Fourier Analysis

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Fourier Analysis can indicate the study of Fourier transformations, Fourier series, and their extensions. One studies e.g. the convergence properties of Fourier series of certain classes of functions. It may also indicate the use of the Fourier mechanism in other subjects, e.g. in differential equations or in signal analysis. This course contains something of both these worlds. The course is intended for the master level mathematics at Dutch universities. Thus we assume some knowledge of elementary Fourier Analysis, Functional Analysis and Integration theory. At the UvA the courses Integration theory and Linear analysis of the bachelor program are more than sufficient. For convenience of the reader the essentials of all this are mentioned in the notes, but we do not dwell on the proofs.

The notes are based on a course that I first gave in 1996 at the UvA. They have been modified and slightly been edited, but there are bound to be many typo's and other errors. I would certainly appreciate it if a reader pointed out some to me.

Everything in these notes can be found in the literature, but one may have to look for it a while.

Jan Wiegerinck

CHAPTER 1

Classical Fourier series

1.1. Introduction and Reminder

In this section we recall a few facts from the linear analysis course of the third year. See the notes by Prof. Koornwinder [**13**] or the books [**20, 12**]. As usual we identify functions on the unit circle T with 2π-periodic functions on $\mathbb{R}$; if f is defined on T, then $g(t) = f(e^{it})$ is the associated 2π-periodic function on $\mathbb{R}$. We denote either by $L^p_{2\pi}$ or by $L^p(T)$, $(1 \leq p < \infty)$ the set of 2π-periodic measurable functions that satisfy

$$\|f\|_p \stackrel{\text{def}}{=} \left(\int_0^{2\pi} |f(t)|^p \frac{dt}{2\pi}\right)^{1/p} < \infty \tag{1.1.1}$$

Notice that we normalized $L^p(T)$ spaces utilizing the measure $\frac{dt}{2\pi}$. The pleasant effect is that the norm of the function 1 equals 1. We know that $\| \; \|_p$ is a norm, which turns L^p into a Banach space, while $L^2(T)$ is even a Hilbert space with inner product

$$\langle f, g\rangle \stackrel{\text{def}}{=} \int_0^{2\pi} f(t)\overline{g(t)}\frac{dt}{2\pi}.$$

Other function spaces on T are $L^\infty(T)$ the space of 2π periodic, essentially bounded measurable functions, and $C(T) \subset L^\infty(T)$ the space of continuous 2π-periodic functions. Both spaces are Banach spaces when equipped with the sup-norm.

We will also have use for sequence spaces: $l^p(\mathbb{Z})$ $(1 \leq p < \infty)$ is the space of sequences

$$\left\{ a = \{a_j\}_{j=-\infty}^{\infty} : |a|_p \stackrel{\text{def}}{=} \left(\sum_{j\in\mathbb{Z}} |a_j|^p\right)^{1/p} < \infty \right\} \tag{1.1.2}$$

Again $l^p(\mathbb{Z})$ is a Banach space with norm $| \; |_p$, and $l^2(\mathbb{Z})$ is a Hilbert space, the inner product being $\sum_{j\in\mathbb{Z}} a_j\bar{b}_j$, $(a, b \in l^2(\mathbb{Z}))$. Other sequence spaces that we will meet are the space of bounded sequences $c(\mathbb{Z})$, and its subspace $c_0(\mathbb{Z})$, which consists of sequences $\{a_j\}$ tending to 0 if $|j| \to \infty$. Both are Banach spaces when equiped with the sup-norm.

The *Fourier series* of $f \in L^1(T)$ is

$$\sum_{-\infty}^{\infty} a_n e^{int}, \tag{1.1.3}$$

where $a_n = \hat{f}(n) = \int_0^{2\pi} f(s)e^{-ins}\frac{ds}{2\pi}$ are the *Fourier coefficients* of f. We know by the *Riemann-Lebesgue Lemma* that $\hat{f}(n) \to 0$ if $|n| \to \infty$, that is, $\hat{f} \in c_0(\mathbb{Z})$. A formal sum of the form (1.1.3) with arbitrary a_n is called a *trigonometric series*. If we start with a Borel measure supported on $[-\pi, \pi)$ we can also form the Fourier coefficients of μ

$$a_n = \hat{\mu}(n) = \int_{-\pi}^{\pi} e^{-int} d\mu(t).$$

The series (1.1.3) is then called a *Fourier-Stieltjes series*. Of course $|\hat{\mu}(n)| \leq \|\mu\|$, but it is in general not true that $\hat{\mu}(n) \to 0$, if $|n| \to \infty$. Taking $\mu = \delta$, point mass at 0, we find $\hat{\delta}(j) = 1/2\pi$ for all j and the series (1.1.3) does not converge in the usual sense

We see that taking a Fourier series can be seen as a map $\hat{}$ from a space of functions, or measures, or more general, to a space of sequences. Natural questions are: For what kind of things can one define a Fourier series? Can you say something about the target space if you start in $L^p(T)$? Is $\hat{}$ surjective to some $l^p(\mathbb{Z})$? Is it maybe even an isometry? Is (1.1.3) convergent in L^p if $f \in L^p$? Is it perhaps convergent in any other sense?

Some of these questions will be answered in the course.

Partial sums of the series (1.1.3) are expressed by means of the *Dirichlet kernels* D_N. These are defined as follows

$$D_N(t) = \sum_{-N}^{N} e^{int} = \begin{cases} \frac{\sin((N+1/2)t)}{\sin(t/2)} & \text{if } t \notin 2\pi\mathbb{Z} \\ 2N+1 & \text{if } t \in 2\pi\mathbb{Z}. \end{cases} \tag{1.1.4}$$

For the N-th partial sum $S_N[f](t) = \sum_{n=-N}^{N} \hat{f}(n)e^{int}$ of the Fourier series of f we find

$$S_N[f](t) = \sum_{n=-N}^{N} \left(\int_{-\pi}^{\pi} f(s)e^{-ins} ds \right) e^{int} = \int_0^{2\pi} f(s)D_N(t-s)\frac{ds}{2\pi} =: f * D_N(t). \tag{1.1.5}$$

Similarly, for the N-th *Césaro sum* σ_N, i.e. the average of the partial sums S_0 upto S_N, there is an expression by means of the N-th *Fejér kernel* K_N. The latter is defined by

$$\begin{aligned} K_N(t) &= \frac{1}{N+1}\sum_{n=0}^{N} D_n(t) = \sum_{n=-N}^{N} \frac{N+1-|n|}{N+1} e^{int} \\ &= \begin{cases} \frac{1}{N+1}\left(\frac{\sin((N+1)t/2)}{\sin(t/2)}\right)^2 & \text{if } t \notin 2\pi\mathbb{Z} \\ N+1 & \text{if } t \in 2\pi\mathbb{Z}. \end{cases} \end{aligned} \tag{1.1.6}$$

The Césaro sum of f is given by

$$\sigma_N[f](t) = \frac{1}{N+1}\sum_{n=0}^{N} S_n[f](t) = f * K_N(t).$$

The Fejér kernels K_N are *good kernels*, they have the three characteristic properties of an *approximate identity*:

- $K_N \geq 0$.
- $\int_0^{2\pi} K_N(t)\, \frac{dt}{2\pi} = 1$.
- For every $0 < \delta < \pi$, $K_N(t) \to 0$ as $N \to \infty$ uniformly on $[\delta, 2\pi - \delta]$.

Let $f \in L^1(T)$. If a family of integral kernels L_N on $[0, 2\pi]$ has these three properties, then at a point of continuity a of f one has that $L_N * f(a) \to f(a)$ and, moreover, for $f \in C(T)$ the convergence of $L_N * f$ to f is uniform on T. We indicate the proof.

(1.1.7)

$$|f(a) - L_N * f(a)| = \left|\int_0^{2\pi} \Big(f(a) - f(a-t)\,)L_N(t)\, dt\right| \leq \left|\int_{[\delta,2\pi-\delta]} \cdots \right| + \left|\int_{[0,\delta]\cup[2\pi-\delta,2\pi]} \cdots \right|$$

The first term is small for small δ by continuity of f at a and property i and ii. Fixing such a small δ, the second term is bounded by $\max_{\delta\leq t\leq 2\pi-\delta} L_N(t)\|f\|_1 + |f(a)| \int_\delta^{2\pi-\delta} L_n(t)dt$. This tends to 0 when $N \to \infty$. Now if f is continuous on T, then it is uniformly continuous on T and δ can be chosen independently of a. Moreover the second term can be estimated uniformly, leading to uniform convergence of $L_N * f$ on T. With a bit more effort, if $f \in L^p(T)$ $(1 \leq p < \infty)$, then $L_N * f$ tends to f in L^p sense as $N \to \infty$. See [**10**] for a clever proof with a slightly weaker condition iii.

In particular these things hold for the Fejér kernel, giving the well-known fact that the Césaro sums of $f \in C(T)$ converge to f uniformly on T. In particular every $f \in C(T)$ can be approximated uniformly by goniometric polynomials, namely by its Césaro sums.

$$f(t) = \lim_{N\to\infty} \sigma_N[f](t). \tag{1.1.8}$$

The exponentials e^{int}, $n \in \mathbb{Z}$, form clearly an orthonormal system in $L^2(T)$. Because $C(T)$ is dense in $L^2(T)$ and the goniometric polynomials are dense in $C(T)$, this orthonormal system is complete, hence an orthonormal *basis* for $L^2(T)$. Observe that for an $f \in L^2(T)$ its Fourier series is the expansion of f on the basis $\{e^{int}\}$. As a consequence for $f, g \in L^2(T)$ *Parseval's formula* holds:

$$\int_{-\pi}^{\pi} f\bar{g}\frac{dt}{2\pi} = \sum_{\mathbb{Z}} \hat{f}(n)\bar{\hat{g}}(n), \tag{1.1.9}$$

Observe that the lefthand side of 1.1.9 is the inner product in $L^2(T)$, while the righthand side is the inner product in $l^2(\mathbb{Z})$. So 1.1.9 expresses that $\hat{}$ is an isometry from $L^2(T) \to l^2(\mathbb{Z})$. In fact it is also surjective. If $\{a_j\}_j \in l^2(\mathbb{Z})$ then the partial sums $\sum_{j=-N}^{N} a_j e^{ijt}$ form a Cauchy sequence in $L^2(T)$ that converges to some $f \in L^2(T)$ with $\hat{f}(j) = a_j$.

We will need a few additional estimates on K_N:

$$K_N(t) \leq \min\{N+1, \frac{\pi^2}{(N+1)t^2}\}, \quad t \in [-\pi, \pi] \tag{1.1.10}$$

and, using Parseval's formula,

$$\|K_N\|_2^2 = \sum_{n=-N}^{N} \left(\frac{N+1-|n|}{N+1}\right)^2 \geq N/2. \tag{1.1.11}$$

The sum in the middle can of course be computed, but the last estimate follows easily by comparison with $\int_0^{N+1}(1 - x/(N+1))^2\,dx$.

Pointwise Convergence of the Fourier series is not nearly as good as L^2-convergence. The classical result is as follows.

THEOREM 1.1.1. *Suppose that* $f \in C(T)$ *is* Hölder continuous*, i.e. there exist* $\alpha, C > 0$ *such that*

$$|f(s) - f(t)| < C|s-t|^\alpha.$$

Then

$$S_N[f](t) \to f(t), \quad \textit{uniformly, as } N \to \infty.$$

However, there exist continuous functions on T, the Fourier series of which does not converge uniformly on T. Indeed, for every $x \in T$ the map $\Lambda_n^x : f \mapsto S_n[f](x)$ is a bounded linear functional on $C(T)$. One can show that $\|\Lambda_n^x\| \geq C \log n$ and in particular tends to ∞. The Banach Steinhaus Theorem then gives that for a dense set of functions $f \in C(T)$ one has

$$\sup_n |\Lambda_n^x f| = |S_n[f](x)| = \infty.$$

See [**16**] for details. A more or less constructive proof can be found in [**20**].

Concerning point wise convergence of the Fourier series of L^p functions we state two classical results. Andrey Kolmogorov constructed $L^1(T)$-functions whose Fourier series does not converge in any point of T, [**11**]. Lennart Carleson, on the other hand, showed that the Fourier series of an f in $L^2(T)$ converges almost everywhere on T, [**1**]. His result was extended by Richard Hunt to L^p for $p > 1$, cf. [**8**].

1.2. Sine versus Cosine Series

If we start with an *even function* f, $(f(t) = f(-t))$, in (1.1.3), we will find that $a_n = a_{-n}$ (the sequence is *even*). Then taking together the n-th and $-n$-th term, we obtain

$$a_0 + \sum_{n=1}^{\infty} a_n(e^{int} + e^{-int}) = a_0 + \sum_{n=1}^{\infty} 2a_n \cos(nt),$$

a *cosine series.* Similarly if f is an *odd function* $(f(t) = -f(-t))$, we find $a_n = -a_{-n}$, (the sequence is *odd*), and we obtain the *sine series*

$$\sum_{n=1}^{\infty} a_n(e^{int} - e^{-int}) = \sum_{n=1}^{\infty} 2ia_n \sin(nt).$$

Notice that every function on $(0, \pi)$ can be extended as an even, but also to an odd 2π-periodic function. In that light it is remarkable that sine and cosine series have different convergence behavior even if the coefficients are the same. (Of course the series belonging to even and odd continuation do not have the same coefficients) In the present section we prove two theorems which enlighten this behavior of sine and cosine series.

THEOREM 1.2.1. *Suppose that $(a_n)_{n=-\infty}^{\infty}$ is an even sequence of positive numbers which tend to 0 if $|n| \to \infty$. If (a_n) satisfies the convexity condition*

$$a_{n-1} + a_{n+1} - 2a_n \geq 0, \quad (n \geq 1),$$

then there exists $f \in L^1_{2\pi}$ such that $\hat{f}(n) = a_n$.

PROOF. The convexity condition implies that $a_n - a_{n+1}$ is monotonically decreasing to 0. From this we have

$$\begin{aligned} n(a_n - a_{n+1}) &\leq (a_k - a_{k+1}) + (a_{k+1} - a_{k+2}) \ldots (a_n - a_{n+1}) + (k-1)(a_n - a_{n+1}) \\ &= a_k - a_{n+1} + (k-1)(a_n - a_{n+1}) \to 0, \end{aligned} \tag{1.2.1}$$

by choosing k fixed and large, so that a_k is small, and then letting $n \to \infty$. By cleverly rearranging of the series, so-called *summation by parts*, we also find

$$\begin{aligned} &\sum_{n=1}^{N} n(a_{n-1} + a_{n+1} - 2a_n) = \sum_{n=1}^{N} n((a_{n+1} - a_n) - (a_n - a_{n-1})) \\ &= \sum_{n=1}^{N} n(a_{n+1} - a_n) - \sum_{n=0}^{N-1} (n+1)(a_{n+1} - a_n) = a_0 - a_N - N(a_N - a_{N+1}) \longrightarrow a_0, \end{aligned} \tag{1.2.2}$$

for $N \to \infty$. Put

$$f_N(t) = \sum_{n=1}^{N} n(a_{n-1} + a_{n+1} - 2a_n) K_{n-1}(t).$$

This series has non-negative terms and is Cauchy in L^1 sense. In view of (1.2.2), for $N > M$

$$\begin{aligned} \int_0^{2\pi} |f_N - f_M|\, dt &= \sum_{n=M+1}^{N} n(a_{n-1} + a_{n+1} - 2a_n) \int_0^{2\pi} K_{n-1}(t) dt \\ &= \sum_{n=M+1}^{N} n(a_{n-1} + a_{n+1} - 2a_n) < \varepsilon \quad \text{if } M \text{ is sufficiently large.} \end{aligned} \tag{1.2.3}$$

Therefore $\lim_N \to \infty f_N = f$ exists in $L^1(T)$. Using that by (1.1.6) $\hat{K}_{n-1}(p) = \frac{n-|p|}{n}$ if $n > |p|$ and a dilated version of (1.2.2), we compute $\hat{f}$.

$$\hat{f}(p) = \sum_{n>|p|} (a_{n-1} + a_{n+1} - 2a_n)(n - |p|) = \sum_{j=1}^{\infty} j(a_{|p|+j-1} + a_{|p|+j+1} - 2a_{|p|+j}) = a_{|p|}.$$

□

THEOREM 1.2.2. *Suppose that f is in $L^1(T)$ and that $\hat{f}(n) = -\hat{f}(-n) \geq 0$ for $n \geq 0$. Then $\sum_{n\neq 0} \frac{\hat{f}(n)}{n}$ converges.*

PROOF. Let $F(t) = \int_0^t f(s)\,ds$, $t \in [-\pi, \pi]$. Then F is continuous and $F(-\pi) = F(\pi)$, because $\hat{f}(0) = 0$, i.e. $F \in C(T)$. The Fourier coefficients of F are

$$\hat{F}(n) = \frac{\hat{f}(n)}{in}, \quad (n \neq 0).$$

The Césaro sums of F will converge uniformly to F, therefore, subtracting $\hat{F}(0)$ from F and evaluating at 0,

$$\lim_{N\to\infty} \sum_{1\leq|n|\leq N} \frac{N+1-|n|}{N+1} \frac{\hat{f}(n)}{n} = i(F(0) - \hat{F}(0)) = -i\hat{F}(0).$$

All terms in the sum are positive, so this sum converges absolutely. Next

$$\sum_{1\leq|n|\leq N} \frac{\hat{f}(n)}{n} \leq 2 \sum_{1\leq|n|\leq N} (1 - \frac{|n|}{2N+1}) \frac{\hat{f}(n)}{n} < i\hat{F}(0),$$

which proves the theorem. □

COROLLARY 1.2.3. *Let $b_n = \frac{1}{\log(n+2)}$, then $\sum b_n \cos(nt)$ is the Fourier series of an L^1 function, but $\sum b_n \sin(nt)$ is not.*

1.3. Weak Topologies

Occasionally we will use weak-* convergence of measures. In this section we recall this notion for readers who are not familiar with it.

1.3.1. Weak-* convergence. A sequence of Borel measures $(\mu_j)_j$ on a compact Hausdorff space X converges weak-* to μ if for every $f \in C(X)$

$$\lim_{j\to\infty} \int f\,d\mu_j = \int f\,d\mu. \tag{1.3.1}$$

Similarly, in a Hilbert space H with inner product $\langle\cdot,\cdot\rangle$, a sequence f_j converges weakly to f if for every $g \in H$

$$\lim_{j\to\infty} \langle f_j, g\rangle = \langle f, g\rangle. \tag{1.3.2}$$

This and the Banach-Alaoglu Theorem below is basically all we need to know. Nevertheless some background may be useful.

1.3.2. The weak topology. Recall that a topology τ_1 on a set X is called *weaker* than τ_2 on X if every τ_1 open set in X is also τ_2 open; then τ_2 is called *stronger* than τ_1. Also recall that the product topology is defined by requiring that it is the weakest topology on the set theoretical product such that all projections are continuous mappings.

We can do something similar in topological vector spaces. Thus let X be a topological vector space such that its dual X^* separates points of X, i.e. for every $x \in X$ there exists a continuous linear functional $L \in X^*$ with $Lx \neq 0$. This is certainly the case if X is a Banach or a Hilbert space. The *weak topology* on X is the weakest topology that makes all $L \in X^*$ continuous. Since they are already continuous in the original topology of X, the weak topology is weaker than the original one.

A local subbasis for the weak topology on X consists of the sets

$$V_L^\varepsilon = \{x \in X : |Lx| < \varepsilon\},$$

where $\varepsilon > 0$ and $L \in X^*$. This means that $U \subset X$ is a neighborhood of 0 if there exist $\varepsilon_i > 0$, $L_i \in X^*$ such that $(\cap_{i=1}^p V_{L_i}^{\varepsilon_i}) \subset U$. How does this relate to convergence? Well, $x_j \to x$ if and only if $x_j - x \to 0$, that is, every neighborhood U of 0 must, for sufficiently large j, contain $x_j - x$. Therefore, for every choice of finitely many $V_{L_i}^{\varepsilon_i}$, it holds that $x_j - x \in (\cap_{i=1}^p V_{L_i}^{\varepsilon_i})$ if j is sufficiently large. This happens if and only if $Lx_j \to Lx$ for every L. Compare this to (1.3.2).

1.3.3. The weak-* topology. Recall that X can be seen as a subset of X^{**} via $xL := Lx$, ($x \in X$, $L \in X^*$) and that the subset $X \subset X^{**}$ already separates points on X^*. The *weak-* topology* on X^* is defined as the weakest topology that makes all $x \in X$ continuous functionals on X^*. We do not require continuity of functionals in $X^{**} \setminus X$ (In many important cases, however, this set is empty, compare [**22**]). Similarly to weak convergence, a sequence $L_j \in X^*$ converges weak-* to $L \in X^*$ if and only if for every $x \in X$ we have for every $x \in X$ that $L_j x \to Lx$.

Finally we quote

THEOREM 1.3.1 (Banach-Alaoglu). *If V is a neighborhood of 0 in a topological vector space X and*

$$K_V = \{L \in X^* : \ |Lx| \le 1 \text{ for every } x \in V\}.$$

Then K_V is weak- compact.*

A proof can be found in [**17**].

EXAMPLE 1.3.2. Let $X = C(T)$, $V = \{f \in C(T) : \ \|f\|_\infty < 1\}$, then $K_V = \{\mu \in M(T) : \|\mu\| \le 1\}$ is compact. Theorem 1.3.1 tells us that every sequence of Borel measures $(\mu_\alpha)_\alpha$ on T with uniformly bounded mass has a weak-* convergent subsequence. In other words, there exists a subsequence $(\mu_j)_j$ and a measure $\mu \in M(T)$ such that (1.3.1) holds.

1.4. Lacunary Series

DEFINITION 1.4.1. A sequence $\{\lambda_j\}$, $j = 1, 2, \ldots$, of positive integers is called (Hadamard) *lacunary* with constant $q > 1$ if $\lambda_{j+1} > q\lambda_j$ for all $j \ge 1$. A power series is called lacunary if it is of the form $\sum c_j z^{\lambda_j}$, while a trigonometric series is called lacunary if it is of the form $\sum c_j e^{i\lambda_j t} + \sum d_j e^{-i\lambda_j t}$ with $\{\lambda_j\}$ lacunary.

LEMMA 1.4.2. *Let $n_0 \in \mathbb{Z}$. Suppose that $f \in L^1_{2\pi}$ and $f(t) = O(t)$ as $t \to 0$. If*

$$\hat{f}(j) = 0, \quad \text{for all } 1 \le |n_0 - j| \le 2N, \tag{1.4.1}$$

then

$$|\hat{f}(n_0)| \le 2\pi^4 (N^{-1} \sup_{|t| \le N^{-1/4}} |f(t)/t| + N^{-2}\|f\|_1).$$

PROOF. If g_N is any trigonometric polynomial of degree $2N$ with $\hat{g}(0) = 1$, then

$$\hat{f}(n_0) = \int_{-\pi}^{\pi} e^{-in_0 t} f(t) g_N(t) \frac{dt}{2\pi},$$

because (1.4.1) expresses that $S_N[e^{-in_0 t} f(t)] = \hat{f}(n_0)$. We take $g_N = K_N^2 / \|K_N\|_2^2$. Then in view of (1.1.10) and (1.1.11) $\int_{-\pi}^{\pi} g_N \frac{dt}{2\pi} = 1$, $g_N \ge 0$, $g_N(t) \le \frac{\pi^4 2}{N(N+1)^2 t^4}$. We use this to estimate

$$|\hat{f}(n_0)| \le \int_{-\pi}^{\pi} |f(t)| g_N(t) \frac{dt}{2\pi} = \int_{|t| \le N^{-1}} \quad + \int_{N^{-1} \le |t| \le N^{-1/4}} \quad + \int_{N^{-1/4} \le |t| \le \pi} .$$

Now these three integrals are estimated as follows.

$$\int_{|t| \le N^{-1}} |f(t)| g_N(t) \frac{dt}{2\pi} \le \frac{1}{N} \sup_{|t| \le N^{-1}} \frac{|f(t)|}{|t|} \int_{-\pi}^{\pi} g_N(t) \frac{dt}{2\pi} = \frac{1}{N} \sup_{|t| \le N^{-1}} \frac{|f(t)|}{|t|}.$$

(1.4.2)
$$\begin{aligned}\int_{\frac{1}{N}\le|t|\le\frac{1}{N^{1/4}}}|f(t)|g_N(t)\frac{dt}{2\pi} &\le \sup_{|t|\le\frac{1}{N^{1/4}}}\frac{|f(t)|}{|t|}\int_{\frac{1}{N}\le|t|\le\frac{1}{N^{1/4}}}\frac{|t|\pi^4 2}{N(N+1)^2t^4}\frac{dt}{2\pi}\\ &\le \sup_{|t|\le\frac{1}{N^{1/4}}}\frac{|f(t)|}{|t|}\frac{\pi^4 2N}{2\pi(N+1)^2}.\end{aligned}$$

$$\int_{N^{-1/4}\le|t|\le\pi}|f(t)|g_N(t)\frac{dt}{2\pi}\le\frac{\pi^4 2}{N(N+1)^2(N^{-1/4})^4}\int_{-\pi}^{\pi}|f(t)|\frac{dt}{2\pi}.$$

These three estimates prove the lemma. □

COROLLARY 1.4.3. *Suppose that* $f=\sum_{n=1}^{\infty}a_n\cos(\lambda_n t)\in L^1_{2\pi}$, *with* $\lambda_{n+1}\ge q\lambda_n$ *and* $q>1$. *If* f *is differentiable at a point* p *then* $a_n=o(\lambda_n^{-1})$.

PROOF. Considering $f_p(t)=f(t+p)$ which has $\hat{f}_p(n)=e^{inp}\hat{f}(n)$ we may assume that $p=0$. Replace f by $f-f(0)-f'(0)\sin t$. This has no effect on the tail of the series and now $f(t)=o(|t|)$ at 0. We have $\hat{f}(j)=0$ for $0<|j-\lambda_n|<(1-1/q)\,\lambda_n$. We apply the Lemma and obtain
$$|\hat{f}(\lambda_n)|\le\frac{o(1)}{\lambda_n}+\frac{C}{\lambda_n^2}=\frac{o(1)}{\lambda_n}).$$
□

COROLLARY 1.4.4 (Weierstrass' nowhere differentiable function).
$$f(t)=\sum_{n=0}^{\infty}\frac{\cos(2^n t)}{2^n}$$
is continuous, but nowhere differentiable.

PROOF. The series is lacunary and uniformly convergent, so f is continuous and the previous corollary gives that f is nowhere differentiable. □

1.5. Riesz products

Let $\{\lambda_n\}$ be lacunary with $q\ge 3$. A trigonometric polynomial of the form
$$P_N(t)=\prod_{n=1}^{N}(1+a_n\cos(\lambda_n t+\varphi_n))$$
is called a (finite) *Riesz product*. Observe that, since $q\ge 3$, an integer M can at most in **one** way be written as
$$M=\sum_{1}^{\infty}c_n\lambda_n,\quad c_n\in\{-1,0,1\}.$$
In fact M will be a finite sum and unless $q=3$, not all M can be expressed as such a sum. We use this when expanding P_N. A typical factor of P_N is $1+(a_ne^{i\varphi_n}/2)e^{i\lambda_n t}+(a_ne^{-i\varphi_n}/2)e^{-i\lambda_n t}$. In the expansion of P_N we will thus find exponentials of the form $e^{ikt}=e^{i(\sum c_n\lambda_n)t}$ and by the preceding observation such an exponential can be obtained in at most one way. It follows that

(1.5.1)
$$\hat{P}(k)=\begin{cases}\prod\left(\frac{a_ne^{i\varphi_n c_n}}{2}\right) & \text{if } k=\sum c_n\lambda_n, \text{ with } c_n\ne 0,\\ 0 & \text{elsewhere.}\end{cases}$$

Also, from $P_{N+1}=P_N+\frac{a_{N+1}e^{i\varphi_{N+1}}}{2}P_Ne^{i\lambda_{N+1}t}+\frac{a_{N+1}e^{i\varphi_{N+1}}}{2}P_Ne^{-i\lambda_{N+1}t}$ we see that the Fourier series of P_{N+1} is obtained from the Fourier series of P_N by adding two copies of P_N multiplied by a constant, one shifted λ_{N+1} to the right, the other shifted λ_{N+1} to the left. As $q\ge 3$ there is no overlap. In particular, whatever the sequence $\{a_n\}$, if $N\to\infty$, then $\hat{P}_N$ becomes stationary on every finite subset of $\mathbb{Z}$. We find that $\lim_{N\to\infty}\hat{P}_N$ is a well-defined trigonometric series. If $P=\lim P_N$ in some sense then $\hat{P}=\lim\hat{P}_N$.

For us there are two cases of interest:

(1) Suppose $-1 \le a_n \le 1$. Then all P_N are nonnegative and $\int_{-\pi}^{\pi} P_N = 1$. Thus the P_N are (densities of) probability measures on $(-\pi, \pi)$. There exists at least one weak-* limit point. If μ_1, μ_2 are two weak-* limit points of $P_N\,dt$, we have $\hat{\mu}_1 = \hat{\mu}_2$, so $\mu_1 = \mu_2$ and P_N converges weak-* to a probability measure with Fourier-Stieltjes series $\lim P_N$.

(2) Suppose that $a_n = ib_n$, $b_n \in \mathbb{R}$ and $\sum_n b_n^2 < \infty$. We have $1 \le |(1 + ib_n \cos(\lambda_n t)| \le (1 + b_n^2)^{1/2}$, therefore, with suitable constant C.

$$1 \le |P_N| \le e^{(\sum_n \log(1+b_n^2))/2} \le e^{\frac{1}{2}\sum_n b_n^2} < C. \tag{1.5.2}$$

Thus $|P_N|$ is uniformly bounded and P_N converges weak-* to (a measure given by) an $L^\infty_{2\pi}$ function. (If O is open in T with Lebesgue measure $|O|$, and f is continuous $0 \le f \le 1$ with support in O, then $|\int f P_N\,dt| \le C|O|$; the same goes for the weak-* limit, giving that the weak-* limit is absolutely continuous with respect to Lebesgue measure and the density is in L^∞.)

LEMMA 1.5.1. *Let $\{\lambda_j\}$ be lacunary with constant q. Put $\lambda_{-j} = -\lambda_j$, $\lambda_0 = 0$. There exist constants A_q, B_q such that if $f(t) = \sum_{-N}^{N} c_j e^{i\lambda_j t}$, then*

$$\begin{aligned} \sum |c_j| &\le A_q \|f\|_\infty, \\ \|f\|_2 &\le B_q \|f\|_1. \end{aligned} \tag{1.5.3}$$

PROOF. Notice that if we prove the Lemma for real valued f, then it follows for complex valued f with the constants A_q and B_q doubled. We first deal with the case $q \ge 3$ and assume that f is real, which means that $c_j = \bar{c}_{-j}$. To prove the first inequality, we set

$$P_N(t) = \prod_{j=1}^{N} (1 + \cos(\lambda_j t + \varphi_j)).$$

We choose $\varphi_j = \arg c_j$, $j \ge 1$. We have

$$\int_{-\pi}^{\pi} P_N \bar{f} \frac{dt}{2\pi} = \sum_{j=-N}^{N} \hat{P}_N(\lambda_j) \bar{\hat{f}}(\lambda_j) = \frac{1}{2} \sum_{j=-N}^{N} e^{i \text{Sign}\,(j) \varphi_{|j|}} \bar{c}_j = \frac{1}{2} \sum_{-N}^{N} |c_j|,$$

Also

$$|\int_{-\pi}^{\pi} P_N \bar{f} \frac{dt}{2\pi}| \le \|f\|_\infty \int_{-\pi}^{\pi} P_N \frac{dt}{2\pi} = \|f\|_\infty.$$

Thus we have proved the first equality for $q \ge 3$ with $A_q = 4$.

For the second inequality we set

$$P_N(t) = \prod_{j=1}^{N} \left(1 + i \left(\frac{|c_j|}{\|f\|_2}\right) \cos(\lambda_j t + \varphi_j)\right).$$

We proceed as above and find with the same choice of φ_j

$$\begin{aligned} \|f\|_2 = \sum_{j=-N}^{N} \frac{|c_j|^2}{\|f\|_2} &= -2i \sum_{j=-N}^{N} i \frac{|c_j|}{\|f\|_2} \frac{e^{i\text{Sign}\,(j)\varphi_{|j|}}}{2} \bar{c}_j \\ &= -2i \sum_{j=-N}^{N} \hat{P}_N(\lambda_j) \bar{\hat{f}}(\lambda_j) = -2i \int P_N \bar{f} \frac{dt}{2\pi} \le 2\|P_N\|_\infty \|f\|_1. \end{aligned} \tag{1.5.4}$$

Since the P_N are uniformly bounded by $e^{1/2}$, compare (1.5.2), we are done. Notice that it is the seemingly artificial factor i that we introduced in P_N, that makes it possible to estimate $\|P_N\|_\infty$.

For $q \ge 3$ we may take $B_q = 4e^{1/2}$.

The general case is done by carefully splitting the lacunary sequence in such a way that the Riesz products associated to the subsequences make sense and will only pick up the terms in the series that we want. Let M be a large integer, depending on q and to be determined in the process. Write for a fixed $0 \le m < M$ $\lambda_j^m = \lambda_{m+jM}$. We want that $\{\lambda_j^m\}_j$ is lacunary with constant ≥ 3. Thus we require that M satisfy

$$q^M \ge 3, \tag{1.5.5}$$

hence a Riesz product associated to $\{\lambda_j^m\}_j$ makes sense. Next, we want that each of the frequencies λ_k of f occurs in precisely one Riesz product. Suppose that $n > 0$ is written as $\sum_{j=0}^J c_j \lambda_j^m$, with $c_j \in \{-1, 0, 1\}$ and $c_J = 1$. Then

$$|n - \lambda_J^m| \le \sum_{j=0}^{J-1} \lambda_j^m \le \lambda_J^m \sum_{j=0}^{J-1} \frac{\lambda_j^m}{\lambda_J^m} \le \lambda_J^m \sum_{j=1}^{J} \frac{1}{q^{jM}} \le \frac{\lambda_J^m}{q^M - 1}.$$

Thus we want $|\lambda_k - \lambda_J^m| > \frac{\lambda_J^m}{q^M - 1}$ for all $\lambda_k \ne \lambda_J^m$. If $\lambda_k \ge q\lambda_J^m$ this leads to

$$q - 1 > \frac{1}{1 - q^M} \quad \text{or} \quad q > 1 + \frac{1}{q^M - 1}, \tag{1.5.6}$$

while, if $\lambda_J^m \ge q\lambda_k$, this leads to

$$1 - 1/q > \frac{1}{1 - q^M} \quad \text{or} \quad 1/q < 1 - \frac{1}{q^M - 1}. \tag{1.5.7}$$

We take M so large that (1.5.5), (1.5.6), (1.5.7) are satisfied. Now let $P_N^m = \prod_1^N (1 + a_{m+jM} \cos(\lambda_j^m t + \varphi_{m+jM}))$ be one of the Riesz products considered in the first part of the proof. Then

$$\frac{1}{2\pi} \int P_N^m(t) \bar{f}(t)\, dt = \frac{1}{2} \sum |a_{m+jM}||c_{m+jM}|.$$

The first part of the proof gives

$$\begin{aligned} \sum |c_{m+jM}| &\le 4\|f\|_\infty, \\ (\sum |c_{m+jM}|^2)^{\frac{1}{2}} &\le 4e^{1/2}\|f\|_1, \end{aligned} \tag{1.5.8}$$

in respectively the first and second case of the lemma. Summing over $m = 1, \ldots, M$ gives the result. □

THEOREM 1.5.2. *Suppose that the Fourier series $\sum_{-\infty}^{\infty} c_j e^{i\lambda_j t}$ of $f \in L^1(T)$ is lacunary, then $f \in L^2(T)$. If f is bounded, then $\sum |c_j| < \infty$.*

PROOF. Let $\sigma_N[f]$ be a Césaro sum. These have L^1 norms, uniformly bounded by $\|f\|_1$. The lemma gives that

$$\sum_{-N}^{N} \left(1 - \frac{|j|}{N+1}\right)^2 |c_j|^2 \le B_q M.$$

This implies, by letting $N \to \infty$, that for fixed J the sum $\sum_{-J}^{J} |c_j|^2 \le B_q M$, thus $f \in L^2(T)$. The proof of the second statement is similar. □

The homogeneity of behavior of lacunary series also appears in the following theorem. We shall denote the length of a subarc Γ of T by $|\Gamma|$.

THEOREM 1.5.3. *Suppose that (λ_j) is lacunary with constant q. For every $\delta > 0$ there exists $j_0 \in \mathbb{N}$ such that for all lacunary $f \in L^2(T)$, $f(t) = \sum_{-\infty}^{\infty} c_j e^{i\lambda_j t}$ with $c_j = 0$ for $|j| < j_0$, the following inequality holds for every subarc Γ of T:*

$$(\frac{1}{2\pi}|\Gamma| - \delta)\|f\|^2 \le \int_\Gamma |f|^2 \frac{dt}{2\pi} \le (\frac{1}{2\pi}|\Gamma| + \delta)\|f\|^2.$$

PROOF. It suffices to prove the theorem for trigonometric polynomials. We shall give a proof for $q \geq 3$. For the general case, see exercise 1.7.7. We have

$$\int_\Gamma |P|^2 \frac{dt}{2\pi} = \int_\Gamma \sum_{n,m} c_n \bar{c}_m e^{i(\lambda_n - \lambda_m)t} \frac{dt}{2\pi} = \frac{1}{2\pi}|\Gamma| \|P\|^2 + \sum_{n \neq m} c_n \bar{c}_m \int_\Gamma e^{i(\lambda_n - \lambda_m)t} \frac{dt}{2\pi}. \tag{1.5.9}$$

Let h be the characteristic function of Γ. Then $\hat{h}(n) = \int_\Gamma e^{-int} dt/2\pi$, therefore (1.5.9) can be rewritten as

$$\frac{1}{2\pi}|\Gamma| \|P\|^2 + \sum_{n \neq m} c_n \bar{c}_m \hat{h}(\lambda_m - \lambda_n). \tag{1.5.10}$$

Now because $q \geq 3$, $\lambda_n - \lambda_m$ assumes any integer value j at most twice (solutions occur in pairs: (n, m) and $(-m, -n)$, and compare the beginning of this section). For general $q > 1$ there exists $K = K(q)$ such that there are at most K solution, Ex. 1.7.7. Moreover, $\min_{m,n \geq j_0} |\lambda_m - \lambda_n|\}$ tends to ∞ with j_0. Thus frequencies j of $\hat{h}$ occur at most twice in the sum in (1.5.10). We apply Cauchy-Schwarz to the sum and obtain that the norm of the sum is less than

$$\left(\sum_{n \neq m} |c_n c_m|^2 2 \sum_{|j| \geq \inf_{n \neq m} \{|(\lambda_n - \lambda_m)|\}} |\hat{h}(j)|^2 \right)^{\frac{1}{2}} \leq \|P\|^2 \delta,$$

if we choose j_0 so large that the 2-norm of the tail of the series of h is less then $\delta/2$. It follows from the explicit form of $\hat{h}(j)$ that this can be done independent of Γ. Combining this with (1.5.9) and (1.5.10) we obtain

$$(\frac{1}{2\pi}|\Gamma| - \delta)\|P\|^2 \leq \int_\Gamma |f|^2 dt \leq (\frac{1}{2\pi}|\Gamma| + \delta)\|P\|^2.$$

$\square$

1.6. Sidon sets

DEFINITION 1.6.1. Let $E \subset \mathbb{Z}$. A function f (or measure, or distribution) on T is called *E-spectral* if $\hat{f}(n) = 0$ if $n \notin E$. Denote by C_E, L^p_E, M_E the respective subspaces of $C(T)$, $L^p(T)$, $M(T)$ consisting of E-spectral elements. These are closed subspaces. A subset E of $\mathbb{Z}$ is called a *Sidon set* if $f \in C_E$ implies $\hat{f} \in l^1(E)$.

EXAMPLE 1.6.2. Of course every finite set is a Sidon set. By Theorem 1.5.2 every lacunary set is a Sidon set.

THEOREM 1.6.3. *The following are equivalent:*

1. *E is a Sidon set.*
2. *There exists $K > 0$ such that $\|\hat{f}\|_1 \leq K\|f\|_\infty$ for all E-spectral trigonometric polynomials f.*
3. *$\|\hat{f}\|_1$ is bounded for every $f \in L^\infty_E$.*

3.a *There exists a K such that $\|\hat{f}\|_1 \leq K\|f\|_\infty$ for every $f \in L^\infty_E$.*

4. *$\widehat{M_E} = l^\infty(E)$.*
5. *$\widehat{L^1}_E = c_0(E)$.*

PROOF. ($1 \implies 2$) If E is Sidon, then the map $f \mapsto \hat{f}$ is linear bijective from C_E to $l^1(E)$. Also its inverse (as a linear map) is continuous. Indeed

$$\|f\|_\infty = \sup_t \left| \sum \hat{f}(n) e^{int} \right| \leq \sum |\hat{f}(n)|.$$

Hence, by the Open Mapping Theorem, $\hat{f} \mapsto f$ is open and thus $f \mapsto \hat{f}$ is continuous, which means that for some $K > 0$ one has $\|\hat{f}\|_1 \leq K\|f\|_\infty$. In particular, this is true for E-spectral trigonometric polynomials.

($2 \implies 3(a)$) If $f \in L^\infty_E$, then $\sigma_N(f)$ is an E-spectral trigonometric polynomial with $|\sigma_N(f)|_\infty \leq \|f\|_\infty$, independent of N. Thus there is a constant K such that $\|\hat{\sigma}_N(f)\|_1 \leq K\|f\|_\infty$. As in the proof of Theorem 1.5.2 we conclude that $\|\hat{f}\|_1 \leq K\|f\|_\infty$.

($3 \implies 1$) is trivial.

Item 4 and 5 both say that a certain bounded linear transformation is surjective. Now recall that the Open Mapping Theorem implies that a surjective bounded linear transformation F between Banach spaces X and Y is *open*. Thus $F(\{\|x\| < 1\})$ contains an open neighborhood of $0 \in Y$. By linearity we obtain that there is a constant $C > 0$ such that for every $y \in Y$ there exist $x \in X$ with $Fx = y$ and $\|x\| < C\|y\|$. This will be used in the last two steps of the proof.

($3(a) \implies 4$) Let $(d_j)_j \in l^\infty(E)$. Then $f \mapsto \sum_j \hat{f}(j)d_j$ is a continuous linear functional on C_E which by Hahn-Banach can be extended to $C(T)$. By the Riesz Representation Theorem there exists a complex regular Borel measure μ which represents this functional. Thus for $f \in C_E$ we have $\int f\,d\mu = \sum_j \hat{f}(j)d_j$. We choose $f(t) = e^{-i\lambda_j t}$, $\lambda \in E$ and find $\hat{\mu}(\lambda_j) = d_j$. Thus $\hat{\bar{\mu}}(\lambda_j) = \bar{d}_j$. Replacing d_j by $\bar{d}_j$ we obtain our result.

($4 \implies 5$) First observe that modification of the Fejér kernels yields that if Λ is a finite set of integers and $\varepsilon > 0$, then there exists a trigonometric polynomial $P = P_{\Lambda,\varepsilon}$ such that $\hat{P}(j) = 1$ and $\|P\|_1 \leq 1 + \varepsilon$. We will use this with $\varepsilon = 1$. Now let $(d_j)_j \in c_0(E)$. We may assume that $|d_j| \leq 1$. Put

$$E_k = \{n : \; 2^{-k} < |d_n| \leq 2^{-k+1}\}.$$

By 4. there exist measures μ_k such that $\hat{\mu}_k(j) = d_j$ if $j \in E_k$, while $\hat{\mu}_k(j) = 0$ if $j \in E \setminus E_k$; moreover $\|\mu_k\| \leq C2^{-k}$. Let T_k be trigonometric polynomials with $\hat{T}(j) = 1$ on E_k and $\|T_k\|_1 \leq 2$. Then $T_k * \mu_k$ is a trigonometric polynomial of L^1 norm less than $2C2^{-k}$ and with Fourier coefficients $\hat{T}_k(j)\hat{\mu}_k(j) = \hat{\mu}_k(j)$ on E. The conclusion is that

$$f(t) = \sum_{k=1}^{\infty} T_k * \mu_k$$

is in $L^1(T)$ because the series converges in $L^1(T)$ and has $f(j) = d_j$ on E.

($5 \implies 2$) Let g be an E-spectral trigonometric polynomial. Define $d_n = |\hat{g}(n)|/\bar{\hat{g}}(n)$ if $\hat{g}(n) \neq 0$ and $d_n = 0$ elsewhere. $(d_n)_n \in c_0(E)$ of norm 1, hence there exists $f \in L^1(T)$ with $\hat{f}\mid_E = d_n$ and $\|f\|_1 \leq C$, where C is a constant only depending on E. Then

$$\sum |\hat{g}(n)| = \int_{-\pi}^{\pi} f(t)\bar{g}(t)\,dt \leq \|g\|_\infty \|f\|_1 \leq C\|g\|_\infty.$$

□

REMARKS 1.6.4. The smallest constant K in Theorem 1.6.3, such that 3a holds, is called the *Sidon constant* of E.

In retrospect, we can understand the (limits of the) Riesz products in Lemma (1.5.1) as explicitly constructed measures of Theorem (1.6.3), statement 4 and 5.

COROLLARY 1.6.5 (to Theorem 1.5.2). *Suppose that $E = \{\lambda_j\}$ is lacunary and that $L = (d_j) \in l^2(E)$. Then there exists a bounded function f on T such that $\hat{f}(\lambda_j) = d_j$.*

PROOF. In view of Riesz Representation Theorem, L is in $l^2(E)^* = (L^2_E)^*$. In fact, $L\varphi = \sum \hat{\varphi}(\lambda_j)\bar{d}_j$. By Theorem 1.5.2 L is also a continuous linear functional on L^1_E. Now L^1_E is a closed subspace of $L_1(T)$. By Hahn Banach we can extend L to all of $L^1(T)$. We denote the extension again by L. Recalling, c.f. [**22**][Ch 7], that the dual space of $L^1(T)$

is $L^\infty(T)$, we find that there exists a function $f \in L^\infty$ such that $L\varphi = \int_0^{2\pi} \varphi \bar{f} \frac{dt}{2\pi}$, for all $\varphi \in L^1(T)$. We now apply this with $\varphi(t) = e^{i\lambda_j t}$ and obtain $\hat{f}(\lambda_j) = d_j$. □

REMARK 1.6.6. One can do the same for general Sidon sets by proving the analogue of Theorem 1.5.2. This requires considerably more effort, cf. [**4**].

1.7. Exercises

1.7.1. Let $(b_n)_{n\in\mathbb{Z}}$ be a sequence of positive numbers such that $b_n \to 0$ as $|n| \to \infty$. Show that there exists a sequence (a_n) with $a_n \geq b_n$ which satisfies the convexity condition of Theorem 3.1.

1.7.2. Suppose that $f \in C_{2\pi}$ is *Hölder continuous* of order α, $0 < \alpha \leq 1$, i.e. there exists $B > 0$ such that for all x, $|f(x+t) - f(x)| \leq B|t|^\alpha$. Show that $|\hat{f}(n)| \leq Cn^{-\alpha}$. Hint: Write the Fourier coefficients as

$$\hat{f} = \frac{1}{2}\left(\int_0^{2\pi} f(s)e^{-ins}\frac{ds}{2\pi} - \int_0^{2\pi} f(s - \pi/n)e^{-ins}\frac{ds}{2\pi}\right).$$

1.7.3. Let (λ_n) be lacunary. Suppose that $f \in L^1(T)$ has Fourier series $\sum a_n \cos(\lambda_n t)$. Suppose that f is Hölder continuous at one point $t_0 \in T$. Show that $a_n = O(\lambda_n^{-\alpha})$. Next show that f is Hölder continuous on T.

1.7.4. With the assumptions as in exercise 1.7.3, show that if f equals 0 on a small interval in T, then $f \in C^\infty(T)$. Can you relax the condition that f be 0 on an interval and still reach the same conclusion?

1.7.5. With the same assumptions as in exercise 1.7.3, $f(t) = \sum c_n e^{i\lambda_n t}$, show that if f is (real) analytic on a subarc Γ of T, then f is real analytic on T by completing the following outline.

(1) Observe that f is C^∞ because of exercise 1.7.4 extended.
(2) Choose a suitable interval Γ, a suitable modification $\tilde{f}$ of f and show that there is a $C > 0$ such that for every k, l

$$|c_k|^2 |\lambda_k|^{2l} \leq C \int_\Gamma |\tilde{f}^{(l)}(t)|^2 dt.$$

(3) Use the Cauchy estimates to make the integrals in (2) bounded by a constant times $(l!\delta^{-l})^2$.
(4) Make a favourable choice of l to have $\limsup_{k\to\infty} |c_k|^{\frac{1}{\lambda_k}} < 1$.
(5) Finish it off!

1.7.6. Prove *Hadamard's Theorem*: If $f(z) = \sum_{n=0}^\infty c_n z^{\lambda_n}$ is a lacunary power series with radius of convergence R then $f(z)$ has no analytic continuation to a domain larger than the disc with radius R. ($C(0, R)$ is a natural boundary.)

1.7.7. Prove Theorem 1.5.3 for general q by showing that now there exists $K > 0$ such that every $j \in \mathbb{Z} \setminus \{0\}$ is assumed at most K times as a value of $\lambda_n - \lambda_m$.

1.7.8. Under the assumptions of Corollary 1.6.5, show that f can in fact be chosen continuous, by completing the following steps.

(1) Find sequences (a_n), (b_n) such that $d_n = a_n b_n$, with $a_n \in l^2(E)$ and $b_n \in c_0(E)$.
(2) Show that the convolution of a bounded and an integrable function is continuous.
(3) Prove the assertion.

1.7.9. Suppose that $f \in C(T)$ has positive Fourier coefficients. Prove that $\hat{f} \in l^1$.

1.7.10. Suppose that for every $d = (d_n)_n \in l^\infty(E)$ with $|d_n| \leq 1$, there exists a measure $\mu \in M(T)$ with

$$|d_n - \hat{\mu}(n)| \leq 1 - \delta.$$

Prove that E is Sidon. (Think of the previous exercise and let $d_n = \frac{|\hat{f}(n)|}{\hat{f}(n)}$.)

1.8. Final remarks, notes, and references

The classical book on trigonometric series is [**23**]. Section 1. is in [**13**], but also can be found e.g. in [**10**]. A well written elaborate introduction to Fourier analysis with many applications in other subjects is [**12**].

Material concerning section 3 can be found in any reasonable book on Functional Analysis, e.g. [**17**], see [**22**] for a more extensive list of references.

Section 2. has been taken from [**10**], but is similarly treated in [**23**].

Section 4,5 and 6 are also mostly taken from [**10**]; [**4**] has a more comprehensive treatment and was also used. Many things can be found in [**23**] too.

CHAPTER 2

Distributions and their Fourier Series

2.1. Introduction

Consider the wave equation on $\mathbb{R}^2$

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0. \tag{2.1.1}$$

Introducing new coordinates $x' = x + t$, $y' = x - t$, we obtain the equation

$$\frac{\partial^2 u}{\partial x' \partial y'} = 0, \tag{2.1.2}$$

which has the classical solution

$$u(x', y') = f(x') + g(y'), \quad f, g \in C^2(\mathbb{R}).$$

Thus (2.1.1) has as classical solutions $f(x+t)+g(x-t)$. Classically, C^2 is required because 2 differentiations are performed on u. Physically, however, there is no reason to ask much more than continuity. Also, from (2.1.2) we see that $\frac{\partial u}{\partial y'} = \tilde{g}(y')$, where the only thing that matters is that this function doesn't depend on x'. If $\tilde{g} \in L^1(\mathbb{R})$ we get a solution $u(x', y') = f(x') + g(y')$ with no stronger conditions on f and g than continuity. There are of course problems with changing the variables and we have a solution which is not symmetric in x and y. The point is that it is at least inconvenient not to be able to differentiate continuous functions.

As far as Fourier analysis is concerned, we know that $f \in C^1_{2\pi}$ has the property that $\hat{f'}(n) = in\hat{f}(n)$. We can formally write down this sequence of Fourier coefficients also if f is no longer differentiable. Can we give meaning to it as the Fourier series of something interesting? Moreover, consider the Fourier transform $\mathcal{F}f$ of $f \in L^1(\mathbb{R})$. If f is C^1 we know that

$$\mathcal{F}f'(\xi) = i\xi\mathcal{F}f(\xi). \tag{2.1.3}$$

Multiplication with ξ is a well defined operation on functions, the righthand side of (2.1.3) is always well defined. A meaningful lefthand side, that is, unlimited differentiability of L^1 functions, is desirable.

One way out is the concept of weak solution. Notice that if $u \in C^2$ solves (2.1.1), then for every compactly supported $\varphi \in C^\infty(\mathbb{R}^2)$ we have

$$\iint \left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2}\right) \varphi(x,t)\, dxdt = \iint u(x,t) \left(\frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial^2 \varphi}{\partial t^2}\right) dxdt = 0. \tag{2.1.4}$$

If the last equality in (2.1.4) holds for all compactly supported $\varphi \in C^\infty(\mathbb{R}^2)$ and $u \in C^2$, then u satisfies (2.1.1). However the last integral in (2.1.4) makes sense for locally integrable u. Thus one calls a locally integrable (sometimes only a continuous) u a weak solution of (2.1.1) if u satisfies

$$\iint \left(\frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial^2 \varphi}{\partial t^2}\right) u(x,t)\, dxdt = 0.$$

We will use this idea to differentiate locally integrable functions arbitrarely often. In fact we will go one step further. Observing that

$$u \mapsto \iint u\varphi \, dxdt$$

is a linear functional on compactly supported C^∞ functions, we will explain how to “differentiate” a large class of such linear functionals, which is determined by a continuity condition, and identify the ones originating from locally integrable functions with a subclass.

2.2. Smooth functions on T

Recall, see [**17**][Section 1.33], that a *seminorm* on a (complex) vector space X is a real valued function p such that

$$\begin{aligned} &p(x) \geq 0, \\ &p(x_1 + x_2) \leq p(x_1) + p(x_2), \\ &p(\lambda x) = |\lambda| p(x). \end{aligned} \tag{2.2.1}$$

Here $x, x_i \in X$, $\lambda \in \mathbb{C}$.

A family $\mathcal{P}$ of seminorms is called *separating* if for every $x \in X$ there is a $p \in \mathcal{P}$ such that $p(x) \neq 0$. On $C^\infty(T) = C^\infty_{2\pi}$ we a separting family of seminorms is given by

$$P_j(\varphi) = \|\varphi^{(j)}\|_\infty, \tag{2.2.2}$$

the maximum of the j-th derivative. These seminorms can be used to define a topology on $C^\infty(T)$ by requiring that they are *continuous*, just as weak topologies were introduced in Chapter 1. Section 1.3 Thus, a local subbasis at 0 consists of sets

$$V_{j,\varepsilon} = \{\varphi : P_j(\varphi) < \varepsilon\}, \quad j \in \mathbb{N},\ \varepsilon > 0.$$

The space $C^\infty(T)$ endowed with this topology is called $\mathcal{D}(T)$. From Chapter 1, Section 1.3 we see that

$$f_j \in \mathcal{D}(T) \to f \in \mathcal{D}(T) \iff \forall k \in \mathbb{N}\ f_j^{(k)} \to f^{(k)} \text{ uniformly.} \tag{2.2.3}$$

So it is pretty hard for functions to converge in $\mathcal{D}(T)$, cf. Exercise 2.7.1. However, Fourier series behave nice in $\mathcal{D}(T)$.

LEMMA 2.2.1. *Let $\varphi \in \mathcal{D}(T)$. Then $S_N[\varphi] \to \varphi$ in $\mathcal{D}(T)$ as $N \to \infty$.*

PROOF. Because φ is smooth, $S_N[\varphi] \to \varphi$ uniformly, but also $S_N[\varphi]' = S_N[\varphi'] \to \varphi'$ uniformly. The same is true for higher derivatives. By (2.2.3) we are done. □

LEMMA 2.2.2. *The space $\mathcal{D}(T)$ is metrizable and complete in the metric*

$$d(\varphi, \psi) = \sum_{j=0}^{\infty} 2^{-j} \frac{P_j(\varphi - \psi)}{1 + P_j(\varphi - \psi)}. \tag{2.2.4}$$

PROOF. See Exercise 2.7.2. □

Just as in (2.2.4) any countable separating family of seminorms on a vector space X gives rise to a metric d. Observe that it is *translation invariant*, that is $d(x, y) = d(x - z, y - z)$ for all $x, y, z \in X$. Moreover, one can show that with the induced topology X becomes a topological vector space, i.e addition and scalar multiplication are continuous operations, and it is locally convex, meaning that it has a local basis of convex sets. In particular this holds for T.

However, $\mathcal{D}(T)$ with the present topology cannot be turned into a Banach space because of the following Lemma, which says that $\mathcal{D}(T)$ has the *Heine-Borel property*. A Banach space can only possess this property if it is finite dimensional. Recall that a set X in a topological vector space is *bounded* if it has the property that for every open neighborhood U of 0, there exists $N > 0$ such that $X \subset NU$. If the topology is determined by seminorms, this just means that every seminorm is bounded on X.

LEMMA 2.2.3. *Every bounded sequence in $\mathcal{D}(T)$ has a convergent subsequence; A closed bounded set in $\mathcal{D}(T)$ is compact.*

PROOF. This is a simple consequence of the Arzela-Ascoli Theorem, cf. Exercise 2.7.3. $\square$

It will be convenient to replace P_n by an equivalent set of seminorms

$$\tilde{P}_n = \sum_{j=0}^{n} P_j.$$

This doesn't change the topology, but it has the advantage that a local basis at 0 of $\mathcal{D}(T)$ is now given by $\{\tilde{P}_n(\varphi) < 1/n\}$, $n \in \mathbb{N}$.

REMARK 2.2.4. A vector space with topology induced by a complete invariant metric, like we have met here, is called a *Fréchet space*. (Sometimes local convexity is required, but that can be shown to hold here too.)

2.3. Distributions on T

Let $\mathcal{D}' = \mathcal{D}'(T)$ denote the dual space of $\mathcal{D}(T)$, that is the space of continuous linear functionals on $\mathcal{D}(T)$. The notation is classical. Elements of $\mathcal{D}'$ are called (periodic) *distributions* or *generalized functions*. The space $\mathcal{D}'$ will naturally be equiped with the *weak-* topology*. We denote the action of $L \in \mathcal{D}'(T)$ on $\varphi \in \mathcal{D}(T)$ usually by

$$\langle L, \varphi \rangle \ (= L\varphi).$$

The following lemma is an extension of a familiar result for Banach spaces, cf. [**22**][thm. 3.2].

LEMMA 2.3.1. *Let X be a vector space with a topology induced by a countable set of seminorms $\{p_1, p_2, \ldots\}$ and let L be a linear functional on X. The following are equivalent.*

i. *L is continuous on X,*
ii. *L is continuous at $x_0 \in X$,*
iii. *There exist $C > 0$, $K \in \mathbb{N}$ such that*

$$|Lx| \leq C \max_{i=1,\ldots,K} p_i(x).$$

PROOF. (i $\Longrightarrow$ ii) is trivial.

(ii $\Longrightarrow$ iii). If L is continuous at x_0, then for every $\varepsilon > 0$ there exist $K \in \mathbb{N}$ and $\delta_j > 0$, ($j = 1, \ldots, K$), such that $p_j(x - x_0) < \delta_j$, ($j = 1, \ldots, K$) implies $|L(x - x_0)| < \varepsilon$. Let $\delta = \min_j\{\delta_j\}$ and denote for $y \in X$, $M_y = \max_{i=1,\ldots,K} p_i(y)$. Then

$$|Ly| = |\frac{M_y}{\delta} L(\frac{\delta y}{M_y})| < \frac{M_y}{\delta}\varepsilon,$$

which proves iii, with $C = \varepsilon/\delta$.

(iii $\Longrightarrow$ i). Let $x, y \in X$.

$$|Lx - Ly| = |L(x - y)| < C \max_{i=1,\ldots,K} p_i(x - y).$$

This is less than ε if $p_i(x-y) < \varepsilon/KC$. Thus we described a small neighborhood of x which is mapped in an ε-neighborhood of Lx. $\square$

Apparently a linear functional L on $\mathcal{D}(T)$ is continuous, i.e. a *distribution*, if and only if there exist $n \in \mathbb{N}$ and $C > 0$ such that

$$|\langle L, \varphi \rangle| \leq C\tilde{P}_n(\varphi), \quad \forall \varphi \in \mathcal{D}(T). \tag{2.3.1}$$

The smallest n that is possible in (2.3.1) is called the *order* of the distribution. Of course the zero distribution has order $-\infty$ assigned to it.

As we have seen in Section 1.3.3 the natural notion of convergence for a sequence of distributions is weak-* convergence. To be specific, for $L_j, L \in \mathcal{D}'(T)$ we have

$$\lim_{j\to\infty} L_j = L \text{ if and only if } \forall\varphi \in \mathcal{D}(T) \quad \lim_{j\to\infty} L_j\varphi = L\varphi.$$

Remark 2.3.2. If the underlying space is not compact, e.g. $\mathbb{R}$ instead of T, distributions may have infinite order.

2.3.1. Examples of distributions.

(1) Every $u \in L^1_{2\pi}$ defines a distribution L_u via

$$\langle L_u, \varphi\rangle = \int_{-\pi}^{\pi} u\varphi \frac{dt}{2\pi}.$$

This functional is indeed continuous: $|\langle L_u, \varphi\rangle| \leq \|u\|_1\|\varphi\|_\infty$.

(2) We denote the set of Borel measures on T by $M(T)$. Every measure $\mu \in M(T)$ defines in the same way a distribution L_μ via

$$\langle L_\mu, \varphi\rangle = \int_{-\pi}^{\pi} \varphi \, d\mu(t).$$

Both examples are distributions of order 0.

Abusing the language we will drop the L and identify a function or measure with the associated distribution, writing e.g. $\langle u, \varphi\rangle$.

(3) The delta-distribution $\delta = \delta_0$ is defined by

$$\langle \delta, \varphi\rangle = \varphi(0).$$

The delta distribution originates from point mass at 0.

Remark 2.3.3. The name test functions for the elements of $\mathcal{D}$ is now understandable, these functions are used to test the action of an L^1 function or distribution.

2.4. Operations on Distributions

2.4.1. Differentiation. Let $L \in \mathcal{D}'(T)$. Define its derivative $L' \in \mathcal{D}'(T)$ by

$$\langle L', \varphi\rangle = -\langle L, \varphi'\rangle.$$

Observe that L' is well-defined because $\varphi' \in \mathcal{D}(T)$; L' is also continuous. Indeed, there exist $C > 0$, $n \in \mathbb{N}$ such that $|\langle L, \varphi\rangle| \leq C\tilde{P}_n(\varphi)$. Then

$$|\langle L', \varphi\rangle| = |\langle L, \varphi'\rangle| \leq C\tilde{P}_n(\varphi') \leq C\tilde{P}_{n+1}(\varphi).$$

We conclude that every distribution is infinitely often differentiable. In general, differentiation increases the order of a distribution by 1.

Examples 2.4.1. Let $f(x)$ be the characteristic function of $(0, \pi)$ viewed as element of $L^1_{2\pi}$. We compute its distributional derivative f'. Let $\varphi \in \mathcal{D}(T)$, then

$$\langle f', \varphi\rangle = -\langle f, \varphi'\rangle = -\int_{-\pi}^{\pi} f(t)\varphi'(t)\frac{dt}{2\pi} = -\int_0^{\pi} \varphi'(t)\frac{dt}{2\pi} = \frac{\varphi(0) - \varphi(\pi)}{2\pi}.$$

We conclude that $f' = \frac{\delta_0 - \delta_\pi}{2\pi}$.

The function $f(x) = \log|x|$ is in $L^1_{2\pi}$. Its distributional derivative is determined by

$$\begin{aligned}\langle f', \varphi\rangle = -\langle f, \varphi'\rangle &= -\int_{-\pi}^{\pi} \log|t|\varphi'(t)\frac{dt}{2\pi} = -\lim_{\varepsilon\to 0}\int_{-\pi}^{-\varepsilon} + \int_{\varepsilon}^{\pi} \log|t|\varphi'(t)\frac{dt}{2\pi} \\ &= \frac{-1}{2\pi}\lim_{\varepsilon\to 0}\Big(\log\varepsilon\varphi(-\varepsilon) - \log\pi\varphi(-\pi) - \log\varepsilon\varphi(\varepsilon) + \log\pi\varphi(\pi) - \\ &\qquad \Big(\int_{-\pi}^{-\varepsilon} + \int_{\varepsilon}^{\pi}\frac{\varphi(t)}{t}dt\Big)\Big) = \text{p(rincipal) v(alue)} \int \frac{\varphi(t)}{t}\frac{dt}{2\pi}.\end{aligned} \tag{2.4.1}$$

Here we have used that φ is periodic, making the stock terms cancel in the limit. The last equality is the definition of the *principal value integral.* We have shown that $f' = \text{p.v.}\frac{1}{x}$.

PROPOSITION 2.4.2. *Differentiation is a continuous operation: If $L_j \to L$ in $\mathcal{D}'(T)$ then $L_j' \to L'$ in $\mathcal{D}'(T)$.*

PROOF. For any $\varphi \in \mathcal{D}(T)$, we have $\langle L_j, \varphi\rangle \to \langle L, \varphi\rangle$ by definition of weak-* convergence. Hence $\langle L_j, \varphi'\rangle \to \langle L, \varphi'\rangle$, which by definition of differentiation gives $\langle L_j', \varphi\rangle \to \langle L', \varphi\rangle$. $\square$

2.4.2. Restricted Multiplication. The price one pays for infinite differentiability is that multiplication of distributions is in general not possible. However, if $f \in C^\infty(T)$, $L \in \mathcal{D}'$ then fL can defined by

$$\langle fL, \varphi\rangle = \langle L, f\varphi\rangle.$$

This is indeed continuous: if $\varphi_n \to \varphi$ in $\mathcal{D}(T)$ then $f\varphi_n \to f\varphi$ in $\mathcal{D}(T)$, therefore

$$\langle fL, \varphi_n\rangle = \langle L, f\varphi_n\rangle \to \langle L, f\varphi\rangle = \langle fL, \varphi\rangle.$$

If $\mu \in M(T)$ and $f \in C(T)$, then $f\mu \in M(T)$. Hence multiplication of a continuous function with a distribution associated with a measure or an L^1 function is also possible. More generally one can prove that a distribution of order k can be multiplied with a function in C^k, cf. ex. 2.7.5.

LEMMA 2.4.3 (Product rule). *If $f \in C^\infty(T)$ and $L \in \mathcal{D}'$, then*

$$(fL)' = f'L + fL'.$$

PROOF. Let $\varphi \in \mathcal{D}$. Then

$$\begin{aligned}\langle (fL)', \varphi\rangle &= -\langle fL, \varphi'\rangle = -\langle L, f\varphi'\rangle \\ &= -\langle L, (f\varphi)' - f'\varphi)\rangle = \langle L', f\varphi\rangle + \langle f'L, \varphi\rangle.\end{aligned} \tag{2.4.2}$$

$\square$

2.4.3. Local Equality. The fact that $L^1_{2\pi}$ is identified with a subset of $\mathcal{D}'$ makes it clear that the "value of a distribution in a point" makes no sense. However, on open sets equality makes sense! Recall that the *support* of a continuous function φ is the closed set $\operatorname{Supp}\varphi = \operatorname{cl}\{t : \varphi(t) \neq 0\}$.

DEFINITION 2.4.4. Two distributions L_1, L_2 on T are called *equal on an open subset* $\Gamma \subset T$ if for every $\varphi \in \mathcal{D}(T)$ with support in Γ, one has $\langle L_1, \varphi\rangle = \langle L_2, \varphi\rangle$.

The *support* of a distribution $L \in \mathcal{D}'$ is the complement of the union of the open sets Γ with $L = 0$ on Γ.

EXAMPLES 2.4.5. The support of δ is $\{0\}$. If L originates from a continuous function then the two notions of support coincide.

REMARK 2.4.6. Multiplication can be localized: If $L \in \mathcal{D}'$ has order k, and on an open $\Gamma \subset T$ is equal to a distribution of order $j < k$ then L can be multiplied with C^j functions, that are C^k in a neighborhood of the complement of Γ. See Exercise 2.7.11. Moreover it turns out that (especially in higher dimensions) a further refinement is possible. This is based on local Fourier analysis of the distribution and takes into account the directions in which the singularities occur. We deal with this topic in a later chapter, but to get a flavour of the problem, consider the distribution L on $\mathbb{R}^2$ given by

$$\langle L, \varphi\rangle = \frac{\partial\varphi}{\partial x_1}(0,0).$$

This has order 1, so we can multiply with $f \in C^1$ but a closer analysis gives that we only need to require something like f continuous and differentiable with respect to x_1.

2.5. Fourier Series of Periodic Distributions

In analogy with $L^1_{2\pi}$ we can define for $L \in \mathcal{D}(T)$ the *Fourier coefficients*

$$\hat{L}(n) = \frac{1}{2\pi}\langle L, e^{-int}\rangle.$$

For example, as we know already $\hat{\delta}(n) = 1/2\pi$.

As one expects, we put

$$S_N[L](t) = \sum_{-N}^{N} \hat{L}(n)e^{int}.$$

For example, $S_N[\delta] = D_N$, the Dirichlet kernel. Let $\varphi \in \mathcal{D}(T)$

$$\langle S_N[\delta], \varphi\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{n=-N}^{N} e^{int}\varphi(t)\,dt = D_N * \varphi(0) \to \varphi(0), \quad \text{as } N \to \infty.$$

We see that the Dirichlet kernels tend weakly to δ.

Lemma 2.5.1. *If $L \in \mathcal{D}'$ and $\varphi \in \mathcal{D}(T)$ then*

$$\langle L, \varphi\rangle = 2\pi \sum_{-\infty}^{\infty} \hat{L}(-n)\hat{\varphi}(n).$$

Proof. Using Lemma 2.2.1 we see that

$$\langle L, \varphi\rangle = \lim_{N\to\infty}\langle L, S_N[\varphi]\rangle = \lim_{N\to\infty} 2\pi \sum_{-N}^{N} \hat{\varphi}(n)\hat{L}(-n).$$

This proves the Lemma. □

Theorem 2.5.2. *If $L \in \mathcal{D}'$, then the Fourier series of L tends weak-* to L.*

Proof. For every $\varphi \in \mathcal{D}(T)$ we find by Lemma 2.5.1, if $N \to \infty$,

$$\langle S_N[L], \varphi\rangle = 2\pi \sum_{-N}^{N} \hat{L}(n)\hat{\varphi}(-n) \to \langle L, \varphi\rangle.$$

□

Corollary 2.5.3. *The map $L \to \hat{L}$ is injective on $\mathcal{D}'$.*

Proof. If $\hat{L}(n) = 0$ for all $n \in \mathbb{Z}$, then $L = 0$ by Theorem 2.5.2. □

Corollary 2.5.4. *Let $L \in \mathcal{D}'$. If $L' = 0$ on T, then L is a constant.*

Proof. The Fourier coefficients of L' are $0 = \widehat{L'}(n) = in\hat{L}(n)$. It follows that the Fourier series of L consists only of the constant term. Theorem 2.5.2 gives that L is a constant. □

Theorem 2.5.5. *The following are equivalent.*

i. $\sum_{n=-\infty}^{\infty} c_n e^{int}$ *is the Fourier series of a periodic distribution.*

ii. *There exist constants N, C such that $|c_n| \le Cn^N$.*

Proof. If $\sum_{n=-\infty}^{\infty} c_n e^{int}$ is the Fourier series of a periodic distribution L then $\sum_{n=-N}^{N} c_n e^{int}$ tends weakly to L by Theorem 2.5.2. It follows that for every k, if $N > k$

$$\langle \sum_{-N}^{N} c_n e^{int} - L, e^{-ikt}\rangle = c_k - \hat{L}(k)$$

tends to 0 if $N \to \infty$. Hence $c_k = \hat{L}(k)$. Now L has finite order, say N. Then $|c_n| = |\langle L, e^{-int}\rangle| \le C\tilde{P}_N(e^{-int}) \le CNn^N$.

In the other direction, if $|c_n| \le Cn^N$, then $(c_n/n^{N+1})_n \in l^2(\mathbb{Z})$. Parseval gives

$$f(t) = \sum_{-\infty}^{\infty} \frac{c_n}{n^{N+1}} e^{int} \in L^2(T).$$

It follows that $f^{(N+1)}$ is a distribution of at most order $N+1$. It has Fourier coefficients $i^{N+1}c_n$ for $n \neq 0$. Dividing by i^{N+1} and adding c_0 we have found a distribution with the prescribed Fourier series. □

COROLLARY 2.5.6. *Every periodic distribution is of the form $F^{(N)}+c$ with F continuous (or in L^2).*

PROOF. This is in the proof of the second part of Theorem 2.5.2 for F in L^2. If F is not continuous we consider

$$F(t) = \sum_{-\infty}^{\infty} \frac{c_n}{n^{N+1}} e^{int} \in C(T),$$

because the series is uniformly convergent and proceed as in the last part of the proof of the Theorem. □

2.6. Convolution and Multiplication

From [**13**] we know that if $f, g \in C_{2\pi}$, then $f * g := \int_{-\pi}^{\pi} f(x-y)g(y)\frac{dy}{2\pi}$ and fg are also in $C_{2\pi}$ and have Fourier series given by

$$(f*g)\hat{}(n) = \iint f(x-y)g(y)e^{-in(x-y)}e^{-iny}\frac{dy}{2\pi}\frac{dx}{2\pi} = \hat{f}(n)\hat{g}(n) \tag{2.6.1}$$

and

$$(fg)\hat{}(n) = \sum_{-\infty}^{\infty} \hat{f}(n-j)\hat{g}(j) =: \hat{f} * \hat{g}(n). \tag{2.6.2}$$

2.6.1. We can use (2.6.1) to define the convolution $L_1 * L_2$ for periodic distributions $L_i \in \mathcal{D}$:

$$L_1 * L_2 \overset{\text{def}}{=} \sum_{-\infty}^{\infty} \hat{L}_1(n)\hat{L}_2(n)e^{inx}. \tag{2.6.3}$$

Application of Theorem 2.5.5 gives that there exist $C, k > 0$ such that $|\hat{L}_1(n)\hat{L}_2(n)| < C|n|^k$, $(n \neq 0)$, and another application of Theorem 2.5.5 shows that the series represents a distribution.

Formula (2.6.2) shows once more why it is difficult to multiply distributions: The Fourier coefficients $(fg)\hat{}(n)$ have to be finite. Suppose f is a distribution of order k. Then $\hat{f}(n)$ behaves like n^k. For (2.6.2) to converge, the $\hat{g}(n)$ have to be something like n^{-k}, i.e. g is fairly smooth.

2.7. Exercises

2.7.1. Prove that, as $n \to \infty$, the sequence $\sin(nt)/n^3$ converges to 0 uniformly, but it does not converge in $\mathcal{D}(T)$.

2.7.2. Prove that (2.2.4) defines a metric and complete the proof of Lemma 2.2.2.

2.7.3. Recall the Arzela-Ascoli theorem: If $\{f_\alpha\}_\alpha$ is an equicontinuous family of pointwise bounded continuous functions on a separable compact metric space, then $\{f_\alpha\}_\alpha$ has a uniformly convergent subsequence. Equicontinuous means

$$\forall\varepsilon\ \exists\delta \text{ such that } |x-y| < \delta \implies |f_\alpha(x) - f_\alpha(y)| < \varepsilon,$$

independent of α.

Prove the following. If $\{f_\alpha\}_\alpha$ is bounded in the C^1-norm on T, that is, the norm $\|\cdot\|_\infty + \|\cdot'\|_\infty$ on T, then $\{f_\alpha\}_\alpha$ is equicontinuous and pointwise bounded. (Apply the mean value theorem.)

2.7.4. Prove the continuity in example 2.3.1 by showing that $\varphi_n \to \varphi$ in $\mathcal{D}(T)$ implies $\langle L_u, \varphi_n\rangle \to \langle L_u, \varphi\rangle$.

2.7.5. Let $L \in \mathcal{D}'(T)$ be of order k and let $f \in C^l(T)$. For which k and l can you give a definition of fL by using (2.6.1) ?

2.7.6. Compute the distribution sum of

$$\sum_{-\infty}^{\infty} ne^{inx}, \tag{2.7.1}$$

$$\sum_{n=1}^{\infty} \sin nx.$$

2.7.7. Compute the distributional derivative of

$$f(x) = \begin{cases} x+\pi & \text{if } -\pi < x < 0, \\ x-\pi & \text{if } 0 < x < \pi. \end{cases} \tag{2.7.2}$$

2.7.8. Let $\varphi \in \mathcal{D}(T)$, $L \in \mathcal{D}'(T)$. Put $\varphi^x(y) = \varphi(x-y)$. Show that

$$L * \varphi(x) \stackrel{\text{def}}{=} \langle L, \varphi^x\rangle$$

is well-defined and coincides with definition (2.6.3). Prove that $L * \varphi$ is a smooth function.

2.7.9. Let $L \in \mathcal{D}(T)'$ have support $\{0\}$.

(1) Use Corollary 2.5.6 to express L as the j'th derivative of a continuous function f on $(-\pi, \pi)$.
(2) Show that there are polynomials P_1, P_2 of degree at most $j-1$ such that $f = P_1$ if $x > 0$ and $f(x) = P_2(x)$ if $x < 0$.
(3) Conclude that we may take $f = PU$ with P a polynomial of degree $j-1$ and U the characteristic function of $(0, \pi)$.
(4) Prove that L is a finite linear combination of derivatives of δ.

2.7.10. For $L \in \mathcal{D}'(T)$, let L_h be defined by $L_h\varphi := L\varphi(.-h)$. Compute the distributional limit

$$\lim_{h\to 0} \frac{L_h - L}{h}.$$

2.7.11. Suppose that $L \in \mathcal{D}'$ has order k, and on an open $\Gamma \subset T$ is equal to a distribution $\tilde{L}$ of order $j < k$. Prove that L can be multiplied with any C^j function F, that is C^k in a neighborhood of the complement of Γ. [First prove the result for $L = 0$ on Γ. Then write $L = (L - \tilde{L}) + \tilde{L}$.]

CHAPTER 3

Distributions on $\mathbb{R}^n$

As before, distributions on $\mathbb{R}^n$ will be continuous linear functionals on test spaces of smooth functions. However, because $\mathbb{R}^n$ is not compact, there is some freedom in the choice of the test space, which leads to different classes of distributions. Also the topology on the test space is more complicated. Therefore we will base our treatment on the semi-norm approach, (2.3.1) of the previous chapter. In particular this makes it clear that advanced knowledge of topological vector spaces is not necessary to work fruitfully with distributions.

We will mention some useful facts about smooth functions in Section 3.1. The proofs will be rather sketchy. Details may be found in [**7**]. In what follows U will be open in $\mathbb{R}^n$ and K will be compact in $\mathbb{R}^n$. Variables in $\mathbb{R}^n$ will generally be written as $x = (x_1, \dots, x_n)$, and $dx = dx_1\, dx_2 \dots dx_n$ will denote the corresponding Lebesgue measure on $\mathbb{R}^n$. Also $B(x, r)$ will be the open ball with radius r centered at x. We write $C_0^k(X) \subset C^k(X)$ for the class of k-times differentiable functions that are supported on compact subsets of X; k may of course be ∞. The space $C_0^\infty(U)$ will be our *test space* and following Schwartz we denote it by $\mathcal{D}(U)$ or $\mathcal{D}$ if no confusion is possible. As usual the elements of $\mathcal{D}(U)$ are called *test functions.* The reader is warned that this notation is **not** the same as in [**13**]. Differential operators will be denoted using multi-index notation: with $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we put $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$, it is a partial derivative of order $|\alpha| = \alpha_1 + \dots + \alpha_n$; also $\alpha! := \alpha_1!\alpha_2! \dots \alpha_n!$.

3.1. Smooth Functions

Lemma 3.1.1. *Let $p \in U$. There exists a non-negative test function φ on $\mathbb{R}^n$ with $\varphi(p) > 0$ and $\operatorname{Supp}\varphi$ is a compact subset of U.*

Proof. Without loss of generality, $p = 0 \in \operatorname{cl} B(0, 1) \subset U$. Let

$$f(t) = \begin{cases} e^{-1/t} & \text{if } t > 0, \\ 0 & \text{elsewhere.} \end{cases} \tag{3.1.1}$$

The function f is smooth on $\mathbb{R}$. Now take $\varphi(x) = f(1 - \|x\|^2)$ on $\mathbb{R}^n$, this function is smooth and its support is $\operatorname{cl} B(0, 1)$ □

Lemma 3.1.2. *There exists a smooth, compactly supported approximate identity on $\mathbb{R}^n$.*

Proof. Let φ be as in Lemma 3.1.1 with $p = 0$. Multiplying with a positive constant if necessary, we can assume that $\int_U \varphi(x)\, dx = 1$. Let

$$\varphi_\varepsilon(x) \stackrel{\text{def}}{=} \frac{1}{\varepsilon^n}\varphi(\frac{x}{\varepsilon}).$$

Then $\varphi_\varepsilon \geq 0$, $\int \varphi_\varepsilon = 1$ and $\varphi_\varepsilon \to 0$ uniformly on $\|x\| \geq \delta > 0$, hence $\{\varphi_\varepsilon\}$ is an approximate identity consisting of compactly supported functions. □

Theorem 3.1.3. *Let $f \in C_0^k(U)$ and let $\{\varphi_\varepsilon\}$ be an approximate identity consisting of compactly supported smooth functions. Then $f * \varphi_\varepsilon$ is C^∞ and for $0 \leq |\alpha| \leq k$ the convolutions $D^\alpha f * \varphi_\varepsilon$ will tend to $D^\alpha f$ uniformly on U if $\varepsilon \to 0$.*

Proof. Writing $f * \varphi_\varepsilon(x) = \int f(y)\varphi_\varepsilon(x - y) dy$, an application of Lebesgue's dominated convergence theorem allows us to differentiate under the integral sign infinitely often. On the other hand we may write $f * \varphi_\varepsilon(x) = \int f(x-y)\varphi_\varepsilon(y)dy$ which yields that for $|\alpha| \le k$, $D^\alpha(f*\varphi_\varepsilon) = (D^\alpha f) * \varphi_\varepsilon$. Hence it suffices to prove that $f * \varphi_\varepsilon \to f$ uniformly if $f \in C_0(U)$. Indeed,

$$|f * \varphi_\varepsilon - f(x)| = |\int (f(x-y) - f(x))\varphi_\varepsilon(y)\,dy| \le \delta \int \varphi_\varepsilon \le \delta,$$

if ε is so small that $|f(x) - f(x-y)| < \delta$ for $y \in \operatorname{Supp}\varphi_\varepsilon$. $\square$

Lemma 3.1.4. *Let K be compact in $\mathbb{R}^n$, $\varepsilon > 0$ and let $K_\varepsilon = \{y : d(y,K) < \varepsilon\}$. There exists $\psi_\varepsilon \in C_0^\infty(K_\varepsilon)$ such that $\psi_\varepsilon = 1$ on K. Moreover the derivatives satisfy*

$$|D^\alpha \psi_\varepsilon(x)| \le C_\alpha \varepsilon^{-|\alpha|}, \tag{3.1.2}$$

for certain constants C_α depending on α but not on ε.

Proof. Let $\{\varphi_\varepsilon\}$ be the compactly supported approximate identity constructed in Lemma 3.1.2. Then φ_ε is supported in $B(0,\varepsilon)$ and its derivatives satisfy $D^\alpha(\varphi_\varepsilon)(x) = \frac{1}{\varepsilon^{|\alpha|}}(D^\alpha\varphi)(\frac{x}{\varepsilon})$. Let χ be the characteristic function of $K_{\varepsilon/2}$ and set

$$\psi_\varepsilon = \chi * \varphi_{\varepsilon/2}.$$

As in the proof of Theorem 3.1.3 we obtain that ψ_ε is smooth. It is obviously supported in K_ε and its derivatives are $\psi * D^\alpha(\varphi_{\varepsilon/2})$ which is bounded by a $\varepsilon^{-|\alpha|}C_\alpha$ by our previous estimate. $\square$

We have used the straightforward but noteworthy fact that for L^1 functions f and g, $\operatorname{Supp}(f*g) \subset \operatorname{Supp} f + \operatorname{Supp} g$, where for $A, B \subset \mathbb{R}^n$ we write $A+B = \{a+b : a \in A,\ b \in B\}$.

Lemma 3.1.5. *Every open U can be written as $U = \cup_1^\infty K_j$ with K_j compact and K_j a subset of the interior of K_{j+1}. In fact we can make the convenient choice*

$$K_j = \{x \in U : \|x\| \le j \text{ and } d(x, \partial U) \ge 2^{-j}\}. \tag{3.1.3}$$

Proof. Immediate. $\square$

Definition 3.1.6. A *partition of unity* of an open set U subordinate to an open cover $\mathcal{U} = \{U_\tau\}_\tau$ of U, (i.e., $U = \cup_\tau U_\tau$), is a countable set of nonnegative smooth functions $\{\varphi_j,\ j = 1,\dots\}$ on $\mathbb{R}^n$ with the following properties:

1. $\forall j \exists \tau_j$ such that $\operatorname{Supp}\varphi_j \subset U_{\tau_j}$.
2. If K is a compact subset of U, then only finitely many of the φ_j are non-zero on K.
3. On U we have $\sum_j \varphi_j \equiv 1$.

Notice that the sum in 3. is well-defined because on every compact K we only have to take into account the finitely many φ_j that are positive on K. We keep the notation of the definition.

Theorem 3.1.7. *For every open cover $\mathcal{U}$ of U, there exists a partition of unity subordinate to $\mathcal{U}$.*

Proof. First remark that every compact $K \subset U$ is contained in one of the K_j of Lemma 3.1.5. Thus we only have to check 2. of Definition 3.1.6 for these K_j.

The important observation is as follows. If $x \notin K_j$, then $d(x,\partial U) < 2^{-j}$ or $\|x\| > j$. In the first case, $d(x, K_{j-1}) > 2^{-j}$, in the latter case $d(x, K_{j-1}) > 1$. The ball $B(x,r)$ with $r = \frac{1}{2}\min\{1, d(x,\partial U)\}$ will have empty intersection with K_{j-1}.

We now define for $x \in U_\tau$ the ball $B_\tau(x) = B(x,t)$ with $t = \frac{1}{2}\min\{1, d(x, \partial U_\tau)\}$. By Lemma 3.1.1 we can find non-negative $\varphi_{\tau,x} \in \mathcal{D}(B_\tau(x))$ with $\varphi_{\tau,x}(x) = 1$. Keeping in mind that $\mathcal{U}$ covers U we introduce for every $x \in U$ (at least one) open neighborhood $D_\tau(x) = \{y \in B_\tau(x) : \varphi_{\tau,x}(y) > 1/2\}$.

We will select a good covering of U. First $K_1 \subset \cup_{x\in K_1} D_\tau(x)$. We can find a finite subcover labeled $D_{1,1},\dots D_{1,j_1}$. Next, $K_2' = K_2 \setminus (\cup_{j=1}^{j_1} D_{1,j})$ is compact; again $K_2' \subset \cup_{x \in K_2'} D_\tau(x)$. We can find a finite subcover labeled $D_{2,1},\dots D_{2,j_2}$. The $D_{1,j}$ and $D_{2,j}$ cover

K_2. Continuing in this way, covering each time $K_i' = K_i \setminus \cup_{l<i} D_{l,j}$ with finitely many $D_{i,j}$, and relabeling, we obtain a countable covering $D_1, D_2, \ldots$ of U. Call the corresponding $B_\tau(x)$'s: $B_1, B_2, \ldots$ and the corresponding $\varphi_{\tau,x}$'s: $\varphi_1, \varphi_2, \ldots$.

Now every φ_i has support in some $B_i = B_\tau(x) \subset U_\tau$, therefore it satisfies 1. For condition 2 notice that there are only finitely many B_i of the form $B(x)$ with x in K_j. By our first observation, if $x \notin K_j$ then $B(x) \cap K_{j-1} = \emptyset$. Thus for every j, K_{j-1} is covered by finitely many B_i and condition 2 is satisfied.

To finish the proof we have to modify φ_i a little. Observe that

$$\varphi = \sum_1^\infty \varphi_i$$

is a well-defined smooth function on U, because on compact sets only finitely many terms are non-zero, and since every $y \in U$ is in some D_i, the sum is at least $1/2$. Now introduce

$$\psi_i = \frac{\varphi_i}{\varphi}.$$

Clearly $\psi_i \in \mathcal{D}(U_\tau)$ and the ψ_i satisfy 1, 2, and 3 and form a partition of unity. □

COROLLARY 3.1.8. *For every $\psi \in C_0^\infty(U)$ and every cover $\mathcal{U}$ of U there exists a finite subcollection U_i of $\mathcal{U}$ such that we can write $\psi = \sum_{i=1}^N \psi_i$ with $\psi_i \in \mathcal{D}(U_i)$.*

PROOF. Let K be the support of ψ. Let φ_j, $j = 1, \ldots N$ be the finitely many φ's that are non zero on K. Then $\psi\varphi_j \in \mathcal{D}(\mathbb{R}^n)$ has compact support in $U_{\tau_j} \cap K$. The U_{τ_j} constitute a finite set $\{U_i' \; i = 1, \ldots N\}$. We have

$$\psi = \psi 1 = \sum_j \psi\varphi_j, \quad \text{on } K.$$

Assembling the φ_j which have support in the same U_i, i.e., setting $\psi_i = \sum_{\tau_j = i} \varphi_{\tau_j}$, we are done. □

REMARK 3.1.9. It is possible to put a topology on the test space $\mathcal{D}(U)$ that makes it a complete locally convex topological vector space. This is not so easy, because we have tried to keep $\mathcal{D}(U)$ as small as possible in order to have as many continuous linear functionals as possible on it. However, occasionally we don't need so many distributions and deal with two "easier" spaces of test functions.

We introduce seminorms $Q_{K,\alpha}$ on $C^\infty(U)$ as follows

$$Q_{K,\alpha}(\varphi) = \sup_{x \in K} |D^\alpha \varphi|, \tag{3.1.4}$$

where K runs over the compacts in U and α over all multiindices. $C^\infty(U)$ equipped with the topology induced by $Q_{K,\alpha}$ is denoted by $\mathcal{E}(U)$ or simply $\mathcal{E}$.

3.1.1. The Schwartz Class. In [**13**] the class $\mathcal{S}$ of Laurent Schwartz was introduced on $\mathbb{R}$. For $\mathbb{R}^n$ the definition is similar

$$\mathcal{S} = \mathcal{S}(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) : \text{ for every } \alpha \in \mathbb{N}^n \text{ and } k \in \mathbb{N},\ \|x\|^k |D^\alpha(f)| \text{ is bounded}\}. \tag{3.1.5}$$

$\mathcal{S}$ is topologized by the seminorms $P_{k,\alpha}$ defined by

$$P_{k,\alpha}(f) = \sup_{\mathbb{R}^n} \|x\|^k |D^\alpha(f)|, \quad a \in \mathbb{N}^n,\ k \in \mathbb{N}. \tag{3.1.6}$$

We say that $f \in C^\infty$ is rapidly decreasing if for every k and every α, $|D^\alpha(f(x))| \|x\|^k$ tends to 0 if $\|x\| \to \infty$. Replacing k by $k + k'$ in the definition we see that in fact $\mathcal{S}$ is the collection of rapidly decreasing functions.

PROPOSITION 3.1.10. *$\mathcal{D}(\mathbb{R}^n)$ is a dense subspace of $\mathcal{S}$.*

PROOF. Let $f \in \mathcal{S}$. We have to show that given $k, l \in \mathbb{N}$, $\varepsilon > 0$, there exists $\varphi \in \mathcal{D}$ with $P_{k,\alpha}(f - \varphi) < \varepsilon$ for $|\alpha| \le l$. Let $\psi \in \mathcal{D}$, $0 \le \psi \le 1$, with $\psi = 1$ on $B(0,1)$. Put $\psi_r(x) = \psi(x/r)$, then $\psi_r = 1$ on $B(0,r)$ and $|D^\alpha(\psi_r))| \to 0$ for $1 \le |\alpha| \le k$ if $r \to \infty$. Put $\varphi_r = f\psi_r$. Also $(1 - \psi_r)f \to 0$ if $r \to \infty$. We have

$$D^\alpha(f - \varphi_r) = \sum_{p+q=\alpha} c_{p,q} D^p(f) D^q(1 - \psi_r),$$

for suitable (fixed) constant $c_{p,q}$ and multiindices p and q. Using that $D^p(f) = O(1/\|x\|^k)$ and that for $|q| > 1$ all $D^q(1-\psi_r)$ can be made arbitrary small by choosing r large enough, while in case $|q| = 0$ this can be done on compact sets, we can bound $D^\alpha(f - vf)$ by $\varepsilon\|x\|^{-k}$. The proposition follows. □

PROPOSITION 3.1.11. *$\mathcal{D}$ is dense in $\mathcal{E}$.*

We leave the proof as an exercise.

3.2. Distributions

DEFINITION 3.2.1. A *distribution* L on U is a linear functional on $\mathcal{D}(U)$ such that for every K compact in U there exist C, N such that

$$|L\varphi| = |\langle L, \varphi\rangle| \le C \max_{|\alpha|<N} \|D^\alpha(\varphi)\|, \quad \text{for } \varphi \in \mathcal{D}(U) \text{ supported in } K. \tag{3.2.1}$$

The set of distributions on U is denoted by $\mathcal{D}'(U)$. If N can be chosen independent of K, the smallest N such that (3.2.1) holds for all K is called the *order* of L.

We give $\mathcal{D}'(U)$ the weak-* topology, that is,

$$L \mapsto L\varphi$$

is a continuous map for every $\varphi \in \mathcal{D}(U)$. Equivalently $L_j \to L$ in $\mathcal{D}'(U)$ if and only if for every test function φ we have

$$L_j\varphi \to L\varphi. \tag{3.2.2}$$

LEMMA 3.2.2. *Let φ and $(\varphi_j)_j$ be in $\mathcal{D}(U)$ and have support in a compact set $K \subset U$. Suppose that for every α, $D^\alpha\varphi_j \to D^\alpha\varphi$ uniformly on K. Then for every $L \in \mathcal{D}(U)$ we have $L\varphi_j \to L\varphi$.*

PROOF. $L(\varphi - \varphi_j) \to 0$ in view of (3.2.1). □

Differentiation, local equality and support of distributions are defined just like in Chapter 2. E.g.,

$$\langle D^\alpha L, \varphi\rangle \overset{\text{def}}{=} (-1)^{|\alpha|}\langle L, D^\alpha\varphi\rangle, \quad L \in \mathcal{D}', \ \varphi \in \mathcal{D}.$$

EXAMPLES 3.2.3. Let δ_j denote point mass at $j \in \mathbb{R}$. The following distributional limit exists $L = \sum_{j=1}^\infty \delta_j^{(j)}$. It has infinite order.

Let $u_t(x) = t^N e^{ixt}$, $x \in \mathbb{R}$, $N \in \mathbb{N}$, then as distributions the limit for $t \to \infty$ exists: using integration by parts,

$$\langle u_t, \varphi\rangle = \int_{\mathbb{R}} t^N e^{itx}\varphi(x)\,dx = i^{N+1}t^{-1}\int_{\mathbb{R}} e^{itx}\varphi^{(N+1)}(x)\,dx \to 0, \quad \text{as } t \to \infty, \quad \varphi \in \mathcal{D}.$$

In addition we define

DEFINITION 3.2.4. The *singular support* of $L \in \mathcal{D}'$ is the complement (in U) of the union of the open sets where L is equal to a C^∞-function.

THEOREM 3.2.5. *Let $V \subset \mathbb{R}^m$ be open. Let $L \in \mathcal{D}'(U)$. If $\varphi \in C^\infty(U \times V)$ is such that $\varphi(\cdot, v)$ has support inside a compact $K \subset U$ for all $v \in V$, then $L\varphi(\cdot, v) \in C^\infty(V)$. If $\varphi \in \mathcal{D}(U \times V)$, then $L\varphi(\cdot, v) \in \mathcal{D}(V)$.*

PROOF. We write down the second order Taylor development of φ with respect to the second variable at $v \in V$:

$$\varphi(u, v+h) = \varphi(u,v) + \nabla_v \varphi(u,v)h + \psi(u,v,h),$$

where, by compactness of K for every α

$$\max_{u \in K} |D_u^\alpha \psi(u,v,h)| < C_\alpha \|h\|^2. \tag{3.2.3}$$

We conclude

$$\frac{L\varphi(\cdot, v+h) - L\varphi(\cdot, v) - L(\nabla_v \varphi(\cdot, v))h}{\|h\|} = \frac{1}{\|h\|}|L\psi(\cdot, v, h)|. \tag{3.2.4}$$

Using that $|L\psi(\cdot, v, h)|$ is bounded by a constant times the maximum over finitely many α's of left hand sides of (3.2.3), we see that (3.2.4) tends to 0 with $\|h\|$. We conclude that $L\varphi$ is C^1, its partial derivatives (in V) are given by $LD_j\varphi(\cdot, v)$. By what we have proved, they are C^1! Continuing in this way we obtain the first statement of the theorem. For the second statement, if $\varphi \in \mathcal{D}(U \times V)$ then $\varphi(\cdot, v)$ will vanish identically for v outside a suitable compact set in V, hence so will $L\varphi(\cdot, v)$. □

THEOREM 3.2.6. *Let $V \subset \mathbb{R}^m$ and let $\varphi = \varphi(u,v) \in \mathcal{D}(U \times V)$. If $L \in \mathcal{D}'(U)$ then $L \int \varphi(\cdot, v)\, dv = \int L\varphi(\cdot, v)\, dv$.*

PROOF. We use Riemann sums. For every α we have

$$\int D_u^\alpha \varphi(u,v)\, dv = \lim_{h \to 0} \sum_{t \in \mathbb{Z}^m} D_u^\alpha \varphi(u, ht) h^m,$$

uniformly on a compact set in U, because of compactness. Thus Lemma 3.2.2 implies that

$$L \int \varphi(\cdot, v)\, dv = L\left(\lim_{h\to 0} \sum_{t \in \mathbb{Z}^m} \varphi(\cdot, ht)h^m\right) = \lim_{h \to 0} \sum_{t \in \mathbb{Z}^m} L(\varphi(\cdot, ht))h^m = \int L\varphi(\cdot, v)\, dv.$$

□

THEOREM 3.2.7. *Every distribution $L \in \mathcal{D}'(U)$ of order k can be extended to a linear form on $C_0^k(U)$, in such a way that* (3.2.1) *holds for all $\varphi \in C_0^k$.*

PROOF. Let $f \in C_0^k(U)$. By Theorem 3.1.3 there exists a sequence of functions $\varphi_j \in \mathcal{D}(U)$ that approximate f in the sense that the partial derivatives of $f - \varphi_j$ of order $\leq k$ tend to 0 uniformly. It follows that for $|\alpha| \leq k$, $D^\alpha(\varphi_j)$ form a Cauchy sequence in the uniform norm on U. The same is then true for $L\varphi_j$, because of the estimate (3.2.1). By usual arguments this implies that

$$Lf = \lim_{j \to \infty} L\varphi_j.$$

is well-defined, independent of the choice of φ_j, and that the estimate (3.2.1) also holds for f. □

COROLLARY 3.2.8. *A distribution of order 0 is a complex Borel measure.*

PROOF. Recalling that complex Borel measures on U are precisely the linear forms that satisfy (3.2.1) for all $\varphi \in C_0(U)$, this is immediate from the theorem. □

A distribution $L \in \mathcal{D}'(U)$ is called *positive* if $L\varphi > 0$ for every non-negative test function φ.

THEOREM 3.2.9. *Positive distributions are measures.*

Proof. Let K be compact in U and let $0 \le \psi \le 1$ be a test function that equals 1 on K. If $\varphi \in C_0^\infty(K)$ is real valued, then

$$\|\varphi\|_\infty \psi \pm \varphi \ge 0.$$

Applying L to these relations, it follows that for real valued test functions supported in K

$$|L\varphi| \le \|\varphi\|_\infty L\psi. \tag{3.2.5}$$

For complex valued testfunctions we write $\varphi = \varphi_1 + i\varphi_2$. Applying (3.2.5) we obtain in this case

$$|L\varphi| \le 2\|\varphi\|_\infty L\psi, \quad \varphi \in \mathcal{D}(K).$$

This means that L has order 0 and thus it is a measure. □

3.3. Distributions with Compact Support

Recall that we introduced $\mathcal{E}$ as the space of smooth functions on U equipped with the topology determined by (3.1.4). Its dual space is denoted by $\mathcal{E}'(U)$. As in Chapters 1 and 2 we see that $L \in \mathcal{E}'$ if and only if there exist a compact set K in U, and constants C, N with

$$|L\varphi| \le C \max_{x\in K,\ |\alpha|\le N} |D^\alpha(\varphi)|, \quad (\varphi \in \mathcal{E}). \tag{3.3.1}$$

Theorem 3.3.1. *Let $L \in \mathcal{D}'(U)$ be compactly supported with* $\operatorname{Supp} L = K$*. There exists precisely one linear functional $\tilde{L}$ on $C^\infty(U)$ with the following properties:*

i. *$\tilde{L} = L$ on $\mathcal{D}(U)$;*
ii. *$\tilde{L}\varphi = 0$ for all φ in C^∞ with support in the complement of K.*

Moreover, $\tilde{L} \in \mathcal{E}'(U)$.

Proof. Let $0 \le u \le 1 \in \mathcal{D}$ be identically 1 on a neighborhood of K and let $S = \operatorname{Supp} u$. We define

$$\tilde{L}\varphi = L(u\varphi), \quad \varphi \in \mathcal{E}.$$

Observe that for $\varphi \in \mathcal{D}$ we have

$$L\varphi = L(u\varphi) + L((1-u)\varphi) = \tilde{L}(\varphi),$$

because $(1-u)\varphi \in \mathcal{D}$ has its support in the complement of K, thus is annihilated by L. Next if $\varphi \in \mathcal{E}$ and $\operatorname{Supp}\varphi$ is in the complement of K, then $u\varphi$ has compact support in the complement of K, so $L(u\varphi) = 0$.

Suppose L_1 is another linear functional satisfying i. and ii. Let $\varphi \in \mathcal{E}$. We write $\varphi = u\varphi + (1-u)\varphi$. The first term in the sum is in $\mathcal{D}$, the second has support in the complement of K, hence is annihilates by L_1. Then

$$L_1\varphi = L_1(u\varphi + (1-u)\varphi) = L_1(u\varphi) + L_1((1-u)\varphi) = L(u\varphi) + 0 = \tilde{L}\varphi.$$

Finally observe that by (3.2.1)

$$|\tilde{L}\varphi| = |L(u\varphi)| \le C \max_{x\in S,\ |\alpha|\le N} |D^\alpha(u\varphi)|. \tag{3.3.2}$$

This implies by Leibnitz rule that

$$|\tilde{L}\varphi| \le C' \max_{x\in S,\ |\alpha|\le N} |D^\alpha(\varphi)|,$$

where the norms of the derivatives of the (fixed) function u are absorbed in C'. In other words, $\tilde{L} \in \mathcal{E}'$. □

The proof of Theorem 3.3.1 shows that for a distribution supported on a compact set K, we can take the maximum in (3.3.1) over an arbitrarily small compact neighborhood of K. Can't we just take K? The following example shows that the answer is in general No. It is known that the supports for which the answer is Yes, are finite unions of compact connected sets such that any two points x, y in the same component can be joined by a rectifiable arc of length $C|x-y|$, cf. [**7**], Th. 2.3.10.

EXAMPLE 3.3.2. Consider the sequence X with elements $x_n = 1/n^3$ in $\mathbb{R}$. Let $K = X \cup \{0\}$ be the closure of X. Define $L \in \mathcal{D}'(\mathbb{R})$ by

$$\langle L, \varphi\rangle = \sum_{n=1}^{\infty} n(\varphi(x_n) - \varphi(0)). \tag{3.3.3}$$

Observe that L is compactly supported (on K). Let us show that it has order 1. By the mean value theorem $\varphi(x_n) - \varphi(0) = 1/n^3\varphi'(\xi_n)$ for certain $\xi_n \in (0, x_n)$. We substitute this in (3.3.3) and find

$$|\langle L, \varphi\rangle| \leq C \max_{0\leq x\leq 1} |\varphi'(x)|. \tag{3.3.4}$$

However, we can not replace this maximum by a maximum over K, even if we were willing to increase the order of L. To see this, let $\varphi_m \in \mathcal{D}(\mathbb{R})$ satisfy

$$\varphi_m(x) = \begin{cases} 0 & \text{for } x \leq x_{m+1}, \\ 1 & \text{for } x_m \leq x \leq 2. \end{cases} \tag{3.3.5}$$

We have $|\varphi_m| \leq 1$ on K and all derivatives of φ vanish on K. If (3.3.4) would hold with the max taken over K, then it would follow that $\langle L, \varphi_m\rangle$ were uniformly bounded. However, $\langle L, \varphi_m\rangle = m$, a contradiction!

REMARK 3.3.3. This example easily generalizes to more general compacts having infinitely many connected components.

THEOREM 3.3.4. *Let $L \in \mathcal{E}'$ be of order k and supported on a compact K. Suppose $\varphi \in \mathcal{E}$ has the property that $D^\alpha\varphi \equiv 0$ on K for all $|\alpha| \leq k$. Then $L\varphi = 0$.*

PROOF. Let $u_\varepsilon \in \mathcal{D}$ equal 1 on K and 0 outside K_ε as constructed in Lemma 3.1.4. Then in view of Theorem 3.3.1 we have $L\varphi = Lu_\varepsilon\varphi$ for all ε. There exist C and k such that

$$|Lu_\varepsilon\varphi| \leq C \max_{x\in K_\varepsilon,\ |\alpha|\leq k} |D^\alpha(u_\varepsilon\varphi)|.$$

Using (3.1.2) we see that

$$|D^\alpha(U_\varepsilon\varphi)| \leq C'_\alpha \sum_{|\beta|\leq|\alpha|} \varepsilon^{|\beta|-|\alpha|}|D^\beta\varphi|.$$

Therefore, it suffices to show that for all $|\beta| \leq k$

$$\varepsilon^{|\beta|-k}D^\beta\varphi \to 0 \quad \text{as } \varepsilon \to 0. \tag{3.3.6}$$

If $|\beta| = k$, then this follows from continuity of $D^\beta\varphi$ and $D^\beta\varphi \equiv 0$ on K. For the general case we apply Taylor's formula to $D^\beta\varphi$ on an interval $[x, y]$, $x \in K$, $y \in K_\varepsilon$:

$$|D^\beta(\varphi(t(y-x)+x)| \leq C|t|^j\|y-x\|^j \sum_{|\beta'|=j} |D^{\beta'+\beta}\varphi(\xi)|. \tag{3.3.7}$$

where $j = k - |\beta|$, $\xi \in [x, y]$. We take $t = 1$ and use again that for $|\alpha| = k$, we have $D^\alpha\varphi(x) = o(1)$ on K_ε for $\varepsilon \to 0$. Hence (3.3.7) becomes

$$|D^\beta(\varphi(y)| = \varepsilon^j o(1) \quad \text{on } K_\varepsilon \quad \text{for } \varepsilon \to 0.$$

Inserting this in (3.3.6) we are done. ☐

Corollary 3.3.5. *If $L \in \mathcal{D}'$ has order N and* $\operatorname{Supp} L = \{0\}$, *then*

$$L\varphi = \sum_{|\alpha| \le N} c_\alpha D^\alpha \varphi(0).$$

That is, L is a linear combination of derivatives of δ.

Proof. Let $\varphi \in \mathcal{E}$. Write $\varphi(x) = \sum_{|\alpha|\le N} \frac{D^\alpha\varphi(0)}{\alpha!} x^\alpha + \psi(x)$, where the derivatives of ψ up to order N at 0 vanish. By Theorem 3.3.4 $L\psi = 0$, hence

$$L\varphi = \sum_{|\alpha|\le N} L(x^\alpha)\frac{D^\alpha\varphi(0)}{\alpha!}.$$

$\square$

3.4. Convolutions and Product Spaces

It is convenient to have some notation for translated and reflected functions. Let φ be a function on $\mathbb{R}^n$.

- Translation: $(\tau_x\varphi)(y) = \varphi_x(y) = \varphi(y - x)$;
- Reflection: $\check{\varphi}(y) = \varphi(-y)$.

Recalling the definition of convolution we see that

$$f * g(x) = \int_{\mathbb{R}^n} f(y)g(x-y)\,dy = \int f(y)\tau_x\check{g}(y)\,dy, \quad f, g \in L^1(\mathbb{R}^n) \tag{3.4.1}$$

It is now natural to make the following

Definition 3.4.1. For $L \in \mathcal{D}'$, $\varphi \in \mathcal{D}$ the *convolution*

$$L * \varphi(x) \stackrel{\text{def}}{=} \langle L, \tau_x\check{\varphi}.\rangle (= \langle L, \varphi(x - .)\rangle).$$

By Theorem 3.2.5, $L * \varphi$ is a smooth function. In view of Theorem 3.3.1, if L has compact support, then $L * \varphi$ is defined for $\varphi \in \mathcal{E}$. Note that $L * \varphi(0) = L\check{\varphi}$.

Theorem 3.4.2. *If $L \in \mathcal{D}'$, and $\varphi, \psi \in \mathcal{D}$ then*

$$(L * \varphi) * \psi = L * (\varphi * \psi).$$

Proof. We use Theorem 3.2.6.

$$\begin{aligned}(L * \varphi) * \psi(t) &= \int (L\tau_x\check{\varphi})\psi(t-x)\,dx = L\int \varphi(x - .)\psi(t-x)\,dx \\ &= L\int \varphi(x)\psi(t - . - x)\,dx = L(\varphi * \psi(t - .)) = L * (\varphi * \psi)(t).\end{aligned} \tag{3.4.2}$$

$\square$

Convolution with a distribution is a translation invariant operator from $\mathcal{D}$ to $\mathcal{E}$. Indeed,

$$\tau_x(L * \varphi)(t) = \tau_x(L(\tau_t\check{\varphi})) = L(\tau_{t-x}\check{\varphi}) = L(\tau_t(\tau_x\varphi)\check{}\,) = L * (\tau_x\varphi)(t).$$

It also has some continuity properties, in the sense that, of course, $\varphi \mapsto L * \varphi(t)$ is for every t a distribution. The converse is also true:

Theorem 3.4.3. *Suppose that $T : C_0^\infty \to C(\mathbb{R}^n)$ is a translation invariant operator with the property that $\varphi \mapsto T\varphi(0)$ is an element of $\mathcal{D}'$. Then there exists a unique $L \in \mathcal{D}'$ such that $T\varphi = L * \varphi$. In particular, T maps C_0^∞ into C^∞.*

Proof. Let $L\varphi \stackrel{\text{def}}{=} T\check{\varphi}(0)$. Then L is a distribution. We compute

$$L * \varphi(x) = L * (\tau_{-x}\varphi)(0) = L(\tau_{-x}\varphi)\check{}\, = T(\tau_{-x}\varphi)(0) = \tau_{-x}T\varphi(0) = T\varphi(x).$$

Thus we have found L. If $\tilde{L} * \varphi = L * \varphi$ for all $\varphi \in \mathcal{D}$ then for test functions φ

$$\langle \tilde{L}, \varphi\rangle = \tilde{L} * \varphi(0) = L * \varphi(0) = \langle L, \varphi\rangle,$$

and $\tilde{L} = L$ by the very definition of distribution. $\square$

Now let $L_1 \in \mathcal{D}'$, $L_2 \in \mathcal{E}'$. Then for $\varphi \in \mathcal{D}$ we find that

$$L_2 * (L_1 * \varphi) \quad \text{and} \quad L_1 * (L_2 * \varphi)$$

are wel defined – for the second expression observe that $L_2 * \varphi$ has compact support. Moreover, both operators are translation invariant. Therefore by Theorem 3.4.3 we can make the following

DEFINITION 3.4.4. $L_1 * L_2$ is the unique distribution L with the property that $L * \varphi = L_1 * (L_2 * \varphi)$. One similarly defines $L_2 * L_1$.

One can show that $L_1 * L_2 = L_2 * L_1$, cf. exercise 3.6.7.

There is another way of looking at convolutions. We start with (3.4.1) and wish to view that as a distribution, thus let it act on $\varphi \in \mathcal{D}$.

$$\langle f * g, \varphi\rangle = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(y)g(x-y)\,dy\varphi(x)\,dx = \iint_{\mathbb{R}^n \times \mathbb{R}^n} f(y)g(t)\varphi(y+t)\,dydt.$$

This explains commutativity of convolution as a direct consequence of commutativity in $\mathbb{R}^n$. To make it work we have to make sense of $u(x)v(y)$ for distributions $u \in \mathcal{D}'(\mathbb{R}^n)$, $v \in \mathcal{D}'(\mathbb{R}^m)$.

Let $X_j \subset \mathbb{R}^{n_j}$, $j = 1, 2$. For $f_j \in C(X_j)$ we define the *tensor product* as follows

$$f_1 \otimes f_2(x_1, x_2) = f_1(x_1)f_2(x_2).$$

What we need is an analogous tensor product of distributions. It should be defined on $X_1 \times X_2$ and it is clear how it must act on tensor products of test functions of the form $\varphi_1 \otimes \varphi_2$.

THEOREM 3.4.5. *Suppose that X_j are open sets in $\mathbb{R}^{n_j}$ ($j = 1, 2$) and that $u_j \in \mathcal{D}'(X_j)$. There exists a unique distribution $u \in \mathcal{D}'(X_1 \times X_2)$ which has the property that*

$$u(\varphi_1 \otimes \varphi_2) = u_1(\varphi_1)u_2(\varphi_2) \quad \varphi_j \in \mathcal{D}(X_j).$$

Moreover,

$$u(\varphi) = u_{1x}[u_{2y}\varphi(x,y)] = u_{2y}[u_{1x}\varphi(x,y)], \quad \varphi \in \mathcal{D}(X_1 \times X_2). \tag{3.4.3}$$

We write $u = u_1 \otimes u_2$, the *tensor product* of u_1 and u_2.

Notice that the expressions in (3.4.3) are well-defined in view of Theorem 3.2.5 L_x is a sloppy notation, indicating that the distribution L acts on the test functions viewed as a function of x, while the other variables are viewed as parameters.

PROOF. Uniqueness follows once we can show that for $L \in \mathcal{D}'(X_1 \times X_2)$ the assumption $L(\varphi_1 \otimes \varphi_2) = 0$ for all $\varphi_j \in \mathcal{D}(X_j)$, implies that $L = 0$. Thus let $\varphi \in \mathcal{D}(X_1 \otimes X_2)$. Choose $\psi_i \in \mathcal{D}(X_i)$ positive with integral equal to 1. Then $\psi_\varepsilon(x,y) = \varepsilon^{-n_1-n_2}\psi_1(\varepsilon^{-n_1}x)\psi_2(\varepsilon^{-n_2}y)$ is a tensor product, as well as an approximate identity on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Therefore

$$L(\varphi * \psi_\varepsilon) = L * (\varphi * \psi_\varepsilon)\check{}(0) = ((L * \psi_\varepsilon) * \varphi)\check{}(0) = 0,$$

by the assumption. Letting $\varepsilon \to 0$ uniqueness follows.

Next we show that

$$\varphi \mapsto u_{1x}[u_{2y}(\varphi(x,y))], \quad \varphi \in \mathcal{D}(X_1 \times X_2) \tag{3.4.4}$$

is in $\mathcal{D}'(X_1 \times X_2)$. We estimate, using Theorem 3.2.5,

$$\begin{aligned} |u_{1x}[u_{2y}(\varphi(x,y))]| &\le C_1 \sup_{\substack{|\alpha| \le m_1 \\ x \in K_1}} |D_x^\alpha(u_{2y}(\varphi(x,y)))| = C_1 \sup_{\substack{|\alpha| \le m_1 \\ x \in K_1}} |(u_{2y}(D_x^\alpha\varphi(x,y)))| \\ &\le C_1C_2 \sup_{\substack{|\alpha| \le m_1 \\ x \in K_1}} \sup_{\substack{|\beta| \le m_2 \\ y \in K_2}} |D_y^\beta D_x^\alpha\varphi(x,y)))|. \end{aligned} \tag{3.4.5}$$

Hence u is in $\mathcal{D}'(X_1 \times X_2)$. Since u has the required behavior on tensor products of smooth functions, the unicity part completes the proof. □

REMARK 3.4.6. We mention without proof the important *kernel theorem*:

THEOREM 3.4.7 (Laurent Schwartz). *There is a one one correspondence between distributions K on $X_1 \times X_2$ and continuous linear maps T from $\mathcal{D}(X_1)$ to $\mathcal{D}'(X_2)$ given by*

$$K \in \mathcal{D}'(X_1 \times X_2) \quad \leftrightarrow \quad T_K, \quad \langle T_K(\varphi_1), \varphi_2 \rangle = \langle K, \varphi_1 \otimes \varphi_2 \rangle.$$

Here continuity means $T\varphi_j \to 0$ in $\mathcal{D}'$ if $\varphi_j \to 0$ in $\mathcal{D}$. A proof can be found in [**7**].

3.5. Tempered distributions

The dual space $\mathcal{S}'$ of $\mathcal{S}$ is called the space of *tempered distributions* (or temperate). It turns out that this is the "best" space to do Fourier analysis on. In this section we only give a few properties of $\mathcal{S}'$. In view of (3.1.6) a linear functional L on $\mathcal{S}$ is in $\mathcal{S}'$ if and only if there exist $C > 0$, $k, N \in \mathbb{N}$, such that

$$|L\varphi| \leq C \sup_{x \in \mathbb{R}^n} \|x\|^k |D^\alpha \varphi(x)|, \quad \text{for all } |\alpha| \leq N.$$

EXAMPLES 3.5.1. The distribution associated to a locally integrable function f with the property that

$$\int (1 + \|x\|^2)^{-N} |f(x)| \, dx \quad \text{is bounded,}$$

for N sufficiently large, is tempered. This explains the word tempered: moderate growth at infinity is allowed. If P is a polynomial and $L \in \mathcal{S}'$ then $PL \in \mathcal{S}'$; also $D^\alpha L \in \mathcal{S}'$; compactly supported distributions are in $\mathcal{S}'$.

3.6. Exercises

3.6.1. Determine the distributional limit of any approximate identity φ_ε.

3.6.2. Suppose that $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and $\varphi(0) = 0$. Prove that there exist $\varphi_j \in \mathcal{D}$ such that $\varphi(x) = \sum_{j=1}^n x_j \varphi_j(x)$. Hint: At 0 write

$$\varphi(x) = \int_0^{x_1} D_1 \varphi(t, x_2, \ldots, x_n) \, dt + \varphi(0, x_2, \ldots, x_n).$$

Continue with respect to the other variables and chanche the interval of integration to $[0, 1]$. get everything in $\mathcal{D}$ by writing far away from 0

$$\varphi = \sum \frac{x_i^2 \varphi(x)}{\|x\|^2},$$

and patch things together with a partition of unity.

3.6.3. Let $u_t(x) = t^N e^{itx}$, for $x \geq 0$ and $u_t(x) = 0$ elsewhere. Determine the distributional limit $\lim_{t \to \infty} u_t$.

3.6.4. Let $u_t(x) = t^{1/k} e^{itx^k}$, $x \in \mathbb{R}$ and k an integer > 1. Determine the distributional limit $\lim_{t \to \infty} u_t$. Hint: Using integration by parts, write

$$\langle u_t, \varphi \rangle = - \int F(x) t^{1/k} \varphi'(x) \, dx = (F(\infty) - F(-\infty)) \varphi(0),$$

with $F(x) = \int_0^x e^{it^k} dt$. Next apply Cauchy's theorem to compute

$$\lim_{x \to \infty} F(x) = e^{\pi i/2k} \int_0^\infty e^{-t^k} dt,$$

$$\lim_{x \to -\infty} F(x) = -e^{(-1)^k \pi i/2k} \int_0^\infty e^{-t^k} dt.$$

Compare with the case $k = 1$ which is in Example 3.3.2.

3.6.5. Show that distributions are locally determined, i.e., if $L \in \mathcal{D}'$ has the property that for every x there is a neighborhood B_x such that $L = 0$ on B_x, then $L \equiv 0$.

3.6.6. (continued) Suppose that U_τ is an open cover of U and that on every U_τ there is given $L_\tau \in \mathcal{D}(U_\tau)$. Show that there exists $L \in \mathcal{D}(U)$ with $L = L_\tau$ on U_τ if and only if $L_\tau = L_\sigma$ on $U_\tau \cap U_\sigma$ whenever this intersection is non empty. Show also that L is unique.

3.6.7. If u_1, u_2 are distributions one of them with compact support, then $u_1 * u_2 = u_2 * u_1$. Prove this by showing first that for any $\varphi_1, \varphi_2 \in \mathcal{D}$ the equality

$$u_1 * u_2 * \varphi_1 * \varphi_2 = u_2 * u_1 * \varphi_1 * \varphi_2$$

holds and that this implies the equality that you are looking for.

3.6.8. (continued) Show that $\operatorname{Supp} u_1 + \operatorname{Supp} u_2$ contains $\operatorname{Supp} u_1 * u_2$.

3.6.9. (continued) Prove that $(u_1 * u_2)(\varphi) = (u_1 \otimes u_2)(\varphi(x+y)$.

3.6.10. Suppose $L \in \mathcal{D}'$ has the property that $D_j L = 0$ for $j = 1, \ldots, n$. Prove that L is a constant.

3.6.11. Let $u, v \in \mathcal{D}'$. Show that $u * \delta = u$. Show that $u * D^\alpha\delta = D^\alpha u$. Finally show that $D^\alpha(u * v) = (D^\alpha u) * v = u * (D^\alpha v)$.

3.6.12. Show that for every compactly supported distribution u there exists a sequence $(\varphi_j)_j$ in $\mathcal{D}$ such that $\varphi_j \to u$ in distribution sense. (Use smart convolutions).

3.6.13. (continued) Show that every $u \in \mathcal{D}'$ is a distributional limit of distributions with compact support. Conclusion?

3.6.14. (Riesz) Let $a \in \mathbb{C}$ and let

$$\chi_a(x) = \begin{cases} x^a, & \text{if } x \geq 0; \\ 0, & \text{if } x \leq 0 \end{cases} \tag{3.6.1}$$

i. Show that for $\mathbf{Re}\, a > -1$ the function χ_a is in $L^1_{loc}(\mathbb{R})$, i.e. integrable over compact sets in $\mathbb{R}$, and satisfies $(a+1)\chi_a = \frac{d}{dx}\chi_{a+1}$.

ii. Let I_a denote the distribution associated to χ_a. Show that it satisfies

$$I_a \varphi' = -a I_{a-1}(\varphi), \quad \text{if } \mathbf{Re}\, a > 0, \quad \varphi \in \mathcal{D}(\mathbb{R}). \tag{3.6.2}$$

iii. Observe that for fixed φ the function $I_a(\varphi)$ is analytic in a for $\mathbf{Re}\, a > -1$ and use (3.6.2) to extend I_a analytically to $\mathbb{C} \setminus \{-1, -2, \ldots\}$. Show that

$$\langle I_a, x\varphi \rangle = \langle I_{a+1}, \varphi \rangle \quad \text{if } \; a \in \mathbb{C} \setminus \{-1, -2, \ldots\}. \tag{3.6.3}$$

iv. Try to compute the residues of I_a in the negative integers. (These must be distributions, of course).

v. Try to compute the distribution I_a minus its principal part in a, where a is a negative integer. Show that (6.2) holds for this distribution too.

3.6.15. Show that the distribution on $\mathbb{R}^n$ induced by $f(x) = e^{|x|^2}$ is not tempered. Construct a function g on $\mathbb{R}$ such that g induces a tempered distribution and

$$\limsup_{t \to \infty} g(t) e^{-t^2} = 1.$$

3.6.16. Show that the compactly supported distributions are dense in $\mathcal{S}'$.

3.7. Final remarks

Distributions were introduced in the 1940's by Laurent Schwartz. His book [**18**] is a classic. Already around 1930 Sobolev came very close to the concept while the physicist Dirac worked with distribution ideas informally. An interesting book about the history of distribution theory is Lutzens [**14**]; Dutch readers may be happily surprized by the relatively large Dutch role in it. Another classical reference is [**5**].

We have followed Hörmander [**7**] quite closely. Because this book is aimed at partial differential operators, it selects the parts of the theory that are really important for other subjects in a way that can be appreciated beyond abstract functional analysis. It is an excellent book anyway and highly recommended.

CHAPTER 4

The Fourier Transform

We collect some results in Fourier analysis in addition to the results in Grubb. This material is taken from a chapter in another course in Fourier Analysis. We have kept Theorem 4.1.3 although convolutions of distributions where not treated in enough depth to appreciate it fully. Notation in this chapter $\mathcal{E} = C^\infty(\mathbb{R}^n)$, so that $\mathcal{E}'$ is the space of compactly supported distributions.

4.1. Fourier Transform on $\mathcal{S}'$

In this section distributions u will act on functions of two variables, where one of the variables is viewed as a parameter. We will write $< u_x, \varphi(x,\xi) >$ to indicate that u "acts on the variable x" and ξ is viewed as a parameter. The result will thus be a function of ξ.

THEOREM 4.1.1. *Let $u \in \mathcal{E}'$. Then $\hat{u}$ can be represented by a smooth function (also denoted by $\hat{u}$) and*

$$\hat{u}(\xi) =< u_x, e^{-ix\cdot\xi} >, \quad \xi \in \mathbb{R}^n. \tag{4.1.1}$$

Moreover, the righthand side of (4.1.1) *defines a holomorphic function on $\mathbb{C}^n$.*

SKETCH OF PROOF. It has been shown in Grubb that the righthand side of (4.1.1) is a smooth function of ξ on $\mathbb{C}^n$ and that differentiation towards the parameter is allowed. It satisfies the *Cauchy-Riemann* equations, i.e.

$$\frac{\partial \hat{u}}{\partial \overline{\xi_j}} = 0, \quad j = 1, \dots, n$$

because $e^{-ix\cdot\xi}$ satisfies these.

This settles holomorphy. Now let $\varphi \in \mathcal{S}$. We find

$$< \hat{u}, \varphi >=< u, \hat{\varphi} >=< u_x, \int e^{-ix\cdot\xi}\varphi(\xi)\, d\xi >= \int < u_x, e^{-ix\cdot\xi}) > \varphi(\xi)\, d\xi. \tag{4.1.2}$$

The latter equality may be justified by approximating the integrals with Riemann sums. □

EXAMPLES 4.1.2. The Fourier transform of δ equals

$$\hat{\delta} =< \delta_x, e^{-ix\xi} >= 1.$$

Its derivatives have Fourier transform

$$\widehat{D_j\delta} =< D_j\delta, e^{-ix\xi} >=< \delta_x, \xi_j e^{-ix\cdot\xi} >= \xi_j. \tag{4.1.3}$$

THEOREM 4.1.3 (Convolutions). *If $u_1 \in \mathcal{S}'$, $u_2 \in \mathcal{E}'$ then $u_1 * u_2 \in \mathcal{S}'$ and $(u_1 * u_2)\hat{} = \hat{u}_1\hat{u}_2$.*

PROOF. Recall that by the definition of convolution

$$(u_1 * u_2)(\varphi) = u_1 * u_2 * \check{\varphi}(0) = u_1(\check{u}_2 * \varphi), \quad \varphi \in \mathcal{D}.$$

The righthand side extends to a continuous form on $\mathcal{S}$: if u_2 has order k and $\varphi \in \mathcal{S}$, then using $\check{u}_s * \varphi = u_2(\varphi(x + .))$, we check that

$$\sum_{|\alpha+\beta|\le j} \sup |x^\beta D^\alpha(\check{u}_2 * \varphi)| \le \sum_{|\alpha+\beta|\le j+k} \sup |x^\beta D^\alpha(\varphi)|.$$

Now we compute the Fourier transform. For $\varphi \in \mathcal{S}$

$$\begin{aligned}(u_1 * u_2)(\hat\varphi) &= u_1(\check{u}_2 * \hat\varphi) = u_{1x}(\check{u}_{2y}(\hat\varphi(x-y)) = u_{1x}\left(\int u_{2y} e^{-i(x+y)\cdot t}\varphi(t)\,dt\right) \\ &= u_{1x}\left(\int \hat{u}_2(t) e^{-ix\cdot t}\varphi(t)\,dt\right) = u_{1x}\left((\hat{u}_2\varphi)\hat{}\right) = \hat{u}_1(\hat{u}_2\varphi) = (\hat{u}_1\hat{u}_2)(\varphi).\end{aligned} \tag{4.1.4}$$

□

Corollary 4.1.4. *For $u \in \mathcal{S}'$ we have*

$$\widehat{D}_j u = \xi_j \hat{u}, \quad \widehat{x}_j u = -D_j \hat{u}.$$

Proof. This follows directly from (4.1.3) and the previous theorem, e.g.,

$$\widehat{D}_j u = \widehat{D}_j \delta * u = \xi_j \hat{u}(\xi).$$

One can also use that the corollary is known for $\mathcal{S}$ a dense subset of $\mathcal{S}'$ in combination with continuity of $\mathcal{F}$ on $\mathcal{S}'$. □

Remark 4.1.5 (Computational tricks). If $f \in \mathcal{S}'$ is associated to a locally integrable function, also denoted by f, its Fourier transform can be computed by approximation:

$$f = \lim_{A\to\infty} f_A \quad \text{in } \mathcal{S}' \text{ with } f_A(x) = \begin{cases} f(x) & \text{if } |x| \le A; \\ 0 & \text{if } |x| > A. \end{cases} \tag{4.1.5}$$

This follows from Lebesgue's dominated convergence theorem. By continuity of $\mathcal{F}$ we find

$$\hat{f} = \lim_{A\to\infty} \hat{f}_A = \lim_{A\to\infty} \int_{|x|\le A} f(x) e^{-ix\cdot\xi} dx. \tag{4.1.6}$$

Similarly, again by Lebesgue's theorem

$$f = \lim_{\varepsilon\downarrow 0} f e^{-\varepsilon|x|} \quad \text{in } \mathcal{S}'$$

and therefore

$$\hat{f} = \lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}^n} f(x) e^{-\varepsilon|x| - ix\cdot\xi} d\xi. \tag{4.1.7}$$

Of course, all limits are in $\mathcal{S}'$ and will in general not exist pointwise or in L^1 sense.

Examples 4.1.6. Applying 4.1.6 and 4.1.7 to the tempered distribution 1 on $\mathbb{R}$ yields

$$\mathcal{F}(1) = \lim_{A\to\infty} \int_{-A}^{A} e^{-ix\xi} dx = \lim_{A\to\infty} 2\frac{\sin A\xi}{\xi} \quad (= 2\pi\delta).$$

$$\mathcal{F}(1) = \lim_{\varepsilon\downarrow 0} \int_0^\infty e^{-i\xi x - \varepsilon x} dx + \int_{-\infty}^0 e^{-i\xi x + \varepsilon x} dx = \lim_{\varepsilon\downarrow 0} i\left(\frac{1}{\xi - i\varepsilon} - \frac{1}{\xi + i\varepsilon}\right) \quad (= 2\pi\delta).$$

The limits as $\varepsilon \to 0$ of the fractions between the big brackets are usually denoted by

$$\frac{1}{\xi - i0} \quad \text{respectively} \quad \frac{1}{\xi + i0}.$$

These two are **not** the same! Indeed, with χ denoting characteristic function, we have by the above

$$\mathcal{F}(\chi_{(0,\infty)}) = \frac{i}{\xi - i0}, \quad \mathcal{F}(\chi_{(-\infty,0)}) = \frac{-i}{\xi + i0}.$$

In the same vein we find for the signum function Sign

$$\mathcal{F}(\text{Sign}) = \frac{i}{\xi - i0} - \frac{-i}{\xi + i0} = \lim_{\varepsilon \downarrow 0} i \left(\frac{1}{\xi - i\varepsilon} + \frac{1}{\xi + i\varepsilon} \right) = \lim_{\varepsilon \downarrow 0} \frac{2i\xi}{\xi^2 + \varepsilon^2} = 2iP.V.\frac{1}{\xi}.$$

The *Airy function* $Ai(x)$ is classically defined as the inverse Fourier transform of $e^{i\xi^3/3}$. Thus $Ai(x)$ is a tempered distribution. In fact, it is in C^∞. We have in analogy with 4.1.7

$$Ai(x) = \frac{1}{2\pi} \lim_{\varepsilon \downarrow 0} \int e^{i(\xi + i\varepsilon)^3/3 + i(\xi + i\varepsilon)x} d\xi \quad \text{in } \mathcal{S}'. \tag{4.1.8}$$

The integrals on the righthand side are (for a fixed positive ε) convergent and yield smooth functions of x. They can even be extended analytically to $x \in \mathbb{C}$, because the leading term in the exponent is $-\xi^2\varepsilon$, independent of $x \in \mathbb{C}$. On the other hand they are independent of ε in view of Cauchy's theorem.

In $\mathbb{R}^n$ we will consider Gaussians. Let A be a positive definite symmetric matrix. Then clearly $e^{-\frac{1}{2}Ax\cdot x}$ is in $\mathcal{S}$. It is called a *Gaussian.* We compute its Fourier transform. We can write $A = R^2$ with R again positive symmetric.

$$\int_{\mathbb{R}^n} e^{-Ax\cdot x - i\xi\cdot x} dx = \int e^{-Rx\cdot Rx - iR^{-1}\xi\cdot Rx} dx. \tag{4.1.9}$$

Now we change coordinates, $Rx = y$ and (4.1.9) equals

$$\int e^{-y\cdot y - iR^{-1}\xi\cdot y} dR^{-1}y = (\det A)^{-1/2} e^{\frac{-1}{4}A^{-1}\xi\cdot\xi} \int e^{-(y + \frac{i}{2}R^{-1}\xi)\cdot(y + \frac{i}{2}R^{-1}\xi)} dy.$$

The last integral is, by an application of *Cauchy's theorem* (compare the proof of the Paley-Wiener-Schwartz theorem), equal to

$$\int e^{-\|y\|^2} dy = \prod_{j=1}^n \int e^{-y_j^2} dy_j = \pi^{n/2}.$$

The conclusion is that

$$\mathcal{F}(e^{-Ax\cdot x}) = \pi^{n/2} (\det A)^{-1/2} e^{\frac{-1}{4}A^{-1}\xi\cdot\xi}.$$

Homogeneous functions can be treated as follows. Let $u(x) = \|x\|^\alpha$ be the homogeneous, radial function on $\mathbb{R}^n$ of degree α. We assume $\alpha > -n$ in order that u be a locally integrable function. Then u may be viewed as a temperate distribution. We set out from the Gamma function and change variables $s \to s\|x\|^2$.

$$\Gamma(z) \stackrel{\text{def}}{=} \int_0^\infty s^{z-1} e^{-s} ds = \|x\|^{2z} \int_0^\infty s^{z-1} e^{-s\|x\|^2} ds.$$

Hence

$$\|x\|^\alpha = \frac{1}{\Gamma(-\alpha/2)} \int_0^\infty s^{-\alpha/2-1} e^{-s\|x\|^2} ds.$$

For $\hat{u}$ we find, if $\alpha < 0$, so that the Fubini and Lebesgue theorem can be applied, that

$$\begin{aligned}
\Gamma(\frac{-\alpha}{2})\hat{u}(\xi) &= \lim_{N\to\infty} \int_{\|x\|<N} \left(\int_0^\infty s^{-\alpha/2-1} e^{-s\|x\|^2} ds \right) e^{-i\xi\cdot x} dx \\
&= \lim_{N\to\infty} \int_0^\infty s^{-\alpha/2-1} \left(\int_{\|x\|<N} e^{-s\|x\|^2} e^{-i\xi\cdot x} dx \right) ds = \int_0^\infty s^{-\alpha/2-1} \left(\frac{\pi}{s}\right)^{n/2} e^{\frac{-1}{4s}\|\xi\|^2} ds \\
&= \|\xi\|^{-\alpha-n} \pi^{n/2} 2^{n+\alpha} \int_0^\infty t^{(\alpha+n)/2-1} e^{-t} dt = \|\xi\|^{-\alpha-n} \pi^{n/2} 2^{n+\alpha} \Gamma(\frac{\alpha+n}{2}).
\end{aligned}$$

where in passing to the last line we made the change of variables $t = \frac{1}{4s}\|\xi\|^2$.

4.2. Poisson Summation

Let $f \in \mathcal{S}$. We construct from f a periodic function with period 2π:

$$Pf(x) = \sum_{n\in\mathbb{Z}} f(x + 2\pi n).$$

The sum is a smooth function because f and its derivatives are rapidly decreasing. We compute the Fourier coefficients $c_n[Pf]$ of Pf.

$$c_n[Pf] = \frac{1}{2\pi}\int_0^{2\pi} Pf(x)e^{-imx}dx = \int_{-\infty}^{\infty} f(x)e^{imx}dx = \frac{1}{2\pi}\hat{f}(m).$$

As a smooth periodic function Pf equals its Fourier series, hence

$$Pf(x) = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} \hat{f}(m)e^{imx}.$$

We plug in the definition of Pf and take $x = 0$ to obtain *Poisson's Summation Formula* for functions f in $\mathcal{S}$

$$\sum_{n\in\mathbb{Z}} f(2\pi n) = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} \hat{f}(m).$$

A similar formula with a similar proof holds in $\mathbb{R}^n$:

$$\sum_{k\in\mathbb{Z}^n} f(2\pi k) = \frac{1}{(2\pi)^n}\sum_{m\in\mathbb{Z}^n} \hat{f}(m).$$

4.3. The Paley-Wiener-Schwartz Theorem

Let K be a compact subset of $\mathbb{R}^n$. The *supporting function* of K is

$$H(x) = \sup_{y\in K}\{x\cdot y\}.$$

PROPOSITION 4.3.1. *Let $\mathcal{H}_x = \{y : y\cdot x \le H(x)\}$. Then the convex hull of K is given by*

$$\operatorname{ch} K = \cap_{x\in\mathbb{R}^n}\mathcal{H}_x.$$

PROOF. Recall that the convex hull of $K \subset \mathbb{R}^n$ is the intersection of all convex sets that contain K. Observe that $y \in K$ implies that for every x, $x\cdot y \le H(x)$, therefore $K \subset \mathcal{H}_x$ for every x. Using the well-known corollary of the Hahn-Banach theorem, cf. [W], that for every closed convex G and $y \notin G$ there exists a hyperplane separating y from G, we find that $\operatorname{ch} K$ equals the intersection of all half-spaces containing K. In particular, if $x_0 \notin \operatorname{ch} K$ then we can find x such that $x\cdot x_0 > H(x)$, thus $x_0 \notin \mathcal{H}_x$. □

THEOREM 4.3.2 (Paley-Wiener-Schwartz). *Let H be the supporting function of a compact set K in $\mathbb{R}^n$.*
1. The following are equivalent:
i. The distribution $u \in \mathcal{E}'$ has order N and support with convex hull $K \subset \mathbb{R}^n$
ii. There exists a constant C such that the Fourier transform $\hat{u}$ is an entire function on $\mathbb{C}^n$ that satisfies the inequality

$$|\hat{u}(z)| \le C(1 + |z|)^N e^{H(\mathbf{Im}\, z)}, \quad (z \in \mathbb{C}^n). \tag{4.3.1}$$

2. The following are equivalent:
i. The function $u \in \mathcal{D}$ has compact support with convex hull $K \subset \mathbb{R}^n$.
ii. For every N there exists a constant C_N such that the Fourier transform $\hat{u}$ is an entire function on $\mathbb{C}^n$ that satisfies the inequality

$$|\hat{u}(z)| \le C_N(1 + |z|)^{-N} e^{H(\mathbf{Im}\, z)}, \quad (z \in \mathbb{C}^n). \tag{4.3.2}$$

PROOF. We begin with 2. If $u \in \mathcal{D}$ then for every α, $D^\alpha u \in \mathcal{D}$ and

$$|z^\alpha \hat{u}(z)| = |\widehat{D^\alpha u}(z)| \le \|D^\alpha u(z)\|_1 \max_{\xi \in \operatorname{Supp} u} |e^{-iz\cdot\xi}| \le C_\alpha e^{H(\mathbf{Im}\, z)}.$$

This shows that (4.3.2) is necessary. Next, suppose that (4.3.2) holds for some entire function U. Then U restricted to $\mathbb{R}^n$ is the Fourier transform of some function u which is in C^∞ because of (4.3.2). We have

$$u(\xi) = (2\pi)^{-n} \int e^{i\xi\cdot x} U(x)\, dx. \tag{4.3.3}$$

Consider the integral in (4.3.3) as a repeated integral, say first integrating with respect to x_1, the other variables being fixed. The integrand is a holomorphic function in $x_1 + iy_1$, and on the strip $\{z_1 : |y_1| \le A_1\}$ it is bounded by a constant times $(1 + |z_1|)^{-N}$. Thus we may apply Cauchy's theorem and replace integration over $\mathbb{R}$ by integration over $\mathbb{R} + iA_1$, for any A_1 in $\mathbb{R}$. We can also do this for the other variables, and obtain that for every $A = (A_1, \ldots, A_n)$

$$u(\xi) = (2\pi)^{-n} \int e^{i\xi\cdot(x+iA)} U(x + iA)\, dx. \tag{4.3.4}$$

Using (4.3.2) with N sufficiently large, we infer from (4.3.4) that for every A

$$|u(\xi)| \le (2\pi)^{-n} e^{-\xi\cdot A + H(A)}.$$

By considering tA, $t > 0$, instead of A, we infer that $u(\xi) \ne 0$ implies that $\xi \cdot A \le H(A)$, that is, $\xi \in \mathcal{H}_A$ for all A. However, by Proposition 4.3.1, this implies that $\xi \in K$.

Next we turn to 1. and show that (4.3.1) is necessary. Suppose that u has order N and compact support in the convex set $K \subset \mathbb{R}^n$. Let $K_\delta = \{x + y : x \in K,\ |y| < \delta\}$ be the δ-neighborhood of K. Let $\chi = \chi_\delta \in \mathcal{D}(K_\delta)$ equal 1 on $K_{\delta/2}$. Observe that $|D^\alpha \chi_\delta| = O(\delta^{-|\alpha|})$. We estimate

$$\begin{aligned} |\hat{u}(z)| = |\langle u, e^{-iz\cdot\cdot}\rangle| = |\langle u, \chi_\delta e^{-iz\cdot\cdot}\rangle| &\le \sum_{|\alpha|\le N} \max_{x\in K_\delta} C_\alpha |D_\xi^\alpha(\chi_\delta(\xi) e^{\xi\cdot z}| \\ &\le C e^{H(y)+\delta|y|} \sum_{|\alpha|\le N} \delta^{-|\alpha|}(1 + |z|)^{N-|\alpha|}. \end{aligned} \tag{4.3.5}$$

For a fixed $|z|$, choose $\delta = (1 + |z|)^{-1}$ and necessity of (4.3.1) follows.

Finally we show that (4.3.1) is sufficient. If U satisfies (4.3.1) then U restricted to $\mathbb{R}^n$ is the Fourier transform of a distribution u, Let (φ_ε) be an approximate identity, which has $\operatorname{Supp}\varphi_\varepsilon \subset B(0, \varepsilon)$, so that the corresponding $H(y) \le \varepsilon|y|$. Then $(u * \varphi_\varepsilon)\hat{} = \hat{u}\hat{\varphi}_\varepsilon$, and in view of (4.3.2) and (4.3.1) for all M there is a C_M with

$$|(u * \varphi_\varepsilon)\hat{}(z)| \le C_M (1 + |z|)^{N-M} e^{H(y)+\varepsilon|y|}.$$

By 2. of the Theorem we conclude that $u * \varphi_\varepsilon$ has support in K_ε. Letting $\varepsilon \to 0$ we obtain that $\operatorname{Supp} u \subset K$. □

4.4. Fourier Transform and L^p Spaces

Because

$$\begin{aligned} &\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \\ &\mathcal{F} : L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n), \end{aligned} \tag{4.4.1}$$

it is reasonable to expect that for $1 < q < 2$ Fourier transform maps L^q to some space between L^2 and L^∞. That this is indeed the case follows from the *Riesz-Thorin Convexity Theorem.*

THEOREM 4.4.1 (Riesz-Thorin). *Let* $1 \le p_j, q_j \le \infty$ *and let* T *be a continuous linear mapping from* $L^{p_1} \cap L^{p_2}$ *to* $L^{q_1} \cap L^{q_2}$, *such that*

$$\|Tf\|_{q_j} \le M_j \|f\|_{p_j}. \tag{4.4.2}$$

Put $\frac{1}{p} = t\frac{1}{p_1} + (1-t)\frac{1}{p_2}$ *and* $\frac{1}{q} = t\frac{1}{q_1} + (1-t)\frac{1}{q_2}$, *where* $0 < t < 1$. *Then*

$$\|Tf\|_q \le M_1^t M_2^{1-t} \|f\|_p. \tag{4.4.3}$$

COROLLARY 4.4.2 (Hausdorff-Young). *If* $f \in L^p$, $1 \le p \le 2$, *then* $\hat{f} \in L^{p'}$, $1/p + 1/p' = 1$ *and*

$$\|\hat{f}\|_{p'} \le (2\pi)^{n/p'} \|f\|_p.$$

PROOF OF THE COROLLARY. $\mathcal{F}$ satisfies the conditions of Riesz-Thorin, with $p_1 = 1$, $q_1 = \infty$, $M_1 = 1$ and $p_2 = q_2 = 2$, $M_2 = (2\pi)^{n/2}$. Then $\frac{1}{p} = t + \frac{1-t}{2}$ and $\frac{1}{q} = \frac{1-t}{2}$ So $q = p'$ and the estimates follow from (4.4.3). □

PROOF OF THE RIESZ-THORIN THEOREM. Recall from functional analysis that $(L^q)^* = L^{q'}$ with $1/q + 1/q' = 1$. Moreover, for any h that is locally integrable we can estimate by duality

$$\|h\|_q = \sup_{\|g\|_{q'}=1} |\langle h, g\rangle| = \sup_{\|g\|_{q'}=1} |\int h(t)g(t)\,dt|.$$

The sup is finite if and only if $h \in L^q$. We can now rewrite (4.4.2) as

$$|\langle Tf, g\rangle| \le M_j \|f\|_{p_j} \|g\|_{q_j'} \qquad \forall g \in L^{q_j'}. \tag{4.4.4}$$

To prove the theorem it suffices to prove

$$|\langle Tf, g\rangle| \le M_1^t M_2^{1-t} \|f\|_p \|g\|_{q'}, \tag{4.4.5}$$

which is the dual version of (4.4.3), for f of the form $f_0 F^{1/p}$ tested against g of the form $g_0 G^{1/q'}$, where f_0, g_0 are in absolute value ≤ 1 and, because step functions are dense in L^r if $r < \infty$, F, G are non negative step functions with integral equal to 1. Now consider, for fixed such f_0, F, g_0, G

$$\Phi(z) = \langle T(f_0 F^{z/p_1 + (1-z)/p_2}), g_0 G^{z/q_1 + (1-z)/q_2}\rangle \dot{M}_1^{-z} \dot{M}_2^{z-1}.$$

This function is continuous on the strip $0 \le \mathbf{Re}\, z \le 1$ and analytic on its interior (The reader may want to check this first for F, G being multiples of characteristic functions). Moreover it is bounded on the strip and has boundary values ≤ 1 on $\mathbf{Re}\, z \in \{0, 1\}$. The *Phragmén-Lindelöf Theorem* states that Φ is in absolute value ≤ 1 and the Theorem is proven. □

4.5. Exercises

4.5.1. Let $T\varphi = \overline{\mathcal{F}}\mathcal{F}\varphi$. Prove that (1): $(TD_j - D_jT)\varphi = 0$ and (2): $T(x_j\varphi) - x_jT(\varphi) = 0$, for $\varphi \in \mathcal{S}$. For $\varphi \in \mathcal{S}$ with $\varphi(y) = 0$ we saw in an earlier exercise that

$$\varphi(x) = \sum_{j=1}^{n} (x_j - y_j)\varphi_j(x),$$

for appropriate $\varphi_j \in \mathcal{S}$. Show that $\varphi(x_0) = 0$ implies $T(\varphi)(x_0) = 0$. Pick a positive function $\varphi_0 \in \mathcal{S}$, and apply this to $\varphi(x_0)\varphi_0 - \varphi_0(x_0)\varphi_0$ to show that there exists a function $c(x)$ so that $(T\varphi)(x) = c(x)\varphi(x)$. Use relation 1 to show that $c(x)$ is a constant c. Take $\varphi = e^{-|x|^2}$ to determine c and derive the inversion formula for $\mathcal{S}$.

4.5.2. Suppose that f is an entire function on $\mathbb{C}^n$ that satisfies the growth condition (4.3.1) for some N and that its restriction to $\mathbb{R}^n$ is in $L^2(\mathbb{R}^n)$. Show that $\hat{f}$ is a compactly supported L^2-function.

4.5.3. Suppose that f is an entire function on $\mathbb{C}^n$ that satisfies for every $\varepsilon > 0$ a growth condition

$$|f(z)| \leq C_\varepsilon (1 + |z|)^{N_\varepsilon} e^{\varepsilon |\mathbf{Im}\, z|}.$$

Show that f is a polynomial. (Hint study $\hat{f}$.)

4.5.4. Prove that the Airy function satisfies the differential equation

$$Ai''(x) - xAi(x) = 0.$$

Let ω be a third root of unity ($\omega^3 = 1$). Show that $Ai(\omega x)$ is another solution of this equation. How many linearly independent solutions are there? Conclusion?

4.5.5. Prove that $Ai(0) = 3^{-1/6}\Gamma(1/3)/2\pi$ and $Ai'(0) = -3^{-1/6}\Gamma(2/3)/2\pi$. (Take $\varepsilon = 1$ in 4.1.8)

4.5.6. Let A be a symmetric matrix, possibly with complex coefficients. Show that $e^{-Ax\cdot x}$ is a tempered distribution if and only if $\mathbf{Re}\, A$ is positive semi-definite. Assuming in addition that A is non-singular, compute the Fourier transform of $e^{-Ax\cdot x}$.

4.5.7. Show that

$$\sum_{k\in\mathbb{Z}} e^{-4\pi^2 t k^2} = \frac{1}{\sqrt{4\pi t}} \sum_{k\in\mathbb{Z}} e^{-k^2/4t}.$$

4.5.8. For $x \in \mathbb{R}^n$ let δ_x be the distribution $\varphi \mapsto \varphi(x)$ ($\varphi \in \mathcal{S}$). Show that $\sum_{k\in\mathbb{Z}^n} \delta_{x-k}$ is a temperate distribution. Next show that

$$\mathcal{F}\left(\sum_{k\in\mathbb{Z}^n} \delta_{x-k}\right) = (2\pi)^n \sum_{k\in\mathbb{Z}^n} \delta_{x-2\pi k}.$$

CHAPTER 5

Applications to partial differential equations

5.1. Introduction

We will work on $\mathbb{R}^n$, $n \geq 1$. Recall the notation $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha^n}$ introduced in [**9**]. We want to solve linear partial differential equations of form

$$P(D)u = f,$$

that is we are looking for a function u on $\mathbb{R}^n$, where f is a given function on $\mathbb{R}^n$, and $P(D)u(x)$ is shorthand for the expression

$$P(D)u(x) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha u(x). \tag{5.1.1}$$

Here the a_α are functions on $\mathbb{R}^n$. Often, but not necessarily, these functions will be smooth. We start with the situation where a_α are constants. Then it is easy to write down a formal solution of (5.1.1). We apply Fourier transformation to both sides of (5.1.1) and obtain

$$P(\xi)\hat{u}(\xi) = f(\xi).$$

Dividing by both sides by the polynomial $P(\xi) = \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha$ and transforming back, we find

$$u(x) = \left(\mathcal{F}^{-1} \frac{\hat{f}}{P} \right)(x). \tag{5.1.2}$$

The problem with (5.1.2) is of course that one can almost never take the inverse Fourier transform.

We call the function $(x, \xi) \mapsto P(\xi) = \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha$ the *symbol* of the *partial differential operator* $P(D)$. It is a polynomial of the variables $\xi_1, \cdots, \xi_n$ with variable coefficients. The *principal symbol* $P_m = P_m(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha$, the highest degree homogeneous polynomial in the symbol. Of course we have assumed that m is chosen as small as possible, so that $P_m \not\equiv 0$.

5.2. Elliptic equations

A linear partial differential is called *elliptic* if its principal symbol satisfies $P_m(x, \xi) = 0$ if and only if $\xi = 0$. It follows immediately that there exist constants $c, R > 0$ such that $|P(\xi)| > c|\xi|^m$ if $|\xi| > R$. For test functions f and an elliptic operator with constant coefficients P, we can solve $P(D)u = f$, and the solution is almost given by (5.1.2)! We write out the formal solution

$$u(x) = \frac{1}{2\pi^n} \int_{\mathbb{R}^n} \frac{\hat{f}(\xi)}{P(\xi)} d\xi. \tag{5.2.1}$$

If the polynomial $P(\xi)$ has zeros on $\mathbb{R}^n$, we have a problem integrating. However, we shall see that we can always change the domain of integration in such a way that it does not meet zeros of P. We have the following theorem.

THEOREM 5.2.1. *Let $P(D)$ be a linear elliptic differential operator with constant coefficients, and let $f \in \mathcal{D}(\mathbb{R}^n)$. Then the equation*

$$P(D)u = f$$

admits a solution $u \in C^\infty(\mathbb{R}^n)$.

Proof. There exist constants $c, R > 0$ such that $|P(\xi)| \geq c|\xi|^m$ if $|\xi| \geq R$. We now take another look at the naive approach (5.1.2). Writing out the inverse Fourier transform in (5.1.2) we arrive at

$$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{\hat{f}(\xi)}{P(\xi)} e^{ix\cdot\xi} d\xi = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} \frac{\hat{f}(\xi_1, \xi')}{P(\xi_1, \xi')} e^{ix_1\xi_1} \right) e^{ix'\cdot\xi'} d\xi'.$$

Here $\xi = (\xi_1, \xi')$ and $x = (x_1, x')$. By the Paley Wiener theorem, $\hat{f}$ extends to an analytic function on $\mathbb{C}^n$. With ξ' fixed we try to integrate in the inner integral the quotient of the analytic functions $\hat{f}(\xi_1, \xi')$ and $P(\xi_1, \xi')$ over $\mathbb{R}$. This may be impossible, but it is possible to find a path $\Gamma_{\xi'}$ depending on ξ', that consists of $\{\xi_1 < -R - m)\} \cup \{\xi_1 > R + m\} \cup \gamma_{\xi'}$, where $\gamma_{\xi'}$ is a bounded path from $-R-m$ to $R+m$ that does not meet any of the m zeros of $P(\xi_1, \xi')$. We end up with a well defined expression

$$u(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{n-1}} \left(\int_{\Gamma_{\xi'}} \frac{\hat{f}(\zeta, \xi')}{P(\zeta, \xi')} e^{ix_1\zeta} d\zeta \right) e^{ix'\cdot\xi'} d\xi'. \tag{5.2.2}$$

We will see in a moment that the paths $\gamma_{\xi'}$ can be chosen in such a way that differentiation under the integral is allowed, so that u will be a smooth function. Assuming this we compute

$$\begin{aligned} P(D)u(x) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}_{n-1}} \left(\int_{\Gamma_{\xi'}} \frac{\hat{f}(\zeta, \xi')}{P(\zeta, \xi')} P(D)[e^{i(x_1\zeta + x'\cdot\xi')}] d\zeta \right) d\xi' \\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{n-1}} \left(\int_{\Gamma_{\xi'}} \hat{f}(\zeta, \xi') e^{i(x_1\zeta + x'\cdot\xi')} d\zeta \right) d\xi'. \end{aligned} \tag{5.2.3}$$

Now the inner integrand in the final integral is analytic on all of $\mathbb{C}$, so by applying Cauchy's theorem we can replace $\Gamma_{\xi'}$ by $\mathbb{R}$. Thus (5.2.3) reduces to $\mathcal{F}^{-1}\hat{f} = f$.

It remains to be shown that $\gamma_{\xi'}$ can be chosen so that (5.2.2) is smooth and differentiation under the integral is allowed. If $|\xi'| > R$ we simply choose $\Gamma_{\xi'} = \mathbb{R}$. For $|\xi'| \leq R$ we proceed as follows. Consider half circles $\gamma_j = C(0, R+j) \cap \{\mathbf{Im}\,\zeta \geq 0\}$, $j = 0, \cdots, m$. Since $P(\zeta, \xi')$ is a polynomial of degree m in ζ it has at most m different zeros, hence there is at least one γ_j such that the distance of γ_j to the zeros of $P(\cdot, \xi')$ is ≥ 1. Choose $\gamma(\xi')$ to be such a a half circle. The location of the zeros of $P(\cdot, \xi')$ depends continuously on ξ'. This can be shown, e.g., by Rouché's theorem. Hence for every ξ' there is an open neighborhood $U_{\xi'}$ such that for $\eta' \in U_{\xi'}$ the distance of the zeros of $P(\cdot, \eta')$ to $\gamma(\xi')$ is at least 1/2. Now by compactness of $\overline{B(0,R)}$ there exists $J \in \mathbb{N}$ with $\overline{B(0,R)} \subset \cup_{j=0}^J U_{\xi'_j}$. Let $V_0 = U_{\xi'_0} \cap \overline{B(0,R)}$ and

$$V_j = (U_{\xi'_j} \cap \overline{B(0,R)}) \setminus (\cup_0^{j-1} U_{\xi'_j}).$$

Then the V_j are measurable and every point $\xi' \in \overline{B(0,R)}$ is element of a unique V_j. For $\xi' \in V_j$ we choose

$$\Gamma_{\xi'} = [-R-m, -R-j] \cap \gamma_{\xi'_j} \cap [R+j, R+m].$$

With this choice the integral (5.2.2) can be written as a finite sum

$$\int_{|\xi'|>R} + \int_{V_0} + \cdots + \int_{V_J}.$$

Because $\hat{f}$ is in $\mathcal{S}$ and $1/P$ is uniformly bounded on $\Gamma_{\xi'\times V_j}$ we can apply the differentiation Lemma 2.8 in [**9**] and the proof is finished. $\square$

5.3. Linear partial differential equations with constant coefficients

In this section we prove the Malgrange–Ehrenpreis theorem, which states that every linear PDE with constant coefficients admits a fundamental solution. The proof we present is essentially the original non constructive one, as presented by Rudin, [**17**]. Recently a constructive proof was found, cf. [**15, 21**], see also [**3**]. We will write $\mathbb{T}^n$ for the torus $\{z : |z_j| = 1, j = 1, \ldots, n\}$ in $\mathbb{C}^n$ and $\sigma_n = \frac{1}{(2\pi)^n}|dz_1|\cdots|dz_n|$, the Haar measure on $\mathbb{T}^n$.

LEMMA 5.3.1 (Malgrange). *Let $P = P_0 + \cdots + P_N$ be a polynomial in $\mathbb{C}^n$ of degree N written as sum of homogeneous polynomials P_j of degree j. Let $A = \int_{\mathbb{T}^n} |P_N(z)|\, d\sigma_n$. Then $A > 0$ and for every $z \in \mathbb{C}^n$ and every $r > 0$*

$$|f(z)| \le \frac{1}{r^N A}\int_{\mathbb{T}^n} |(fP)(z + rw)|\, d\sigma_n(w). \tag{5.3.1}$$

PROOF. Let F be an entire function and let $Q(\zeta) = c\prod_{j=1}^N(\zeta + a_j)$ be a polynomial of exact degree N on $\mathbb{C}$. Let $Q_0(\zeta) = c\prod_{j=1}^N(1 + \overline{a_j}\zeta)$. If $|\zeta| = 1$ then $|\zeta + a_j| = |1 + \overline{a_j}\zeta|$. It follows from Cauchy's theorem that

$$|cF(0)| = |(Q_0F)(0)| \le \frac{1}{2\pi}\int |(FQ_0)(e^{i\theta})|\, d\theta = \frac{1}{2\pi}\int |(FQ)(e^{i\theta})|\, d\theta. \tag{5.3.2}$$

We apply this with $F(\zeta) = f(z + r\zeta w)$ and $Q(\zeta) = P(z + r\zeta w)$. Note that then $c = Q_0(0) = \lim_{\zeta\to\infty} Q(\zeta)/\zeta^N = r^N P_N(w)$. We plug this in (5.3.2) and integrate with respect to w over $\mathbb{T}^n$:

$$|f(z)|\int_{\mathbb{T}^n} r^N|P_N(w)|\, d\sigma_n(w) \le \int_{\mathbb{T}^n}\frac{1}{2\pi}\int_0^{2\pi} |f(z + re^{i\theta}w)P(z + re^{i\theta}w)|\, d\theta d\sigma_n(w) \tag{5.3.3}$$

Changing the order of integration, and making the substitution $w \to e^{i\theta}w$ we find that the final integral equals $\int_{\mathbb{T}^n} |f(z + rw)P(z + rw)|\, d\sigma_n(w)$. Note that an n-fold application of Cauchy's theorem gives for $z \in \mathbb{C}^n$, $|z_j| < 1$,

$$P_N(z) = \frac{1}{(2\pi i)^n}\int_{\mathbb{T}^n}\frac{P_N(\zeta)}{(\zeta_1 - z_1)\cdots(\zeta_n - z_n)} d\zeta_1\cdots d\zeta_n.$$

Therefore $A = \int_{\mathbb{T}^n} |P_N|\, d\sigma_n > 0$ and (5.3.1) is proved. □

THEOREM 5.3.2. *Let P be a polynomial in n variables and $P(D)$ the associated linear PDO. Let $v \in \mathcal{E}'(\mathbb{R}^n)$ be a compactly supported distribution. Then the equation*

$$P(D)u = v \tag{5.3.4}$$

has a compactly supported distribution u as solution if and only if there exists an entire function g such that

$$Pg = \hat{v} \tag{5.3.5}$$

PROOF. If (5.3.4) has a compactly supported distribution as solution u, Fourier transformation yields $P\hat{u} = \hat{v}$. By the Paley–Wiener Theorem, $\hat{u}$ and $\hat{v}$ are entire functions of exponential type, hence (5.3.5).

If (5.3.5) holds for some g, we wish to apply the Paley–Wiener Theorem to the effect that we can take $u = \mathcal{F}^{-1}g$. We have to prove that g satisfies the desired growth conditions. Let $r > 0$ be such that $\operatorname{supp} v \subset B(0, r)$. By Lemma 5.3.1, with $A = \int_{\mathbb{T}^n} |P_n| d\sigma_n$

$$|g(z)| \le \frac{1}{A}\int_{\mathbb{T}^n} |\hat{v}(z + w)| d\sigma_n(w), \quad z \in \mathbb{C}^n. \tag{5.3.6}$$

By the Paley–Wiener Theorem there exist C and N such that

$$|\hat{v}(z + w)| \le C(1 + |z + w|)^N e^{r|\operatorname{Im}(z+w)|} \tag{5.3.7}$$

For $|w| = 1$

$$1 + |z + w| \le 2(1 + |z|)$$

and

$$|\operatorname{Im}(z+w)| \leq 1 + |\operatorname{Im} z|.$$

This gives

$$|g(z)| \leq \frac{1}{A} C'(1+|z|)^N e^{r(1+|\operatorname{Im} z|)} \leq C''(1+|z|)^N e^{r|\operatorname{Im} z|}, \quad z \in \mathbb{C}^n, \tag{5.3.8}$$

which had to be proved. □

Apparently equations like (5.3.4) seldomly admit solutions with compact support.

THEOREM 5.3.3 (Malgrange–Ehrenpreis). *Let P be a polynomial in n variables. The associated partial differential operator $P(D)$ admits a fundamental solution. That is, The equation*

$$P(D)u = \delta \tag{5.3.9}$$

has a solution $u \in \mathcal{D}'$.

PROOF. Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and suppose

$$\psi = P(-D)\varphi.$$

Then $\hat{\psi} = P\hat{\varphi}$ and both $\hat{\psi}$ and $\hat{\varphi}$ are entire functions. Hence by the uniqueness theorem for holomorphic functions, ψ determines φ. Thus $\psi \mapsto \varphi(0)$ is a well defined linear functional on the range of $P(D)$, which is a subspace of $\mathcal{D}(\mathbb{R}^n)$. We will show below that this functional is continuous. Then by the Hahn–Banach Theorem, it extends to a contnious linear functional on $\mathcal{D}(\mathbb{R}^n)$ hence it can be represented by a distribution E with the property that

$$\langle E, P(-D)\varphi \rangle = \varphi(0), \quad \varphi \in \mathcal{D}(\mathbb{R}^n).$$

Then we find

$$\langle P(D)E, \varphi \rangle = \langle E, P(-D)\varphi \rangle = \varphi(0) = \langle \delta, \varphi \rangle,$$

proving that E is a fundamental solution.

To show that our linear functional is continuous, we introduce explicitly the *characters* on $\mathbb{R}^n$, that is the homomorphisms $\mathbb{R}^n \to \mathbb{T}$ given by the exponential functions $e_s : x \mapsto e^{is\cdot x}$. We need to show that $\varphi(0)$ tends to zero if $\psi = P(-D)\varphi$ tends to zero. Lemma 5.3.1 relates $\hat{\varphi}$ to $\hat{\psi}$. Hence, fixing $r > 0$, we are led to

(5.3.10)

$$\begin{aligned} |\varphi(0)| &= \left| \int_{\mathbb{R}^n} \hat{\varphi}(\xi) d\xi \right| \leq \int_{\mathbb{R}^n} d\xi \frac{1}{A} \int_{\mathbb{T}^n} \left| \hat{\psi}(\xi + rw) \right| d\sigma_w \\ &= \int_{\mathbb{R}^n} d\xi \frac{1}{A} \int_{\mathbb{T}^n} \left| \widehat{\psi e_{-rw}}(\xi) \right| d\sigma_w \leq \int_{\mathbb{T}^n} \left\| \frac{1}{(1+|\xi|^2)^n} \right\|_2 \left\| (1+|\xi|^2)^n \widehat{\psi e_{-rw}}(\xi)) \right\|_2 d\sigma_w. \end{aligned}$$

The final inequality is Cauchy–Schwarz. Recall that ψ tends to 0 implies that there is a fixed ball $B(0,R)$ that contains the supports of all ψ. The characters e_{-rw} are uniformly bounded by a constant $C > 0$, independently of $w \in \mathbb{T}$ on $B(0,R)$. Leibnitz' formula then gives

$$|D^\alpha(e_{-rw}\psi)| \leq C' \max_{\beta \leq \alpha, \xi \in B(0,R)} \left| D^\beta \psi(\xi) \right|,$$

where C' is a positive constant depending only on R and α. It follows that for suitable constant C'' we also have for the 2-norm

$$\|(I-\Delta)^n(e_{-rw}\psi)\|_2 \leq C'' \max_{|\beta| \leq n, \xi \in B(0,R)} \left| D^\beta \psi(\xi) \right|.$$

Application of Plancherel's formula gives

$$\|(1+|\xi|^2)^n \widehat{e_{-rw}\psi}(\xi)\|_2 \leq C'' \max_{|\beta| \leq n, \xi \in B(0,R)} \left| D^\beta \psi(\xi) \right|.$$

Inserting this in the final term of (5.3.10) we have proven continuity. □

Bibliography

[1] L. Carleson *On convergence and growth of partial sums of Fourier series.* Acta Math. 116 1966 135157.
[2] I. Daubechies *Ten lectures on Wavelets* CBMS-NSF Regional conference series in Applied Math. SIAM, 1992.
[3] J.J. Duistermaat, J. Kolk *Distributions. Theory and applications.* Translated from the Dutch by J. P. van Braam Houckgeest. Cornerstones. Birkhäuser Boston, Inc., Boston, MA, 2010.
[4] R.E. Edwards *Fourier series*, Vol. 1 and 2, Springer GTM 1979 and 1981.
[5] I. Gelfand and G.E. Shilov *Generalized functions 1-3* Academic press 1977 (translated from the Russian)
[6] Holschneider and Tchamitchian
[7] L. Hörmander, *The analysis of Linear Partial differential operators, 1*, Grundl. Math, Wiss., Springer 198
[8] O.G. Jorsboe and L. Mejlbro, *The Carleson–Hunt Theorem on Fourier Series*, 1982, SLM 911.
[9] G. Grubb *Distributions and Operators*, 2009, GTM 252.
[10] Y. Katznelson, *An introduction to harmonic analysis*, Dover, 1976
[11] A. Kolmogoroff *Une serie de Fourier-Lebesgue divergente presque partout.* (French) Fundamenta math. 4, 324-328 (1923).
[12] J. Korevaar *Fourier analysis and related topics* (Electronic: https://staff.fnwi.uva.nl/j.korevaar/Foubook.pdf) 2011
[13] T. H. Koornwinder, *Fourier analysis*, UvA-Notes, 1996 (these notes has been reviewed since then)
[14] J. Lützen,*The prehistory of the theory of distributions.* Studies in the History of Mathematics and Physical Sciences, 7. Springer-Verlag, 1982.
[15] N. Ortner, P. Wagner, *A short proof of the Malgrange-Ehrenpreis theorem.* Functional analysis (Trier, 1994), 343352, de Gruyter, Berlin, 1996.
[16] W. Rudin, *Real and Complex Analysis*, McGraw-Hill 1987 (3d edition)
[17] W. Rudin, *Functional Analysis*, McGraw-Hill 1991 (2d edition)
[18] L. Schwartz, *Théorie des distributions* , Hermann 1966
[19] E.M. Stein and G. Weiss *Introduction to Fourier analysis on Euclidean spaces*, Princeton UP 1971
[20] E.M. Stein and R. Shakarchi, *Fourier analysis. An introduction*, Princeton Lectures in Analysis, 1. Princeton University Press, Princeton, NJ, 2003.
[21] P. Wagner, *A new constructive proof of the Malgrange-Ehrenpreis theorem.* Amer. Math. Monthly 116 (2009), no. 5, 457462.
[22] J. Wiegerinck, *Functionaal analyse* UvA-Notes, 1992 (in Dutch)
[23] A. Zygmund, *Trigonometric series*, Cambridge Univ. Press 1959 (2d edition)

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