(\lambda_j) \bar{\hat f}(\lambda_j) = -2i \int P_N \bar f \frac{dt}{2\pi} \le 2\|P_N\|_\infty \|f\|_1. \end{aligned} \tag{1.5.4}$$

Since the P_N are uniformly bounded by $e^{1/2}$, compare (1.5.2), we are done. Notice that it is the seemingly artificial factor i that we introduced in P_N, that makes it possible to estimate $\|P_N\|_\infty$.

For $q \ge 3$ we may take $B_q = 4e^{1/2}$.

The general case is done by carefully splitting the lacunary sequence in such a way that the Riesz products associated to the subsequences make sense and will only pick up the terms in the series that we want. Let M be a large integer, depending on q and to be determined in the process. Write for a fixed $0 \le m < M$ $\lambda_j^m = \lambda_{m+jM}$. We want that $\{\lambda_j^m\}_j$ is lacunary with constant ≥ 3. Thus we require that M satisfy

$$q^M \ge 3, \tag{1.5.5}$$

hence a Riesz product associated to $\{\lambda_j^m\}_j$ makes sense. Next, we want that each of the frequencies λ_k of f occurs in precisely one Riesz product. Suppose that $n > 0$ is written as $\sum_{j=0}^{J} c_j \lambda_j^m$, with $c_j \in \{-1, 0, 1\}$ and $c_J = 1$. Then

$$|n - \lambda_J^m| \le \sum_{j=0}^{J-1} \lambda_j^m \le \lambda_J^m \sum_{j=0}^{J-1} \frac{\lambda_j^m}{\lambda_J^m} \le \lambda_J^m \sum_{j=1}^{J} \frac{1}{q^{jM}} \le \frac{\lambda_J^m}{q^M - 1}.$$

Thus we want $|\lambda_k - \lambda_J^m| > \frac{\lambda_J^m}{q^M-1}$ for all $\lambda_k \ne \lambda_J^m$. If $\lambda_k \ge q\lambda_J^m$ this leads to

$$q - 1 > \frac{1}{1 - q^M} \quad \text{or} \quad q > 1 + \frac{1}{q^M - 1}, \tag{1.5.6}$$

while, if $\lambda_J^m \ge q\lambda_k$, this leads to

$$1 - 1/q > \frac{1}{1 - q^M} \quad \text{or} \quad 1/q < 1 - \frac{1}{q^M - 1}. \tag{1.5.7}$$

We take M so large that (1.5.5), (1.5.6), (1.5.7) are satisfied. Now let $P_N^m = \prod_1^N (1 + a_{m+jM} \cos(\lambda_j^m t + \varphi_{m+jM}))$ be one of the Riesz products considered in the first part of the proof. Then

$$\frac{1}{2\pi} \int P_N^m(t) \bar{f}(t)\, dt = \frac{1}{2} \sum |a_{m+jM}||c_{m+jM}|.$$

The first part of the proof gives

$$\begin{aligned} \sum |c_{m+jM}| &\le 4\|f\|_\infty, \\ (\sum |c_{m+jM}|^2)^{\frac{1}{2}} &\le 4e^{1/2}\|f\|_1, \end{aligned} \tag{1.5.8}$$

in respectively the first and second case of the lemma. Summing over $m = 1, \dots, M$ gives the result. ☐

THEOREM 1.5.2. *Suppose that the Fourier series $\sum_{-\infty}^{\infty} c_j e^{i\lambda_j t}$ of $f \in L^1(T)$ is lacunary, then $f \in L^2(T)$. If f is bounded, then $\sum |c_j| < \infty$.*

PROOF. Let $\sigma_N[f]$ be a Césaro sum. These have L^1 norms, uniformly bounded by $\|f\|_1$. The lemma gives that

$$\sum_{-N}^{N} \left(1 - \frac{|j|}{N+1}\right)^2 |c_j|^2 \le B_q M.$$

This implies, by letting $N \to \infty$, that for fixed J the sum $\sum_{-J}^{J} |c_j|^2 \le B_q M$, thus $f \in L^2(T)$. The proof of the second statement is similar. ☐

The homogeneity of behavior of lacunary series also appears in the following theorem. We shall denote the length of a subarc Γ of T by $|\Gamma|$.

THEOREM 1.5.3. *Suppose that (λ_j) is lacunary with constant q. For every $\delta > 0$ there exists $j_0 \in \mathbb{N}$ such that for all lacunary $f \in L^2(T)$, $f(t) = \sum_{-\infty}^{\infty} c_j e^{i\lambda_j t}$ with $c_j = 0$ for $|j| < j_0$, the following inequality holds for every subarc Γ of T:*

$$(\frac{1}{2\pi}|\Gamma| - \delta)\|f\|^2 \le \int_\Gamma |f|^2 \frac{dt}{2\pi} \le (\frac{1}{2\pi}|\Gamma| + \delta)\|f\|^2.$$

PROOF. It suffices to prove the theorem for trigonometric polynomials. We shall give a proof for $q \geq 3$. For the general case, see exercise 1.7.7. We have

$$\int_\Gamma |P|^2 \frac{dt}{2\pi} = \int_\Gamma \sum_{n,m} c_n \bar{c}_m e^{i(\lambda_n - \lambda_m)t} \frac{dt}{2\pi} = \frac{1}{2\pi}|\Gamma| \|P\|^2 + \sum_{n \neq m} c_n \bar{c}_m \int_\Gamma e^{i(\lambda_n - \lambda_m)t} \frac{dt}{2\pi}. \tag{1.5.9}$$

Let h be the characteristic function of Γ. Then $\hat{h}(n) = \int_\Gamma e^{-int} dt/2\pi$, therefore (1.5.9) can be rewritten as

$$\frac{1}{2\pi}|\Gamma| \|P\|^2 + \sum_{n \neq m} c_n \bar{c}_m \hat{h}(\lambda_m - \lambda_n). \tag{1.5.10}$$

Now because $q \geq 3$, $\lambda_n - \lambda_m$ assumes any integer value j at most twice (solutions occur in pairs: (n, m) and $(-m, -n)$, and compare the beginning of this section). For general $q > 1$ there exists $K = K(q)$ such that there are at most K solution, Ex. 1.7.7. Moreover, $\min_{m,n \geq j_0} |\lambda_m - \lambda_n|\}$ tends to ∞ with j_0. Thus frequencies j of $\hat{h}$ occur at most twice in the sum in (1.5.10). We apply Cauchy-Schwarz to the sum and obtain that the norm of the sum is less than

$$\left(\sum_{n \neq m} |c_n c_m|^2 2 \sum_{|j| \geq \inf\limits_{n \neq m} \{|(\lambda_n - \lambda_m)|\}} |\hat{h}(j)|^2 \right)^{\frac{1}{2}} \leq \|P\|^2 \delta,$$

if we choose j_0 so large that the 2-norm of the tail of the series of h is less then $\delta/2$. It follows from the explicit form of $\hat{h}(j)$ that this can be done independent of Γ. Combining this with (1.5.9) and (1.5.10) we obtain

$$(\frac{1}{2\pi}|\Gamma| - \delta)\|P\|^2 \leq \int_\Gamma |f|^2 dt \leq (\frac{1}{2\pi}|\Gamma| + \delta)\|P\|^2.$$

□

1.6. Sidon sets

DEFINITION 1.6.1. Let $E \subset \mathbb{Z}$. A function f (or measure, or distribution) on T is called *E-spectral* if $\hat{f}(n) = 0$ if $n \notin E$. Denote by C_E, L^p_E, M_E the respective subspaces of $C(T)$, $L^p(T)$, $M(T)$ consisting of E-spectral elements. These are closed subspaces. A subset E of $\mathbb{Z}$ is called a *Sidon set* if $f \in C_E$ implies $\hat{f} \in l^1(E)$.

EXAMPLE 1.6.2. Of course every finite set is a Sidon set. By Theorem 1.5.2 every lacunary set is a Sidon set.

THEOREM 1.6.3. *The following are equivalent:*

1. *E is a Sidon set.*
2. *There exists $K > 0$ such that $\|\hat{f}\|_1 \leq K\|f\|_\infty$ for all E-spectral trigonometric polynomials f.*
3. *$\|\hat{f}\|_1$ is bounded for every $f \in L^\infty_E$.*

3.a *There exists a K such that $\|\hat{f}\|_1 \leq K\|f\|_\infty$ for every $f \in L^\infty_E$.*

4. *$\widehat{M_E} = l^\infty(E)$.*
5. *$\widehat{L^1}_E = c_0(E)$.*

PROOF. ($1 \implies 2$) If E is Sidon, then the map $f \mapsto \hat{f}$ is linear bijective from C_E to $l^1(E)$. Also its inverse (as a linear map) is continuous. Indeed

$$\|f\|_\infty = \sup_t \left| \sum \hat{f}(n) e^{int} \right| \leq \sum |\hat{f}(n)|.$$

Hence, by the Open Mapping Theorem, $\hat{f} \mapsto f$ is open and thus $f \mapsto \hat{f}$ is continuous, which means that for some $K > 0$ one has $\|\hat{f}\|_1 \leq K\|f\|_\infty$. In particular, this is true for E-spectral trigonometric polynomials.

($2 \implies 3(a)$) If $f \in L^\infty_E$, then $\sigma_N(f)$ is an E-spectral trigonometric polynomial with $|\sigma_N(f)|_\infty \leq \|f\|_\infty$, independent of N. Thus there is a constant K such that $\|\hat{\sigma}_N(f)\|_1 \leq K\|f\|_\infty$. As in the proof of Theorem 1.5.2 we conclude that $\|\hat{f}\|_1 \leq K\|f\|_\infty$.

($3 \implies 1$) is trivial.

Item 4 and 5 both say that a certain bounded linear transformation is surjective. Now recall that the Open Mapping Theorem implies that a surjective bounded linear transformation F between Banach spaces X and Y is *open*. Thus $F(\{\|x\| < 1\})$ contains an open neighborhood of $0 \in Y$. By linearity we obtain that there is a constant $C > 0$ such that for every $y \in Y$ there exist $x \in X$ with $Fx = y$ and $\|x\| < C\|y\|$. This will be used in the last two steps of the proof.

($3(a) \implies 4$) Let $(d_j)_j \in l^\infty(E)$. Then $f \mapsto \sum_j \hat{f}(j)d_j$ is a continuous linear functional on C_E which by Hahn-Banach can be extended to $C(T)$. By the Riesz Representation Theorem there exists a complex regular Borel measure μ which represents this functional. Thus for $f \in C_E$ we have $\int f\, d\mu = \sum_j \hat{f}(j)d_j$. We choose $f(t) = e^{-i\lambda_j t}$, $\lambda \in E$ and find $\hat{\mu}(\lambda_j) = d_j$. Thus $\hat{\bar{\mu}}(\lambda_j) = \bar{d}_j$. Replacing d_j by $\bar{d}_j$ we obtain our result.

($4 \implies 5$) First observe that modification of the Fejér kernels yields that if Λ is a finite set of integers and $\varepsilon > 0$, then there exists a trigonometric polynomial $P = P_{\Lambda,\varepsilon}$ such that $\hat{P}(j) = 1$ and $\|P\|_1 \leq 1 + \varepsilon$. We will use this with $\varepsilon = 1$. Now let $(d_j)_j \in c_0(E)$. We may assume that $|d_j| \leq 1$. Put

$$E_k = \{n :\ 2^{-k} < |d_n| \leq 2^{-k+1}\}.$$

By 4. there exist measures μ_k such that $\hat{\mu}_k(j) = d_j$ if $j \in E_k$, while $\hat{\mu}_k(j) = 0$ if $j \in E \setminus E_k$; moreover $\|\mu_k\| \leq C2^{-k}$. Let T_k be trigonometric polynomials with $\hat{T}(j) = 1$ on E_k and $\|T_k\|_1 \leq 2$. Then $T_k * \mu_k$ is a trigonometric polynomial of L^1 norm less than $2C2^{-k}$ and with Fourier coefficients $\hat{T}_k(j)\hat{\mu}_k(j) = \hat{\mu}_k(j)$ on E. The conclusion is that

$$f(t) = \sum_{k=1}^{\infty} T_k * \mu_k$$

is in $L^1(T)$ because the series converges in $L^1(T)$ and has $f(j) = d_j$ on E.

($5 \implies 2$) Let g be an E-spectral trigonometric polynomial. Define $d_n = |\hat{g}(n)|/\bar{\hat{g}}(n)$ if $\hat{g}(n) \neq 0$ and $d_n = 0$ elsewhere. $(d_n)_n \in c_0(E)$ of norm 1, hence there exists $f \in L^1(T)$ with $\hat{f}\mid_E = d_n$ and $\|f\|_1 \leq C$, where C is a constant only depending on E. Then

$$\sum |\hat{g}(n)| = \int_{-\pi}^{\pi} f(t)\bar{g}(t)\, dt \leq \|g\|_\infty \|f\|_1 \leq C\|g\|_\infty.$$

□

REMARKS 1.6.4. The smallest constant K in Theorem 1.6.3, such that 3a holds, is called the *Sidon constant* of E.

In retrospect, we can understand the (limits of the) Riesz products in Lemma (1.5.1) as explicitly constructed measures of Theorem (1.6.3), statement 4 and 5.

COROLLARY 1.6.5 (to Theorem 1.5.2). *Suppose that $E = \{\lambda_j\}$ is lacunary and that $L = (d_j) \in l^2(E)$. Then there exists a bounded function f on T such that $\hat{f}(\lambda_j) = d_j$.*

PROOF. In view of Riesz Representation Theorem, L is in $l^2(E)^* = (L^2_E)^*$. In fact, $L\varphi = \sum \hat{\varphi}(\lambda_j)\bar{d}_j$. By Theorem 1.5.2 L is also a continuous linear functional on L^1_E. Now L^1_E is a closed subspace of $L_1(T)$. By Hahn Banach we can extend L to all of $L^1(T)$. We denote the extension again by L. Recalling, c.f. [**22**][Ch 7], that the dual space of $L^1(T)$

is $L^\infty(T)$, we find that there exists a function $f \in L^\infty$ such that $L\varphi = \int_0^{2\pi} \varphi \bar{f} \frac{dt}{2\pi}$, for all $\varphi \in L^1(T)$. We now apply this with $\varphi(t) = e^{i\lambda_j t}$ and obtain $\hat{f}(\lambda_j) = d_j$. $\square$

REMARK 1.6.6. One can do the same for general Sidon sets by proving the analogue of Theorem 1.5.2. This requires considerably more effort, cf. [**4**].

1.7. Exercises

1.7.1. Let $(b_n)_{n\in\mathbb{Z}}$ be a sequence of positive numbers such that $b_n \to 0$ as $|n| \to \infty$. Show that there exists a sequence (a_n) with $a_n \geq b_n$ which satisfies the convexity condition of Theorem 3.1.

1.7.2. Suppose that $f \in C_{2\pi}$ is *Hölder continuous* of order α, $0 < \alpha \leq 1$, i.e. there exists $B > 0$ such that for all x, $|f(x+t) - f(x)| \leq B|t|^\alpha$. Show that $|\hat{f}(n)| \leq Cn^{-\alpha}$. Hint: Write the Fourier coefficients as

$$\hat{f} = \frac{1}{2}\left(\int_0^{2\pi} f(s)e^{-ins}\frac{ds}{2\pi} - \int_0^{2\pi} f(s-\pi/n)e^{-ins}\frac{ds}{2\pi}\right).$$

1.7.3. Let (λ_n) be lacunary. Suppose that $f \in L^1(T)$ has Fourier series $\sum a_n \cos(\lambda_n t)$. Suppose that f is Hölder continuous at one point $t_0 \in T$. Show that $a_n = O(\lambda_n^{-\alpha})$. Next show that f is Hölder continuous on T.

1.7.4. With the assumptions as in exercise 1.7.3, show that if f equals 0 on a small interval in T, then $f \in C^\infty(T)$. Can you relax the condition that f be 0 on an interval and still reach the same conclusion?

1.7.5. With the same assumptions as in exercise 1.7.3, $f(t) = \sum c_n e^{i\lambda_n t}$, show that if f is (real) analytic on a subarc Γ of T, then f is real analytic on T by completing the following outline.

(1) Observe that f is C^∞ because of exercise 1.7.4 extended.
(2) Choose a suitable interval Γ, a suitable modification $\tilde{f}$ of f and show that there is a $C > 0$ such that for every k, l

$$|c_k|^2|\lambda_k|^{2l} \leq C\int_\Gamma |\tilde{f}^{(l)}(t)|^2 dt.$$

(3) Use the Cauchy estimates to make the integrals in (2) bounded by a constant times $(l!\delta^{-l})^2$.
(4) Make a favourable choice of l to have $\limsup_{k\to\infty} |c_k|^{\frac{1}{\lambda_k}} < 1$.
(5) Finish it off!

1.7.6. Prove *Hadamard's Theorem*: If $f(z) = \sum_{n=0}^\infty c_n z^{\lambda_n}$ is a lacunary power series with radius of convergence R then $f(z)$ has no analytic continuation to a domain larger than the disc with radius R. ($C(0,R)$ is a natural boundary.)

1.7.7. Prove Theorem 1.5.3 for general q by showing that now there exists $K > 0$ such that that every $j \in \mathbb{Z} \setminus \{0\}$ is assumed at most K times as a value of $\lambda_n - \lambda_m$.

1.7.8. Under the assumptions of Corollary 1.6.5, show that f can in fact be chosen continuous, by completing the following steps.

(1) Find sequences (a_n), (b_n) such that $d_n = a_n b_n$, with $a_n \in l^2(E)$ and $b_n \in c_0(E)$.
(2) Show that the convolution of a bounded and an integrable function is continuous.
(3) Prove the assertion.

1.7.9. Suppose that $f \in C(T)$ has positive Fourier coefficients. Prove that $\hat{f} \in l^1$.

1.7.10. Suppose that for every $d = (d_n)_n \in l^\infty(E)$ with $|d_n| \leq 1$, there exists a measure $\mu \in M(T)$ with

$$|d_n - \hat{\mu}(n)| \leq 1 - \delta.$$

Prove that E is Sidon. (Think of the previous exercise and let $d_n = \frac{|\hat{f}(n)|}{\hat{f}(n)}$.)

1.8. Final remarks, notes, and references

The classical book on trigonometric series is [**23**]. Section 1. is in [**13**], but also can be found e.g. in [**10**]. A well written elaborate introduction to Fourier analysis with many applications in other subjects is [**12**].

Material concerning section 3 can be found in any reasonable book on Functional Analysis, e.g. [**17**], see [**22**] for a more extensive list of references.

Section 2. has been taken from [**10**], but is similarly treated in [**23**].

Section 4,5 and 6 are also mostly taken from [**10**]; [**4**] has a more comprehensive treatment and was also used. Many things can be found in [**23**] too.

CHAPTER 2

Distributions and their Fourier Series

2.1. Introduction

Consider the wave equation on $\mathbb{R}^2$

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0. \tag{2.1.1}$$

Introducing new coordinates $x' = x + t$, $y' = x - t$, we obtain the equation

$$\frac{\partial^2 u}{\partial x' \partial y'} = 0, \tag{2.1.2}$$

which has the classical solution

$$u(x', y') = f(x') + g(y'), \quad f, g \in C^2(\mathbb{R}).$$

Thus (2.1.1) has as classical solutions $f(x+t)+g(x-t)$. Classically, C^2 is required because 2 differentiations are performed on u. Physically, however, there is no reason to ask much more than continuity. Also, from (2.1.2) we see that $\frac{\partial u}{\partial y'} = \tilde{g}(y')$, where the only thing that matters is that this function doesn't depend on x'. If $\tilde{g} \in L^1(\mathbb{R})$ we get a solution $u(x', y') = f(x') + g(y')$ with no stronger conditions on f and g than continuity. There are of course problems with changing the variables and we have a solution which is not symmetric in x and y. The point is that it is at least inconvenient not to be able to differentiate continuous functions.

As far as Fourier analysis is concerned, we know that $f \in C^1_{2\pi}$ has the property that $\hat{f'}(n) = in\hat{f}(n)$. We can formally write down this sequence of Fourier coefficients also if f is no longer differentiable. Can we give meaning to it as the Fourier series of something interesting? Moreover, consider the Fourier transform $\mathcal{F}f$ of $f \in L^1(\mathbb{R})$. If f is C^1 we know that

$$\mathcal{F}f'(\xi) = i\xi \mathcal{F}f(\xi). \tag{2.1.3}$$

Multiplication with ξ is a well defined operation on functions, the righthand side of (2.1.3) is always well defined. A meaningful lefthand side, that is, unlimited differentiability of L^1 functions, is desirable.

One way out is the concept of weak solution. Notice that if $u \in C^2$ solves (2.1.1), then for every compactly supported $\varphi \in C^\infty(\mathbb{R}^2)$ we have

$$\iint \left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2}\right) \varphi(x,t)\, dxdt = \iint u(x,t) \left(\frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial^2 \varphi}{\partial t^2}\right) dxdt = 0. \tag{2.1.4}$$

If the last equality in (2.1.4) holds for all compactly supported $\varphi \in C^\infty(\mathbb{R}^2)$ and $u \in C^2$, then u satisfies (2.1.1). However the last integral in (2.1.4) makes sense for locally integrable u. Thus one calls a locally integrable (sometimes only a continuous) u a weak solution of (2.1.1) if u satisfies

$$\iint \left(\frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial^2 \varphi}{\partial t^2}\right) u(x,t)\, dxdt = 0.$$

We will use this idea to differentiate locally integrable functions arbitrarely often. In fact we will go one step further. Observing that

$$u \mapsto \iint u\varphi \, dxdt$$

is a linear functional on compactly supported C^∞ functions, we will explain how to "differentiate" a large class of such linear functionals, which is determined by a continuity condition, and identify the ones originating from locally integrable functions with a subclass.

2.2. Smooth functions on T

Recall, see [**17**][Section 1.33], that a *seminorm* on a (complex) vector space X is a real valued function p such that

$$\begin{aligned} & p(x) \geq 0, \\ & p(x_1 + x_2) \leq p(x_1) + p(x_2), \\ & p(\lambda x) = |\lambda| p(x). \end{aligned} \tag{2.2.1}$$

Here $x, x_i \in X$, $\lambda \in \mathbb{C}$.

A family $\mathcal{P}$ of seminorms is called *separating* if for every $x \in X$ there is a $p \in \mathcal{P}$ such that $p(x) \neq 0$. On $C^\infty(T) = C^\infty_{2\pi}$ we a separting family of seminorms is given by

$$P_j(\varphi) = \|\varphi^{(j)}\|_\infty, \tag{2.2.2}$$

the maximum of the j-th derivative. These seminorms can be used to define a topology on $C^\infty(T)$ by requiring that they are *continuous*, just as weak topologies were introduced in Chapter 1. Section 1.3 Thus, a local subbasis at 0 consists of sets

$$V_{j,\varepsilon} = \{\varphi : P_j(\varphi) < \varepsilon\}, \quad j \in \mathbb{N}, \ \varepsilon > 0.$$

The space $C^\infty(T)$ endowed with this topology is called $\mathcal{D}(T)$. From Chapter 1, Section 1.3 we see that

$$f_j \in \mathcal{D}(T) \to f \in \mathcal{D}(T) \iff \forall k \in \mathbb{N} \ f_j^{(k)} \to f^{(k)} \text{ uniformly.} \tag{2.2.3}$$

So it is pretty hard for functions to converge in $\mathcal{D}(T)$, cf. Exercise 2.7.1. However, Fourier series behave nice in $\mathcal{D}(T)$.

LEMMA 2.2.1. *Let $\varphi \in \mathcal{D}(T)$. Then $S_N[\varphi] \to \varphi$ in $\mathcal{D}(T)$ as $N \to \infty$.*

PROOF. Because φ is smooth, $S_N[\varphi] \to \varphi$ uniformly, but also $S_N[\varphi]' = S_N[\varphi'] \to \varphi'$ uniformly. The same is true for higher derivatives. By (2.2.3) we are done. □

LEMMA 2.2.2. *The space $\mathcal{D}(T)$ is metrizable and complete in the metric*

$$d(\varphi, \psi) = \sum_{j=0}^{\infty} 2^{-j} \frac{P_j(\varphi - \psi)}{1 + P_j(\varphi - \psi)}. \tag{2.2.4}$$

PROOF. See Exercise 2.7.2. □

Just as in (2.2.4) any countable separating family of seminorms on a vector space X gives rise to a metric d. Observe that it is *translation invariant*, that is $d(x, y) = d(x - z, y - z)$ for all $x, y, z \in X$. Moreover, one can show that with the induced topology X becomes a topological vector space, i.e addition and scalar multiplication are continuous operations, and it is locally convex, meaning that it has a local basis of convex sets. In particular this holds for T.

However, $\mathcal{D}(T)$ with the present topology cannot be turned into a Banach space because of the following Lemma, which says that $\mathcal{D}(T)$ has the *Heine-Borel property*. A Banach space can only possess this property if it is finite dimensional. Recall that a set X in a topological vector space is *bounded* if it has the property that for every open neighborhood U of 0, there exists $N > 0$ such that $X \subset NU$. If the topology is determined by seminorms, this just means that every seminorm is bounded on X.

LEMMA 2.2.3. *Every bounded sequence in $\mathcal{D}(T)$ has a convergent subsequence; A closed bounded set in $\mathcal{D}(T)$ is compact.*

PROOF. This is a simple consequence of the Arzela-Ascoli Theorem, cf. Exercise 2.7.3. $\square$

It will be convenient to replace P_n by an equivalent set of seminorms

$$\tilde{P}_n = \sum_{j=0}^{n} P_j.$$

This doesn't change the topology, but it has the advantage that a local basis at 0 of $\mathcal{D}(T)$ is now given by $\{\tilde{P}_n(\varphi) < 1/n\}$, $n \in \mathbb{N}$.

REMARK 2.2.4. A vector space with topology induced by a complete invariant metric, like we have met here, is called a *Fréchet space.* (Sometimes local convexity is required, but that can be shown to hold here too.)

2.3. Distributions on T

Let $\mathcal{D}' = \mathcal{D}'(T)$ denote the dual space of $\mathcal{D}(T)$, that is the space of continuous linear functionals on $\mathcal{D}(T)$. The notation is classical. Elements of $\mathcal{D}'$ are called (periodic) *distributions* or *generalized functions*. The space $\mathcal{D}'$ will naturally be equiped with the *weak-* topology.* We denote the action of $L \in \mathcal{D}'(T)$ on $\varphi \in \mathcal{D}(T)$ usually by

$$\langle L, \varphi \rangle \ (= L\varphi).$$

The following lemma is an extension of a familiar result for Banach spaces, cf. [**22**][thm. 3.2].

LEMMA 2.3.1. *Let X be a vector space with a topology induced by a countable set of seminorms $\{p_1, p_2, \ldots\}$ and let L be a linear functional on X. The following are equivalent.*

i. *L is continuous on X,*
ii. *L is continuous at $x_0 \in X$,*
iii. *There exist $C > 0$, $K \in \mathbb{N}$ such that*

$$|Lx| \leq C \max_{i=1,\ldots,K} p_i(x).$$

PROOF. (i $\Longrightarrow$ ii) is trivial.

(ii $\Longrightarrow$ iii). If L is continuous at x_0, then for every $\varepsilon > 0$ there exist $K \in \mathbb{N}$ and $\delta_j > 0$, $(j = 1, \ldots, K)$, such that $p_j(x - x_0) < \delta_j$, $(j = 1, \ldots, K)$ implies $|L(x - x_0)| < \varepsilon$. Let $\delta = \min_j\{\delta_j\}$ and denote for $y \in X$, $M_y = \max_{i=1,\ldots,K} p_i(y)$. Then

$$|Ly| = |\frac{M_y}{\delta} L(\frac{\delta y}{M_y})| < \frac{M_y}{\delta}\varepsilon,$$

which proves iii, with $C = \varepsilon/\delta$.

(iii $\Longrightarrow$ i). Let $x, y \in X$.

$$|Lx - Ly| = |L(x - y)| < C \max_{i=1,\ldots,K} p_i(x - y).$$

This is less than ε if $p_i(x-y) < \varepsilon/KC$. Thus we described a small neighborhood of x which is mapped in an ε-neighborhood of Lx. $\square$

Apparently a linear functional L on $\mathcal{D}(T)$ is continuous, i.e. a *distribution*, if and only if there exist $n \in \mathbb{N}$ and $C > 0$ such that

$$|\langle L, \varphi \rangle| \leq C\tilde{P}_n(\varphi), \quad \forall \varphi \in \mathcal{D}(T). \tag{2.3.1}$$

The smallest n that is possible in (2.3.1) is called the *order* of the distribution. Of course the zero distribution has order $-\infty$ assigned to it.

As we have seen in Section 1.3.3 the natural notion of convergence for a sequence of distributions is weak-* convergence. To be specific, for $L_j, L \in \mathcal{D}'(T)$ we have

$$\lim_{j\to\infty} L_j = L \text{ if and only if } \forall \varphi \in \mathcal{D}(T) \quad \lim_{j\to\infty} L_j\varphi = L\varphi.$$

Remark 2.3.2. If the underlying space is not compact, e.g. $\mathbb{R}$ instead of T, distributions may have infinite order.

2.3.1. Examples of distributions.

(1) Every $u \in L^1_{2\pi}$ defines a distribution L_u via

$$\langle L_u, \varphi\rangle = \int_{-\pi}^{\pi} u\varphi \frac{dt}{2\pi}.$$

This functional is indeed continuous: $|\langle L_u, \varphi\rangle| \le \|u\|_1\|\varphi\|_\infty$.

(2) We denote the set of Borel measures on T by $M(T)$. Every measure $\mu \in M(T)$ defines in the same way a distribution L_μ via

$$\langle L_\mu, \varphi\rangle = \int_{-\pi}^{\pi} \varphi \, d\mu(t).$$

Both examples are distributions of order 0.

Abusing the language we will drop the L and identify a function or measure with the associated distribution, writing e.g. $\langle u, \varphi\rangle$.

(3) The delta-distribution $\delta = \delta_0$ is defined by

$$\langle \delta, \varphi\rangle = \varphi(0).$$

The delta distribution originates from point mass at 0.

Remark 2.3.3. The name test functions for the elements of $\mathcal{D}$ is now understandable, these functions are used to test the action of an L^1 function or distribution.

2.4. Operations on Distributions

2.4.1. Differentiation. Let $L \in \mathcal{D}'(T)$. Define its derivative $L' \in \mathcal{D}'(T)$ by

$$\langle L', \varphi\rangle = -\langle L, \varphi'\rangle.$$

Observe that L' is well-defined because $\varphi' \in \mathcal{D}(T)$; L' is also continuous. Indeed, there exist $C > 0$, $n \in \mathbb{N}$ such that $|\langle L, \varphi\rangle| \le C\tilde{P}_n(\varphi)$. Then

$$|\langle L', \varphi\rangle| = |\langle L, \varphi'\rangle| \le C\tilde{P}_n(\varphi') \le C\tilde{P}_{n+1}(\varphi).$$

We conclude that every distribution is infinitely often differentiable. In general, differentiation increases the order of a distribution by 1.

Examples 2.4.1. Let $f(x)$ be the characteristic function of $(0, \pi)$ viewed as element of $L^1_{2\pi}$. We compute its distributional derivative f'. Let $\varphi \in \mathcal{D}(T)$, then

$$\langle f', \varphi\rangle = -\langle f, \varphi'\rangle = -\int_{-\pi}^{\pi} f(t)\varphi'(t)\frac{dt}{2\pi} = -\int_0^{\pi} \varphi'(t)\frac{dt}{2\pi} = \frac{\varphi(0) - \varphi(\pi)}{2\pi}.$$

We conclude that $f' = \frac{\delta_0 - \delta_\pi}{2\pi}$.

The function $f(x) = \log|x|$ is in $L^1_{2\pi}$. Its distributional derivative is determined by

$$\langle f', \varphi\rangle = -\langle f, \varphi'\rangle = -\int_{-\pi}^{\pi} \log|t|\varphi'(t)\frac{dt}{2\pi} = -\lim_{\varepsilon\to 0}\int_{-\pi}^{-\varepsilon} + \int_{\varepsilon}^{\pi} \log|t|\varphi'(t)\frac{dt}{2\pi}$$

$$= \frac{-1}{2\pi}\lim_{\varepsilon\to 0}\Bigg(\log\varepsilon\varphi(-\varepsilon) - \log\pi\varphi(-\pi) - \log\varepsilon\varphi(\varepsilon) + \log\pi\varphi(\pi) - \tag{2.4.1}$$

$$\left(\int_{-\pi}^{-\varepsilon} + \int_{\varepsilon}^{\pi} \frac{\varphi(t)}{t}dt\right)\Bigg) = \text{p(rincipal) v(alue)} \int \frac{\varphi(t)}{t}\frac{dt}{2\pi}.$$

Here we have used that φ is periodic, making the stock terms cancel in the limit. The last equality is the definition of the *principal value integral*. We have shown that $f' = \text{p.v.}\frac{1}{x}$.

PROPOSITION 2.4.2. *Differentiation is a continuous operation: If $L_j \to L$ in $\mathcal{D}'(T)$ then $L_j' \to L'$ in $\mathcal{D}'(T)$.*

PROOF. For any $\varphi \in \mathcal{D}(T)$, we have $\langle L_j, \varphi\rangle \to \langle L, \varphi\rangle$ by definition of weak-* convergence. Hence $\langle L_j, \varphi'\rangle \to \langle L, \varphi'\rangle$, which by definition of differentiation gives $\langle L_j', \varphi\rangle \to \langle L', \varphi\rangle$. □

2.4.2. Restricted Multiplication. The price one pays for infinite differentiability is that multiplication of distributions is in general not possible. However, if $f \in C^\infty(T)$, $L \in \mathcal{D}'$ then fL can defined by

$$\langle fL, \varphi\rangle = \langle L, f\varphi\rangle.$$

This is indeed continuous: if $\varphi_n \to \varphi$ in $\mathcal{D}(T)$ then $f\varphi_n \to f\varphi$ in $\mathcal{D}(T)$, therefore

$$\langle fL, \varphi_n\rangle = \langle L, f\varphi_n\rangle \to \langle L, f\varphi\rangle = \langle fL, \varphi\rangle.$$

If $\mu \in M(T)$ and $f \in C(T)$, then $f\mu \in M(T)$. Hence multiplication of a continuous function with a distribution associated with a measure or an L^1 function is also possible. More generally one can prove that a distribution of order k can be multiplied with a function in C^k, cf. ex. 2.7.5.

LEMMA 2.4.3 (Product rule). *If $f \in C^\infty(T)$ and $L \in \mathcal{D}'$, then*

$$(fL)' = f'L + fL'.$$

PROOF. Let $\varphi \in \mathcal{D}$. Then

$$\begin{aligned} \langle (fL)', \varphi\rangle &= -\langle fL, \varphi'\rangle = -\langle L, f\varphi'\rangle \\ &= -\langle L, (f\varphi)' - f'\varphi)\rangle = \langle L', f\varphi\rangle + \langle f'L, \varphi\rangle. \end{aligned} \tag{2.4.2}$$

□

2.4.3. Local Equality. The fact that $L^1_{2\pi}$ is identified with a subset of $\mathcal{D}'$ makes it clear that the "value of a distribution in a point" makes no sense. However, on open sets equality makes sense! Recall that the *support* of a continuous function φ is the closed set $\operatorname{Supp}\varphi = \operatorname{cl}\{t : \varphi(t) \neq 0\}$.

DEFINITION 2.4.4. Two distributions L_1, L_2 on T are called *equal on an open subset* $\Gamma \subset T$ if for every $\varphi \in \mathcal{D}(T)$ with support in Γ, one has $\langle L_1, \varphi\rangle = \langle L_2, \varphi\rangle$.

The *support* of a distribution $L \in \mathcal{D}'$ is the complement of the union of the open sets Γ with $L = 0$ on Γ.

EXAMPLES 2.4.5. The support of δ is $\{0\}$. If L originates from a continuous function then the two notions of support coincide.

REMARK 2.4.6. Multiplication can be localized: If $L \in \mathcal{D}'$ has order k, and on an open $\Gamma \subset T$ is equal to a distribution of order $j < k$ then L can be multiplied with C^j functions, that are C^k in a neighborhood of the complement of Γ. See Exercise 2.7.11. Moreover it turns out that (especially in higher dimensions) a further refinement is possible. This is based on local Fourier analysis of the distribution and takes into account the directions in which the singularities occur. We deal with this topic in a later chapter, but to get a flavour of the problem, consider the distribution L on $\mathbb{R}^2$ given by

$$\langle L, \varphi\rangle = \frac{\partial\varphi}{\partial x_1}(0,0).$$

This has order 1, so we can multiply with $f \in C^1$ but a closer analysis gives that we only need to require something like f continuous and differentiable with respect to x_1.

2.5. Fourier Series of Periodic Distributions

In analogy with $L^1_{2\pi}$ we can define for $L \in \mathcal{D}(T)$ the *Fourier coefficients*

$$\hat{L}(n) = \frac{1}{2\pi}\langle L, e^{-int}\rangle.$$

For example, as we know already $\hat{\delta}(n) = 1/2\pi$.

As one expects, we put

$$S_N[L](t) = \sum_{-N}^{N} \hat{L}(n)e^{int}.$$

For example, $S_N[\delta] = D_N$, the Dirichlet kernel. Let $\varphi \in \mathcal{D}(T)$

$$\langle S_N[\delta], \varphi\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{n=-N}^{N} e^{int}\varphi(t)\,dt = D_N * \varphi(0) \to \varphi(0), \quad \text{as } N \to \infty.$$

We see that the Dirichlet kernels tend weakly to δ.

LEMMA 2.5.1. *If $L \in \mathcal{D}'$ and $\varphi \in \mathcal{D}(T)$ then*

$$\langle L, \varphi\rangle = 2\pi \sum_{-\infty}^{\infty} \hat{L}(-n)\hat{\varphi}(n).$$

PROOF. Using Lemma 2.2.1 we see that

$$\langle L, \varphi\rangle = \lim_{N\to\infty}\langle L, S_N[\varphi]\rangle = \lim_{N\to\infty} 2\pi \sum_{-N}^{N} \hat{\varphi}(n)\hat{L}(-n).$$

This proves the Lemma. □

THEOREM 2.5.2. *If $L \in \mathcal{D}'$, then the Fourier series of L tends weak-* to L.*

PROOF. For every $\varphi \in \mathcal{D}(T)$ we find by Lemma 2.5.1, if $N \to \infty$,

$$\langle S_N[L], \varphi\rangle = 2\pi \sum_{-N}^{N} \hat{L}(n)\hat{\varphi}(-n) \to \langle L, \varphi\rangle.$$

□

COROLLARY 2.5.3. *The map $L \to \hat{L}$ is injective on $\mathcal{D}'$.*

PROOF. If $\hat{L}(n) = 0$ for all $n \in \mathbb{Z}$, then $L = 0$ by Theorem 2.5.2. □

COROLLARY 2.5.4. *Let $L \in \mathcal{D}'$. If $L' = 0$ on T, then L is a constant.*

PROOF. The Fourier coefficients of L' are $0 = \widehat{L'}(n) = in\hat{L}(n)$. It follows that the Fourier series of L consists only of the constant term. Theorem 2.5.2 gives that L is a constant. □

THEOREM 2.5.5. *The following are equivalent.*

i. $\sum_{n=-\infty}^{\infty} c_n e^{int}$ *is the Fourier series of a periodic distribution.*
ii. *There exist constants N, C such that $|c_n| \le Cn^N$.*

PROOF. If $\sum_{n=-\infty}^{\infty} c_n e^{int}$ is the Fourier series of a periodic distribution L then $\sum_{n=-N}^{N} c_n e^{int}$ tends weakly to L by Theorem 2.5.2. It follows that for every k, if $N > k$

$$\langle\sum_{-N}^{N} c_n e^{int} - L, e^{-ikt}\rangle = c_k - \hat{L}(k)$$

tends to 0 if $N \to \infty$. Hence $c_k = \hat{L}(k)$. Now L has finite order, say N. Then $|c_n| = |\langle L, e^{-int}\rangle| \le C\tilde{P}_N(e^{-int}) \le CNn^N$.

In the other direction, if $|c_n| \le Cn^N$, then $(c_n/n^{N+1})_n \in l^2(\mathbb{Z})$. Parseval gives

$$f(t) = \sum_{-\infty}^{\infty} \frac{c_n}{n^{N+1}} e^{int} \in L^2(T).$$

It follows that $f^{(N+1)}$ is a distribution of at most order $N+1$. It has Fourier coefficients $i^{N+1}c_n$ for $n \neq 0$. Dividing by i^{N+1} and adding c_0 we have found a distribution with the prescribed Fourier series. □

COROLLARY 2.5.6. *Every periodic distribution is of the form $F^{(N)}+c$ with F continuous (or in L^2).*

PROOF. This is in the proof of the second part of Theorem 2.5.2 for F in L^2. If F is not continuous we consider

$$F(t) = \sum_{-\infty}^{\infty} \frac{c_n}{n^{N+1}} e^{int} \in C(T),$$

because the series is uniformly convergent and proceed as in the last part of the proof of the Theorem. □

2.6. Convolution and Multiplication

From [**13**] we know that if $f, g \in C_{2\pi}$, then $f * g := \int_{-\pi}^{\pi} f(x-y)g(y)\frac{dy}{2\pi}$ and fg are also in $C_{2\pi}$ and have Fourier series given by

$$(f*g)\hat{}(n) = \iint f(x-y)g(y)e^{-in(x-y)}e^{-iny}\frac{dy}{2\pi}\frac{dx}{2\pi} = \hat{f}(n)\hat{g}(n) \tag{2.6.1}$$

and

$$(fg)\hat{}(n) = \sum_{-\infty}^{\infty} \hat{f}(n-j)\hat{g}(j) =: \hat{f} * \hat{g}(n). \tag{2.6.2}$$

2.6.1. We can use (2.6.1) to define the convolution $L_1 * L_2$ for periodic distributions $L_i \in \mathcal{D}$:

$$L_1 * L_2 \stackrel{\text{def}}{=} \sum_{-\infty}^{\infty} \hat{L}_1(n)\hat{L}_2(n)e^{inx}. \tag{2.6.3}$$

Application of Theorem 2.5.5 gives that there exist $C, k > 0$ such that $|\hat{L}_1(n)\hat{L}_2(n)| < C|n|^k$, $(n \neq 0)$, and another application of Theorem 2.5.5 shows that the series represents a distribution.

Formula (2.6.2) shows once more why it is difficult to multiply distributions: The Fourier coefficients $(fg)\hat{}(n)$ have to be finite. Suppose f is a distribution of order k. Then $\hat{f}(n)$ behaves like n^k. For (2.6.2) to converge, the $\hat{g}(n)$ have to be something like n^{-k}, i.e. g is fairly smooth.

2.7. Exercises

2.7.1. Prove that, as $n \to \infty$, the sequence $\sin(nt)/n^3$ converges to 0 uniformly, but it does not converge in $\mathcal{D}(T)$.

2.7.2. Prove that (2.2.4) defines a metric and complete the proof of Lemma 2.2.2.

2.7.3. Recall the Arzela-Ascoli theorem: If $\{f_\alpha\}_\alpha$ is an equicontinuous family of pointwise bounded continuous functions on a separable compact metric space, then $\{f_\alpha\}_\alpha$ has a uniformly convergent subsequence. Equicontinuous means

$$\forall \varepsilon\ \exists \delta \text{ such that } |x-y| < \delta \implies |f_\alpha(x) - f_\alpha(y)| < \varepsilon,$$

independent of α.

Prove the following. If $\{f_\alpha\}_\alpha$ is bounded in the C^1-norm on T, that is, the norm $\|\cdot\|_\infty + \|\cdot\,'\|_\infty$ on T, then $\{f_\alpha\}_\alpha$ is equicontinuous and pointwise bounded. (Apply the mean value theorem.)

2.7.4. Prove the continuity in example 2.3.1 by showing that $\varphi_n \to \varphi$ in $\mathcal{D}(T)$ implies $\langle L_u, \varphi_n\rangle \to \langle L_u, \varphi\rangle$.

2.7.5. Let $L \in \mathcal{D}'(T)$ be of order k and let $f \in C^l(T)$. For which k and l can you give a definition of fL by using (2.6.1) ?

2.7.6. Compute the distribution sum of

$$\sum_{-\infty}^{\infty} n e^{inx}, \qquad \sum_{n=1}^{\infty} \sin nx. \tag{2.7.1}$$

2.7.7. Compute the distributional derivative of

$$f(x) = \begin{cases} x+\pi & \text{if } -\pi < x < 0, \\ x-\pi & \text{if } 0 < x < \pi. \end{cases} \tag{2.7.2}$$

2.7.8. Let $\varphi \in \mathcal{D}(T)$, $L \in \mathcal{D}'(T)$. Put $\varphi^x(y) = \varphi(x-y)$. Show that

$$L * \varphi(x) \stackrel{\text{def}}{=} \langle L, \varphi^x\rangle$$

is well-defined and coincides with definition (2.6.3). Prove that $L * \varphi$ is a smooth function.

2.7.9. Let $L \in \mathcal{D}(T)'$ have support $\{0\}$.

(1) Use Corollary 2.5.6 to express L as the j'th derivative of a continuous function f on $(-\pi, \pi)$.
(2) Show that there are polynomials P_1, P_2 of degree at most $j-1$ such that $f = P_1$ if $x > 0$ and $f(x) = P_2(x)$ if $x < 0$.
(3) Conclude that we may take $f = PU$ with P a polynomial of degree $j-1$ and U the characteristic function of $(0, \pi)$.
(4) Prove that L is a finite linear combination of derivatives of δ.

2.7.10. For $L \in \mathcal{D}'(T)$, let L_h be defined by $L_h\varphi := L\varphi(.-h)$. Compute the distributional limit

$$\lim_{h\to 0} \frac{L_h - L}{h}.$$

2.7.11. Suppose that $L \in \mathcal{D}'$ has order k, and on an open $\Gamma \subset T$ is equal to a distribution $\tilde{L}$ of order $j < k$. Prove that L can be multiplied with any C^j function F, that is C^k in a neighborhood of the complement of Γ. [First prove the result for $L = 0$ on Γ. Then write $L = (L - \tilde{L}) + \tilde{L}$.]

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