EXTREME AND EXPOSED REPRESENTING MEASURES OF THE DISK ALGEBRA

ALEX HEINIS AND JAN WIEGERINCK

Abstract. In this paper we study the extreme and exposed points of the convex set consisting of representing measures of the disk algebra, supported in the closed unit disk. A boundary point of this set is shown to be extreme (and even exposed) if its support inside the open unit disk consists of two points that do not ly on a radius of the disk. If its support inside the unit disk consists of 3 or more points, it is very seldom an extreme point. We give a necessary condition for extreme points to be exposed and show that so-called BSZ-measures are never exposed.

1. Introduction and notational matters

In a previous paper [1] Brummelhuis and the second author studied representing measures for the disk-algebra

$$A = A(D) = \{f \in C(\overline{D}) | \, f \text{ is holomorphic on } D\},$$

where D is the unit disk in $\mathbf{C}$. A representing measure μ for $A(D)$ is positive Borel measure on $\overline{D}$ such that

$$\int f \, d\mu = f(x), \quad \forall f \in A.$$

Setting $f = 1$, we find that μ is a probability measure. Let M_0 denote the set of positive representing measures for 0 for $A(D)$. Clearly, M_0 is a convex and weak*-compact subset of M.

Let E be a relatively closed subset of D and let T denote the boundary of D. By M_E we denote the subset of M_0 of representing measures μ with $T \subset \operatorname{supp} \mu \subset T \cup E$. Clearly, M_E is an extreme set of M_0, i.e. if a point in M_E is interior point of a line segment in M_0, this line segment is contained in M_E. M_0 is very large and it is reasonable, cf. [1], to consider only representing measures μ with $T \subset \operatorname{supp} \mu$.

The following result was obtained in [1].

Theorem 1.1. *Let μ be an extreme point of M_0. Suppose that $\mu \in M_E$ where E is a compact subset of D. Then E is a finite set.*

Date: June 2,1998.

Key words and phrases. extreme point, exposed point, representing measure, disk algebra.

Also, it was shown that for each $z_0 \in D \setminus \{0\}$ there exists one extreme point $\mu \in M_0$ with $\operatorname{supp}\mu = T \cup \{z_0\}$. Recently Izzo [5] showed that these μ are also exposed in M_0.

Most of this paper is concerned with the case that E is finite. We thoroughly investigate the situation where E consists of two points. Then M_E is a strictly convex subset of M_0 if and only if these points are not lying on a diameter of D; moreover extreme points in M_E are exposed.

When E consists of more than 2 points, the situation becomes much more complicated. We give examples of sets E consisting of n points in general position in D that are subsets of the support of an extremal element of M_0 and examples of sets E for which this is not the case. We present a sufficient condition for extreme points of M_E to be exposed and give examples of extreme points of M_E that are not exposed.

In passing we show that measures μ in M_D such that $\mu(T) = 0$ can not be exposed. From [1] we know that such measures can be extreme points of M_0.

Acknowledgments. We are grateful to Erik Hendriksen and Chris Klaassen for their useful remarks on this work.

2. Preliminary Observations.

The *Poisson transform* or *balayage* or *sweep* of ν is defined for any finite measure ν on D as the L^1-function on T given by

$$\langle \nu, P(\zeta) \rangle = \int_D \frac{1-|z|^2}{|\zeta - z|^2} d\nu(z). \tag{1}$$

Here $P(\zeta) = P(\zeta, z) = \frac{1-|z|^2}{|\zeta-z|^2}$, the Poisson kernel. The notation is chosen to stress the fact that for fixed $\zeta \in T$, $P(\zeta)$ may be viewed as a functional on spaces of compactly supported complex Borel measures on D. See e.g. [7] or [4] for more details. We recall from [1]

Theorem 2.1. *If $\mu \in M_0, \nu = \mu|_D$ then*

$$\mu = \nu + (1 - \langle \nu, P(\zeta) \rangle) \frac{d\theta}{2\pi} \qquad (\zeta = e^{i\theta}) \tag{2}$$

and $0 \le \langle \nu, P(\zeta) \rangle \le 1$
Conversely, if ν is a positive Borel measure such that $0 \le \langle \nu, P(\zeta) \rangle \le 1$ then the measure defined by (2) is in M_0.

The function $f = \frac{1}{2\pi}(1 - \langle \nu, P(\zeta) \rangle)$ will be referred to as the trace of μ, $f = \operatorname{Tr}\mu$. For $\mu \in M_0$ we will write

$$\mu = \mu_0 + \mu_1, \quad \mu_0 = \mu|_D,\ \mu_1 = f\frac{d\theta}{2\pi},$$

to indicate the decomposition (2)
From all this it follows that $M_0 \leftrightarrow \{\nu \in M_{\ge 0}(D) | 0 \le \langle \nu, P(\zeta) \rangle \le 1\}$, which

describes M_0 more concretely. Moreover this shows that even M_D is huge. We quote from [1]:

Theorem 2.2. *If $\mu \in M_0$ is extreme, $T \subset \operatorname{supp}\mu$ and $Tr\ \mu$ has a real-analytic minorant h which is ≥ 0 but not identically 0, then $E := \operatorname{supp}\mu \cap D$ is finite.*

Corollary 2.3. *If $\mu \in M_E$ is extreme with E compact, then E is finite.*

Indeed, P_ν is clearly real-analytic and if $\mathrm{Tr}\ \mu \equiv 0$ then μ would be supported by a compact subset of D, contradicting $T \subset \operatorname{supp}\mu$.
The following trivial observations will be used later on.

Lemma 2.4. *Let μ be a positive measure with bounded Poisson Transform. Suppose that $\operatorname{supp}\mu \subset [0, e^{i\theta_0}]$. If μ has support outside 0, then $\langle \mu, P(e^{i\theta})\rangle$ is monotonically decreasing on $(\theta_0, \theta_0+\pi)$ and increasing on $(\theta_0-\pi, \theta_0)$. Moreover, with σ denoting reflection in the line $te^{i\theta_0}$, we have*

$$\langle \mu, P(\zeta)\rangle = \langle \mu, P(\sigma(\zeta))\rangle.$$

Lemma 2.5. *If $\mu(0) > 0$ and μ is extreme then $\mu = \delta_0$.*

Proof. Suppose that $0 < \mu(0) < 1$, put $\mu(0) = c, \nu = \mu - c\delta_0, \nu^* = \frac{\nu}{1-c}$. Then ν^* is representing, $\nu^* \neq \delta_0, 0 < c < 1$ and $\mu = (1-c)\nu^* + c\delta_0$, hence μ is not extremal. $\square$

Lemma 2.6. *If $E \subset [0,1]$ and μ is extreme, then $|E| \leq 1$.*

Proof. Suppose there exist two relatively open sets $U, V \subset E$ with $U \cap V = \emptyset$ and U, V having positive $\mu-$measure.
Let $\delta_1 = \langle \mu|_U, P(1)\rangle$, $\delta_2 = \langle \mu|_V, P(1)\rangle$, $W = D \setminus (U \cup V)$ and define

$$\nu_t = (1+t\delta_2)\mu|_U + (1-t\delta_1)\mu|_V + \mu|_W, \quad |t| \leq \frac{1}{\delta_1}, \frac{1}{\delta_2}.$$

Then ν_t is a positive measure on D and

$$\begin{aligned} \langle \nu_t, P(\zeta)\rangle &\leq \langle \nu_t, P(1)\rangle = (1+t\delta_2)\delta_1 + (1-t\delta_1)\delta_2 + \langle \mu|_W, P(1)\rangle \\ &= (\langle \mu|_U, P\rangle + \langle \mu|_V, P\rangle + \langle \mu|_W, P\rangle)(1) = \langle \nu, P(1)\rangle \\ &\leq 1 \Rightarrow \mu_t = \nu_t + \frac{1}{2\pi}(1 - \langle \nu_t, P(\zeta)\rangle d\theta \in M_0. \end{aligned} \tag{3}$$

With $|t| \leq \frac{1}{\delta_1}, \frac{1}{\delta_2}$ we have $\mu = \mu_0 = \frac{\mu_t + \mu_{-t}}{2}$ and since $\mu_t \neq \mu_{-t}$, we find that μ is not an extreme point of M_0 $\square$

Notation 2.7. For finite E we will use the following notation. Let $E = \{a_1, \dots, a_n\} \subset D$. To a positive measure $\mu = \sum_1^n c_i\delta_{a_i}$ on E associate $c = c(\mu) = (c_1, \dots, c_n) \in \mathbf{R}^n$. We write $p_i(\theta) = P(e^{i\theta}, a_i)$, $p(\theta) := (p_1(\theta), \dots, p_n(\theta))$. Then $\langle \mu, P(\zeta)\rangle = \sum_{i=1}^n c_i P(\zeta, a_i) := \langle c, p(\theta)\rangle$, $\zeta = e^{i\theta} \in T$. Let $\Lambda_E = \{(c_1, \dots, c_n) \in \mathbf{R}^n | c_i \geq 0, \langle c, p(\theta)\rangle \leq 1\, \forall \zeta = e^{i\theta} \in T\}$.

The set M_E is mapped by c isomorphically to Λ_E and, denoting by $\mathrm{Ex}(A)$ the set of extreme points of a set A,

$$\mathrm{Ex}(M_E) = \mathrm{Ex}(M_0) \cap M_E \quad \leftrightarrow \quad \mathrm{Ex}(\Lambda_E) \subset \mathbf{R}^n.$$

We conclude this section with the case where E consists of one point, cf. [1]. If $E = \{z\}$ then such a measure is completely determined by $\mu(z) = c$ and $\mu \in M_0$ if and only if $0 \leq c\frac{1-|z|^2}{|\zeta - z|^2} \leq 1$, for all $\zeta \in T$ if and only if $0 \leq c \leq \frac{(1-|z|)^2}{1-|z|^2} = \frac{1-|z|}{1+|z|}$. Then $c = 0$ gives us our well-known $\mu = \frac{d\theta}{2\pi}$ and $c = \frac{1-|z|}{1+|z|}$ gives us an extreme μ with support $T \cup \{z\}$.

3. Two point interior supports

Suppose that $E = \{a_1, a_2\} \subset D \setminus \{0\}$. Without loss of generality we will assume that $a_1 = x$ and $a_2 = ye^{i\alpha}$, $0 < x, y < 1$, $0 \leq \alpha \leq \pi$. In the previous section M_E was identified with

$$\Lambda = \Lambda_E = \{c \in \mathbf{R}^2_{\geq 0} : \langle c, p(\theta)\rangle \leq 1, \quad \forall \theta \in [0, 2\pi)\}. \tag{4}$$

In view of Lemma 2.4 the function $\langle c, p(\theta)\rangle$ assumes its maximum on the interval $S = [0, \alpha]$. It follows that

$$\Lambda = \{c \in \mathbf{R}^2_{\geq 0} : \forall \theta \in S \ \langle c, p(\theta)\rangle \leq 1\}.$$

Let

$$A(\theta) = \begin{pmatrix} p_1(\theta) & p_2(\theta) \\ p_1'(\theta) & p_2'(\theta) \end{pmatrix}.$$

Then

$$\det A(\theta) = \frac{2(1-x^2)^2(1-y^2)^2}{|\zeta - x|^2 |\zeta - a_2|^2} \left(\frac{y \sin(\alpha - \theta)}{|\zeta - a_2|^2} + \frac{x \sin\theta}{|\zeta - x|^2} \right), \tag{5}$$

with $\zeta = e^{i\theta}$. For θ in the interior of S (5) is positive, while if $a_2 \notin \mathbf{R}$, i.e. $\alpha \neq 0, \pi$, (5) is positive on all of S. Thus a real analytic map $\gamma(\theta)$ can be defined on the interior of S (and if $a_2 \notin \mathbf{R}$ even on a neighborhood of S) by solving

$$A(\theta)\gamma(\theta) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{6}$$

For points where $\gamma'(\theta) \neq 0$ the map γ defines a smooth curve. We will identify γ with its image in $\mathbf{R}^2$. In fact γ describes the envelope of the family of lines

$$l_\theta = \{c \in \mathbf{R}^2_{\geq 0} | \ \langle c, p(\theta)\rangle = 1\}. \tag{7}$$

Thus if $\gamma'(\theta) \neq 0$, then l_θ is the tangent to γ at $\gamma(\theta)$. From (4) it is clear that boundary points $(c_1, c_2) \in \mathbf{R}^2_{>0}$ of Λ are contained in γ.

Lemma 3.1. *The derivative of γ is given by*

$$\gamma_i'(\theta) = \frac{(-1)^i p_{3-i}(\theta) \det A'}{\det A^2} \tag{8}$$

If $0 < \alpha < \pi$ then (a) γ' has no zeros on S or (b) γ' has precisely two zeros on S (counting multiplicity). Moreover γ is strictly concave where $\gamma_1' < 0$ and strictly convex where $\gamma_1' > 0$. If $\gamma_1' \leq 0$ then γ is a concave curve. Otherwise there exist $0 < s < \theta_1 < \theta_2 < t < \alpha$ such that γ is concave on $[0, \theta_1) \cup (\theta_2, \alpha]$ and convex on (θ_1, θ_2), while $\gamma(s) = \gamma(t)$.

Proof. Differentiating (6), we obtain the following system for γ and γ'

$$\begin{pmatrix} A(\theta) & 0 \\ A'(\theta) & A(\theta) \end{pmatrix} \begin{pmatrix} \gamma(\theta) \\ \gamma'(\theta) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{9}$$

From Cramers rule we obtain the expression for γ_i'.

We now determine the number of zeros of $\det A'$. After some computations we find

$$\begin{aligned} \det A' &= \frac{4xy}{|\zeta - x|^4 |\zeta - a_2|^4} \times \\ &\times \left[\sin\alpha - 4\sin\theta \sin(\alpha - \theta)\left(\frac{y \sin(\alpha-\theta)}{|\zeta - a_2|^2} + \frac{x \sin\theta}{|\zeta - x|^2}\right)\right]. \end{aligned} \tag{10}$$

As the other factors are positive, we only have to study the number of zeros of the factor $[\ldots]$ of (10). Let $u = 2x/(1+x^2) \in (0,1)$, $v = 2y/(1+y^2) \in (0,1)$. Multiply $[\ldots]$ with its denominators and call the result $\psi = \psi_{u,v,a}$.

$$\begin{aligned} \psi(\theta) = \sin\alpha \; & -u[\sin\alpha\cos\theta + 2\sin^2\theta \sin(\alpha - \theta)] \\ & -v[\sin\alpha \cos(\alpha - \theta) + 2\sin\theta \sin^2(\alpha - \theta)] \\ & +uv \sin\alpha[\cos\theta\cos(\alpha-\theta) + 2\sin\theta\sin(\alpha - \theta)]. \end{aligned} \tag{11}$$

Inspection shows that $\psi(0) > 0$, $\psi(\alpha) > 0$. Furthermore ψ' is given by

$$3\sin(\alpha - 2\theta)[-u\sin\theta - v\sin(\alpha-\theta) + uv\sin\alpha].$$

The first factor has one zero at $\theta = \alpha/2$, the final factor is convex and easily seen to be negative. So ψ' has one zero on S. The conclusion is that ψ and therefore $\det A'$ both have on S either none or two zeros, counting multiplicity. We compute the slope of the tangent l_θ to γ at γ_θ. It equals

$$\operatorname{slope} l_\theta = \gamma_2'(\theta)/\gamma_1'(\theta) = -p_1(\theta)/p_2(\theta). \tag{12}$$

We see that $\operatorname{slope} l_\theta$ is negative and strictly increasing on S. Keeping in mind that $\gamma_1'(\theta)$ is negative for $\theta \in S$ close enough to 0 or α the lemma follows. $\square$

Remark 3.2. The sign of the second derivative of $\langle\gamma(\theta_0), p(\theta)\rangle$ at θ_0 equals the sign of $\gamma_1'(\theta_0)$. A maximum (resp. minimum) of $\langle\gamma(\theta_0), p(\theta)\rangle$ at θ_0 corresponds to concavity (convexity) of γ at θ_0. Indeed (11) gives $\langle\gamma(\theta), p''(\theta)\rangle = -\langle\gamma'(\theta), p'(\theta)\rangle \stackrel{\text{def}}{=} g(\theta)$. This gives the matrix equality $A(\gamma, \gamma') = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -g(\theta) \end{smallmatrix}\right)$.

Since $\det A > 0$ we have for every $\theta \in S$ $\operatorname{sign} \det(\gamma, \gamma') = -\operatorname{sign} \gamma_1(\theta)' = -\operatorname{sign} g(\theta)$.

Theorem 3.3. *We keep the notation as above. If $0 < \alpha < \pi$ and γ' is zero free, then the extreme points of Λ are formed by the origin and the smooth curve γ, while in case γ' has two zeros on S, the extreme points are formed by the origin and a piecewise smooth curve contained in γ which connects $(\frac{1-x}{1+x}, 0)$ and $(0, \frac{1-y}{1+y})$ and consists of two smooth parts. If $\alpha = \pi$, then M_E is a quadrangle in $\mathbf{R}^2_{\geq 0}$ with boundary determined by the two axes and the lines l_0 and l_π.*

Proof. We keep $a_1 > 0$ and assume $0 < \alpha < \pi$ first. Let $\partial\Lambda$ denote the intersection of the boundary of Λ and $\mathbf{R}^2_{>0}$. Observe that by (6) $\gamma(0) = (1/p_1(0), 0)$ and $\gamma(\alpha) = (0, 1/p_2(\alpha))$ in conformity with the $|E| = 1$ situation; moreover these are the only points of intersection of γ with the positive axes. We know that Λ is a convex set in the first quadrant with $\partial\Lambda$ contained in γ. If $\det A' \geq 0$ on S then γ is a concave curve in the first quadrant connecting $\gamma(0)$ and $\gamma(\alpha)$ and strictly concave except for maybe one point θ_0 with $\det A'(\theta_0) = 0$. We conclude that $\overline{\partial\Lambda} = \gamma$ and γ consists of extreme points.

On the other hand if $\det A'$ is negative somewhere, then, keeping the notation of Lemma 3.1 we have $\gamma(s) = \gamma(t)$ and

$$\overline{\partial}\Lambda = \{\gamma(\theta) : \ \theta \in 0, s \cup [t, \alpha]\}.$$

This part of γ is strictly concave, so again all points of $\overline{\partial\Lambda} = \gamma$ are extreme. If $\alpha = \pi$ then it is clear that $\langle c, p(\theta)\rangle$ has extreme points at $\theta = 0, \pi$. Because of the remark after (3.1), γ is convex and so, using symmetry, a stationary point $\theta \notin \{0, \pi\}$ of $\langle c, p(\theta)\rangle$ must be a minimum. It follows that at $0, \pi$ the function $\langle c, p(\theta)\rangle$ has two maxima. They must both be ≤ 1, so M_E is indeed described by $\{c \in \mathbf{R}_{\geq 0} : \ l_0(c) \leq 1, \ l_\pi(c) \leq 1\}$. ☐

The next theorem will show that both cases of Theorem 3.3 can occur.

Definition 3.4. Let $a_1, a_2 \in D \setminus \{0\}$. The pair $\{a_1, a_2\}$ is called *good* if a_1, a_2 are positively dependent or if they are not collinear and $\det(\dot{A}) \geq 0$ on S, *bad* otherwise.

Theorem 3.5. *a) (Non-collinear) bad pairs exist.*
b) If (a_1, a_2) is good then $(\lambda a_1, \mu a_2)$ is good for $\lambda, \mu \in (0, 1]$.
c) For every $\alpha \in (0, \pi)$ there is a constant $0 < \delta < 1$, such that all pairs $\{a_1, a_2\}$ in a sector of aperture α and radius δ are good. Hence (nontrivial) good pairs exist.

Proof. We may assume $a_1 = x > 0$, $a_2 = ye^{i\alpha}$, $0 < \alpha < \pi$.
a) Suppose that all pairs were good, then (10) would be nonnegative for all $0 \leq \theta \leq \alpha, 0 < x, y < 1$. Then this is also valid for $x = 1 = y$, $\theta = \frac{\alpha}{2}$ by continuity. Substituting in (11) gives, however,

$$\psi_{1,1,\alpha}(\alpha/2) = \sin\alpha - 4\sin(\alpha/2) + \sin(\alpha)(1 + \sin^2(\alpha/2))$$

which is negative if $0 < \alpha < \pi$. Therefore, there exist bad pairs.
b) This follows from (10) since the factor $[\dots]$ is decreasing as x, y increase.
c) This follows from (10) since the terms other than $\sin\alpha$ in $[\dots]$ can be made arbitrarily small.

$\square$

4. Finite interior support

In case $|E| \geq 3$ we have only fragmentary results.

Definition 4.1. A finite subset $\{a_1, \dots, a_n\}$ of D is called *good* if every pair $\{a_i, a_j\}$, $1 \leq i < j \leq n$ is good.

One might wonder if every finite set in D that is not contained in an interval $[0, e^{i\theta}]$, cf., Lemma 2.6, is full support for some extreme measure $\mu \in M_0$. The answer is no, as we will now see.

Theorem 4.2. *Let $n \geq 3$ and let $E = \{a_1, \dots, a_n\} \subset D \setminus \{0\}$ be good and suppose E is contained in a sector of aperture $\leq \pi$. Then there is no extreme $\mu \in M_0$ with* $\operatorname{supp}\mu_0 = E$.

Proof. Let $\mu \in M_E$ be extreme. Put $c = c(\mu)$ as introduced in 2.7 and suppose $c_i > 0$ for all i. Let $\theta_i = \arg a_i$. We may assume that $0 = \theta_1 \leq \theta_2 \leq \dots \leq \theta_n < \pi$. Using Lemma 2.4 one easily sees that $\langle c, p(\theta)\rangle$ assumes its maximum only on the arc $[0, \theta_n]$. Moreover, since μ is a boundary point of M_0 this maximum equals 1. Also if $\langle c, p(\theta)\rangle = 1$ for $c \in \Lambda_E$, then $\langle c, p'(\theta)\rangle = 0$ and $\langle c, p''(\theta)\rangle \leq 0$.

Let V_θ be the part of the hyperplane $\langle x, p(\theta)\rangle = 1$, $\langle x, p'(\theta)\rangle = 0$ defined by $V_\theta \subset \mathbf{R}^n_{\geq 0}$. It is clear that there exists a $\theta \in [0, \theta_n]$ such that $c \in V_\theta$. If we are able to show that $V_\theta \subset \Lambda_E$ then $c \in \mathrm{Ex}(\Lambda_E) \Rightarrow c \in \mathrm{Ex}(V_\theta)$ which implies that at least one of the c_i vanishes, leading to a contradiction. The proof is thus completed by the following

Lemma 4.3. *$V_\theta \subset \Lambda_E$ if $\zeta \in [0, \theta_n]$.*

Proof. (of the lemma) For $n = 2$ this follows from the previous theory by distinguishing two cases: If $\theta_2 = 0$, then $\theta = 0$, and $p'(0) = 0$ so $V_0 = \mathbf{R}^2_{\geq 0} \cap \langle c, p(0)\rangle = 1\} \subset \Lambda_E$. In the second case we have $V_\theta = \{\gamma(\theta)\} \subset \Lambda_E$ because every pair is good by hypothesis. We will proceed by induction, supposing $n \geq 3$. Note that $V_\theta \subset \Lambda_E \iff \mathrm{Ex}(V_\theta) \subset \Lambda_E$.

Let q be an extreme point of V_θ, then $q_k = 0$ for a certain $1 \leq k \leq n$. We can apply the induction hypothesis as soon as we know that θ is in the interval determined by the remaining θ_j, $(j \neq k)$. This is certainly the case if $1 < k < n$. We next deal only with $k = n$, observing that $k = 0$ can be treated similarly.

Assume that $q_n = 0$. In view of Lemma 2.4 $q_j p_j(\theta)$ is decreasing on $(\theta_j, \pi + \theta_j)$, hence $\langle q, p(\theta)\rangle = \sum_{j=0}^{n-1} q_j p_j(]th)$ is decreasing on $(\theta_{n-1}, \pi]$. Therefore $\langle q, p'(\varphi)\rangle \neq 0$ for $\varphi \in (\theta_{n-1}, \pi]$. Thus $q \in V_\theta \Rightarrow \theta \in [0, \theta_{n-1}]$. $\square$

$\square$

On the other hand there may also be extreme points supported on all of E, if $|E| = 3$.

Theorem 4.4. *Let $E = \{a_1, a_2, a_3\} \subset D \setminus \{0\}$ with 0 in the interior of* $\mathrm{CH}(E)$. *Then there exists an extreme $\mu \in M_0$ with* $\mathrm{supp}(\mu) = T \cup E$.

Proof. We may assume that $\arg a_1 = 0 < \arg a_2 < \arg a_3 < 2\pi$. Let F be the orthogonal complement of $p(0)$ and $p'(0)$ in $\mathbf{R}^3$. Then F is one-dimensional, spanned by a single vector f. Because $p_1'(0) = 0$ and $p_2'(0)p_3'(0) < 0$, we may assume that $f_2, f_3 > 0$ and, consequently, $f_1 < 0$.

Define $\tilde{c} = (\frac{1-a_1}{1+a_1}, 0, 0)$, the extreme point of Λ_E on the positive c_1 axis. Let $c_\lambda = \tilde{c} + \lambda f$ for $\lambda \geq 0$. Notice that all $c \in \mathbf{R}^3$ with the property that $\langle c, p(0)\rangle = 1$, $\langle c, p'(0)\rangle = 0$ are of the form c_λ. Now we have

$$c_\lambda \in \Lambda_E \iff \forall\theta \in [0, 2\pi) : \lambda\langle f, p(\theta)\rangle \leq 1 - \langle \tilde{c}, p(\theta)\rangle.$$

Observe that the right-hand side of the second inequality is positive on $(0, 2\pi)$ and has a double zero at $\theta = 0$. Its left-hand side also has (at least) a double zero at $\theta = 0$. Thus for sufficiently small but positive λ the last inequality is satisfied and c_λ belongs to Λ_E. If we let λ increase, then $c_{\lambda,1}$ decreases while $c_{\lambda,2}$, $c_{\lambda,3}$ increase. Let λ_1 be the largest value such that $c_{\lambda_1} \in \Lambda_E$. Then all coordinates of c_{λ_1} are positive. Otherwise, $c_{\lambda_1,1}$ would equal 0. We show that that this would imply $c_{\lambda_1} \notin \Lambda_E$. Indeed, $\langle c_{\lambda_1}, p(0)\rangle = 1$, but the maximum of $\langle c_{\lambda_1}, p(\theta)\rangle$ is attained on $(\arg a_2, \arg a_3)$ which does not contain 0, because $\arg a_3 - \arg a_2 < \pi$ by the convexity assumption. The point c_{λ_1} is an extreme point of Λ_E. If c_{λ_1} could be written as a convex combination of $d, e \in \Lambda_E$, then d, e would be of the form c_λ because of the behavior at 0 and this implies that $d = e = c_{\lambda_1}$. $\square$

Remark 4.5. The preceding theorem is based on the fact that there is a θ such that the half-line V_θ introduced in the proof of theorem 4.2, has a segment in common with Λ_E, such that one of the endpoints is in $\mathbf{R}^3_{>0}$. This happens precisely in the following situation. Let $E_i = E \setminus \{a_i\}$. Theorem 3.3 describes the extreme point of Λ_{E_i} as points of the form $\gamma_i(\theta)$, $\theta \in A_i \subset [0, 2\pi)$. If we can choose $\theta \in A_i$ but outside the other A_j's, then V_θ has the required property. Such a choice of θ is certainly possible if all 3 pairs are bad.

5. Exposed points of M_0

Recall that if $p \in C$ is a boundary point of a convex set C in some topological vector space, then there exists a supporting hyperplane at that point p. If the hyperplane H can be chosen with $H \cap C = \{p\}$ then p is called an exposed point of C. In our case the vector space is M, the space of complex Borel measures on $\overline{D}$ endowed with the weak-* topology and hyperplanes are sets of the form $H = \{\mu : \int f\, d\mu = c\}$ where $c \in \mathbf{C}$ and $f \in C(\overline{D}) \setminus \{0\}$.

Again we consider relatively closed subsets E of D and the subsets M_E of M_0 of representing measures defined in the introduction.

Let X be the closure of the convex hull of $\{P(\zeta)|\, |\zeta| > 1\} \cup \{f \in C(\overline{D})|\, f \leq 0\}$ in $C(\overline{D})$. Introduce $\mathcal{M}_0 = \{\mu \in M|\, \mu(T) = 0,\ \langle \mu, x\rangle \leq 1,\ x \in X\}$ and its weak-* closure is $\bar{\mathcal{M}} = \{\mu \in M|\ \langle \mu, x\rangle \leq 1,\ x \in X\}$.

Lemma 5.1. *Suppose that $h \in C(\overline{D})$ has the property that $\langle \mu, h\rangle \leq 1$ for all $\mu \in \mathcal{M}_0$. Then $h \in X$.*

Proof. This is a simple application of the Hahn-Banach theorem. X is a closed convex neighborhood of 0. If $h \notin X$, then the Minkowski functional p of X is larger than 1 at h. Its restriction to the halfline $\{th|\, t > 0\}$ can be extended to a linear functional $\mu \in M$ that is bounded by p. Then $\mu \in \bar{\mathcal{M}}$ and $\langle \mu, h\rangle > 1$, a contradiction. $\square$

Because $\mathcal{M}_0$ is starshaped with respect to 0 (i.e. $\mu \in \mathcal{M}_0$ implies $t\mu \in \mathcal{M}_0$ for $0 \leq t \leq 1$) a supporting hyperplane for a non zero measure $\mu \in \mathcal{M}_0$ cannot pass through 0. We can thus assume $c = 1$ and H is of the form $\langle \mu, f\rangle = 1$, $f \in C(\overline{D})$, and $\langle \nu, f\rangle \leq 1$ for $\nu \in \mathcal{M}_0$. We call f the supporting functional.

Lemma 5.2. *Suppose $\mu \in \mathcal{M}_0$ is a (non-zero) boundary point with supporting functional $h \in C(\overline{D})$. Then there exists a probability measure σ such that $h \leq P[\sigma]$ with equality on $D \cap \operatorname{supp}\mu$ and σ is concentrated on $\{\zeta \in T|\, \langle \mu, P(\zeta)\rangle = 1\}$.*

Proof. By Lemma 5.1 we see that h is a uniform limit on $\overline{D}$ of continuous functions G_N, where $G_N = H_N - f_N$ with $f_N \in C(\overline{D})$ nonnegative and

$$H_N(z) = \sum_{j=1}^{M} c_j P(\zeta_j, z), \tag{13}$$

with $c_j > 0$, $\sum c_j \leq 1$, $|\zeta_j| > 1$ and all depending on N. Observe that H_N are positive harmonic functions on $\overline{D}$ with $H_N(0) = \int_T H_N(e^{i\theta})\frac{d\theta}{2\pi} \leq 1$. After taking a subsequence if necessary, we may assume that H_N converges weak-* to some positive measure σ on T. The harmonic functions H_N will then converge uniformly on compact sets in D to a continuous function. Uniform convergence of the G_N then implies that f_N converges weak-* to some nonnegative measure on $\overline{D}$ and uniformly on compact sets to some continuous $f \geq 0$ on D. We have

$$\begin{aligned} 1 = \int_D \lim_{N\to\infty} G_N\, d\mu &= \lim_{N\to\infty}\left(\int H_N\, d\mu - \int f_N\, d\mu\right) \\ &\leq \liminf_{N\to\infty}\left(H_N(0) - \int f_N\, d\mu\right) \\ &\leq \lim_{N\to\infty} H_N(0) - \limsup_{N\to\infty}\int f_N\, d\mu \leq 1. \end{aligned} \tag{14}$$

We conclude that $H_N(0) \to 1$, which implies $|\sigma| = 1$, and $f_N \to 0$ in $L^1(\mu)$ and uniformly on compact sets in $D \cap \operatorname{supp}\mu$ as $N \to \infty$. It also follows that $h \le \lim H_N$ as measures on $\overline{D}$ with equality at least on $D \cap \operatorname{supp}\mu$.

Now recall that $\langle \mu, P(\zeta)\rangle$ is a nonnegative lower semicontinuous function ≤ 1 which gives meaning to $\int \langle \mu, P(\zeta)\rangle \, d\sigma$. Using that $h = P[\sigma]$ on $D \cap \operatorname{supp}\mu$ we obtain

$$1 = \int_D P[\sigma]\, d\mu = \int_T \langle \mu, P(\zeta)\rangle \, d\sigma. \tag{15}$$

From this we infer that σ is concentrated on $\{\zeta \in T | \langle \mu, P(\zeta)\rangle = 1\}$. $\square$

Lemma 5.3. *If $\mu = \mu_0 + \mu_1$ is an exposed point of M_0 with $\mu_0 \neq 0$, then μ_0 is an exposed point of $\mathcal{M}_0$. The exposing functional can be chosen to be a continuous function f on $\overline{D}$, such that $f = 0$ on T and $f = P[\sigma]$ on $D \cap \operatorname{supp}\mu_0$ for some probability measure σ concentrated on $\{\zeta \in T | \langle \mu, P(\zeta)\rangle = 1\}$.*

Proof. Let h be an exposing functional for μ. Then

$$\langle \mu, P[h]\rangle = \langle \frac{d\theta}{2\pi}, P[h]\rangle = \langle \frac{d\theta}{2\pi}, h\rangle < \langle \mu, h\rangle$$

Set $c = 1/\langle \mu, h - P[h]\rangle$ and

$$f(z) = c(h - P[h])(z)$$

Then for $\nu \neq \mu$ in M_0

$$\begin{aligned} \langle \nu, f\rangle &= c(\langle \nu, h\rangle - \langle \nu, P[h]\rangle) = c(\langle \nu, h\rangle - P[h](0)) \\ &< c(\langle \mu, h\rangle - \langle \mu, P[h]\rangle) = \langle \mu, f\rangle. \end{aligned}$$

Thus $\langle \mu, f\rangle = 1$, f is an exposing functional, and $f|_T = 0$. Now

$$\langle \mu_0, f\rangle = \langle \mu, f\rangle > \langle \nu, f\rangle = \langle \nu_0, f\rangle,$$

for all $\nu_0 \neq \mu_0$ in $\mathcal{M}_0$. This shows that μ_0 is exposed in $\mathcal{M}_0$ and the final statements follow from Lemma 5.2. $\square$

Let $A = \{a_n\}_n$ be a sequence in D without interior limit points and with the property that almost every point of T is non-tangential limit point of A. Brown Shields and Zeller showed that there exist representing measures μ with $\mu(A) = 1$, see [2]. Let us call such representing measures BSZ measures. In [1] it was shown that there exist BSZ measures that are extreme in M_0. However:

Proposition 5.4. *No BSZ measure is exposed in M_0, nor in $\mathcal{M}_0$.*

Proof. To reach a contradiction, suppose that μ is a BSZ measure with mass on the sequence A that is exposed in M_0. Then it would be exposed in $\mathcal{M}_0$ by the previous lemma. If μ is exposed in $\mathcal{M}_0$, then the exposing functional may be assumed of the form $h = P[\sigma]$ on a. On a small enough tail of A there exists a BSZ measure μ' not equal to μ. However $\langle h, \mu'\rangle = \langle P[\sigma], \mu'\rangle = 1$, contradicting that h is the exposing functional. $\square$

Remark 5.5. A little more generally we may consider a closed subarc J of T of length $\leq \pi$ and a sequence A in D without limit points in $\overline{D} \setminus J$ and such that almost every point of J is a non-tangential limit point of a subsequence of A. Then one can see that the proof of [2] gives representing measures μ such that $\mu(J) = 0$ and $\mu(a \cup T) = 1$. If A is contained in the sector determined by J, then μ_0 is real analytic and zero free on $T \setminus J$. The proof of Proposition 5.4 applies (with the note that $l = \sigma$ on $T \setminus J$) and shows that such measures cannot be exposed in M_0.

We will now turn to the case where E is finite and study M_E and the associated subsets $\Lambda = \Lambda_E$ of $\mathbf{R}^n$. Izzo proved the following Theorem 5.8 for the case where E consists of 1 point, cf., [5].

Proposition 5.6. *Let E be finite in D and let ν_0 be a positive measure on E. The following are equivalent*
i. *ν_0 is an exposed point of Λ_E.*
ii. *ν_0 is an exposed point of $\mathcal{M}_0$.*
iii. *$\nu_0 + (\frac{1}{2\pi} - P_{\nu_0})d\theta$ is an exposed point of M_0*

Proof. i$\Rightarrow$ iii. The methods of Lemma's 5.1, 5.2 readily give that the exposing functional h on E must be of the form $P[\sigma]$, with $\operatorname{supp} \sigma \subset \{\zeta | \, P(\zeta, \nu_0) = 1\}$. This is a finite set, therefore σ is singular and $P[\sigma] = 0$ a.e. on T. We now choose $\tilde{h}$ in $C(\overline{D})$ such that $\tilde{h} \leq P[\sigma]$ on D with equality only on E, while $\tilde{h} = 0$ on T. We claim that $\tilde{h}$ exposes ν in M_0. Indeed, for $\mu \in M_0$

$$\langle \tilde{h}, \mu \rangle = \langle \tilde{h}, \mu_0 \rangle \leq \langle P[\sigma], \mu_0 \rangle \leq 1.$$

To have equality in the first inequality, we must have that $\operatorname{supp} \mu_0 \subset E$. But then equality in the last inequality can only occur if $\mu_0 = \nu_0$. It follows that $\mu = \nu$. Lemma 5.3 states that iii$\Rightarrow$ii; finally ii$\Rightarrow$i is obvious. $\square$

Definition 5.7. We will call an extreme point $\mu_0 \in M_E$ *regular* if μ_1 has only zeros of order 2.

Theorem 5.8. *Suppose that E is finite. Then every regular extreme point of M_E is an exposed point of M_E.*

Proof. Let $E = \{a_1, \ldots, a_n\} \subset D$. As in the previous section we may identify M_E with a convex subset $\Lambda = \Lambda_E$ in the positive octant:

$$\Lambda_E = \{c \in \mathbf{R}^n_{\geq 0} | \, \langle c, p(\theta) \rangle \leq 1 \, \forall \theta \in [0, 2\pi)\}. \tag{16}$$

The proof is by induction on n, starting at $n = 3$. For $n = 1$ it is clear that an extreme m_0 is exposed in M_E, if $n = 2$ it follows from the description of the boundary of M_E in Theorem 3.3 as a concave curve which does not contain intervals, that every extreme point is exposed in Λ_E and by Proposition 5.6 also of M_E. Suppose c_0 is an extreme point of Λ. We may assume that all components of c_0 are positive, otherwise we could reduce n and apply Proposition 5.6 to show that such a point is exposed within M_0, hence also in M_E. Consider the set

$$S = \{\zeta \in T | \, \langle c_0, P(\zeta) \rangle = 1\}.$$

Clearly S is a finite subset of T since $\langle c_0, P(\zeta)\rangle$ is real analytic and non constant on T. Also S contains at least one point, otherwise c_0 is not even in the boundary of M_E. Now let

$$X_S = \{c \in \Lambda_E : \langle c, P(\zeta)\rangle = 1, \zeta \in S\}.$$

We claim that $X_S = \{c_0\}$. Obviously $c_0 \in X_S$. If there exists an additional point $c \in X_S$, then X_S would be a convex subset of Λ_E that contains the interval $[c_0, c]$ Consider $c_\epsilon = c_0 + \epsilon(c_0 - c)$, where ϵ will be chosen positive. For $\zeta \in S$, $\zeta = e^{i\theta}$ we have $\langle c_\epsilon, p(\theta)\rangle = 1$ and $\langle c_\epsilon, p'(\theta)\rangle = 0$ Moreover, since c_0 is regular, we have $\langle c_0, p''(\theta)\rangle > \delta > 0$ on a neighborhood of S. Therefore, for $0 < \epsilon < \epsilon_0$ sufficiently small, $\langle c_\epsilon, p''(\theta)\rangle > \delta/2$ for $\zeta = e^{i\theta}$ in a smaller neighborhood V only depending on ϵ_0. Thus on V we have $\langle c_\epsilon, P(\zeta)\rangle < 1$. On $T \setminus V$ we have $\langle c_0, P(\zeta)\rangle \leq 1 - \eta$, hence for ϵ sufficiently small we have $\langle c_\epsilon, P(\zeta)\rangle < 1$ on $T \setminus V$ The conclusion is that $c_\epsilon \in \Lambda_E$. We can now write c_0 as a convex combination of c_ϵ and c, contradicting that c_0 is extreme. For the exposing functional for c_0 we may now take

$$w = \frac{1}{n}\sum_{\zeta \in S} P(\zeta),$$

where $n = \#S$. We have $\langle c_0, w\rangle = 1$ and $\langle c, w\rangle = 1$ implies that for all $\zeta \in S$ $\langle c, P(\zeta)\rangle = 1$, hence $c \in X_S$, that is, $c = c_0$. $\square$

Remark 5.9. What will happen if c_0 is extreme but not regular? We keep the notation of the previous theorem. It is clear from the proof and the fact that exposing functionals must be of the form $P[\sigma]$ with σ supported in S, that $\{c_0\} = X_S$ is equivalent with c_0 being exposed. If $n = 2$ this will always be the case, although c_0 be non-regular precisely in the borderline case when γ' has one zero with multiplicity 2, cf., Lemma 3.1. On the other hand, a little experimenting with maple gives the following example of an extreme point which is not exposed. Take three points $a_1 = 1/10$, $a_2 = \frac{3}{20}e^{i\pi/3}$, $a_3 = \bar{a}_2$. We will put weight $c_1 = 1710234/42508549$ at a_1 and equal weights $c_2 = c_3 = 1$ at a_2 and a_3. We obtain a point in the boundary of $\Lambda = \Lambda_{\{a_1,a_2,a_3\}}$ by multiplying with a suitable positive constant. Call this point c. Then the Poisson transform $\langle c, p(\theta)\rangle$ will have its only maximum at $\theta = 0$ with the property that the first three θ-derivatives vanish at $\theta = 0$. The point c is an extreme point of Λ. It is, however, not exposed, because X_S is the convex hull of c and $(\frac{1-a_1}{1+a_1}, 0, 0)$.

References

[1] **Raymond Brummelhuis** and **Jan Wiegerinck**, *Representing measures for the disc algebra and the ball algebra,* Annales Polonici Mathematici 55 (1991), 19-35.

[2] **L. Brown, A. Shields** and **K. Zeller**, *On absolutely convergent exponential sums,* Trans. Am. Soc. 96 (1960), 162-183.

[3] **T. Gamelin,** *Uniform Algebras,* Prentice-Hall, Englewood Cliffs, N.J., 1969.

[4] **J.B. Garnett,** *Bounded Analytic Functions,* Academic Press, New York, 1981

[5] **A.J. Izzo,** *Exposed points in the set of representing measures for the disc algebra,* Annales Polonici Mathematici 61 (1995), 59-62.

[6] **J. Ryff,** *The support of representing measures for the disc algebra,* in: Function Algebras, F. Birtel (ed.), Scott, Foresman and Company, Glenview, 1966.

[7] **P. Koosis,** *Theory of H^p spaces,* Cambridge Univ. Press, Cambridge, 1980

Mathematical Institute Univ. of Leiden, P.O. Box 9512 2300 RA Leiden, the Netherlands

E-mail address, Alex Heinis: `heinis@wi.leidenuniv.nl`

Korteweg de Vries Institute, Univ. of Amsterdam, Plantage Muidergracht 24, 1018 TV Amsterdam, the Netherlands

E-mail address, Jan Wiegerinck: `janwieg@wins.uva.nl`