THE PLURIPOLAR HULL OF $\{w = e^{-1/z}\}$

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Abstract. In this paper we show that the pluripolar hull of $E = \{(z,w) \in \mathbb{C}^2 : w = e^{-1/z},\ z \neq 0\}$ is equal to E. This implies that E is pluri-thin at 0, which answers a question of Sadullaev. The result remains valid if $e^{-1/z}$ is replaced by certain other holomorphic functions with an essential singularity at 0.

1. Introduction

In [5] Sadullaev poses the following question. Consider the sets

$$E_1 = \{(x,y) \in \mathbb{C}^2 : y = x^\alpha,\ x \in (0,1)\}$$

with α irrational, and

$$E_2 = \{(x,y) \in \mathbb{C}^2 : y = e^{-1/x},\ x \in (0,1)\}.$$

Are the sets E_j pluri-thin at the origin? That is, does there exist a plurisubharmonic (PSH) function h on a neighborhood V of $\overline{E_j}$ such that

$$h(0) > \limsup_{\substack{x \to 0 \\ x \in E_j}} h(x).$$

See also Bedford's survey [1]. In [3] Levenberg and Poletsky show that E_1 is pluri-thin at 0. In fact, they show something stronger: there exists a negative PSH function on the polydisk such that $h|_{E_1} = -\infty$, while $h(0) > -1$.

In this note we will prove the same result for E_2 and a related class of pluripolar sets. Our proof follows the line of argument of [3]. For convenience of the reader we summarize in the next section the relevant parts of [3], where all proofs may be found.

2. Preliminaries

2.1. **Pluripolar Hulls.** A set $E \subset \mathbb{C}^N$ is called *pluripolar* if there exists a PSH function $u \not\equiv -\infty$, defined on a neighborhood of E such that $E \subset \{z : u(z) = -\infty\}$.

There are two kinds of *pluripolar hulls* of a pluripolar set E relative to a neighborhood D of E. One defines

$$E_D^* = \bigcap_{u \in \mathcal{F}} \{z \in D : u(z) = -\infty\},$$

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where $\mathcal{F}$ is the set of all PSH functions on D which are $-\infty$ on E; next one defines

$$E_D^- = \bigcap_{u \in \mathcal{F}^-} \{z \in D : \ u(z) = -\infty\},$$

where $\mathcal{F}^-$ stands for the set of all negative PSH functions on D which are $-\infty$ on E. Of course, $E_D^* \subset E_D^-$. Moreover, these hulls are related as follows, see [3],

Theorem 1. *Let D be pseudoconvex in $\mathbb{C}^N$ and $E \subset D$ pluripolar. Suppose $D = \cup_j D_j$, where D_j form an increasing sequence of relatively compact subdomains of D. Then*

$$E_D^* = \bigcup_j (E \cap D_j)_{D_j}^-.$$

Moreover, if D is hyperconvex, that is, D admits a bounded PSH exhaustion function, then

$$E_D^- = \bigcup_j (E \cap D_j)_{D_j}^-.$$

2.2. **Harmonic Measure.** Let E be a subset of a domain $D \subset \mathbb{C}^n$. The *harmonic measure* at $z \in D$ of E relative to D is the number

$$\omega(z, E, D) = -\sup\{u(z) : \ u \text{ PSH on } D \text{ and } u \leq -\chi_E\}.$$

Here χ_E is the characteristic function of E on D.

Notice that this boils down to the usual concept of harmonic measure if $N = 1$. The harmonic measure has the following properties:

$$\omega(z, E, D) = \inf_{\substack{V \text{open} \\ E \subset V \subset D}} \omega(z, V, D)$$

and

$$\omega(z, E_1 \cup E_2, D) \leq \omega(z, E_1, D) + \omega(z, E_2, D). \tag{2.1}$$

Moreover, there is the following important duality result due to Poletsky [4]:

$$\omega(z, V, D) = \frac{1}{2\pi} \sup_f \{m(\{t \in (0, 2\pi) : \ f(e^{it}) \in V\})\},$$

where f runs over all analytic disks $f : \{|\zeta| \leq 1\} \to D$ with $f(0) = z$, while m denotes Lebesgue measure.

The harmonic measure is connected to pluripolar hulls.

Proposition 2. *Let D be a hyperconvex domain in $\mathbb{C}^N$ and let $E \subset D$ be pluripolar. Then*

$$E_D^- = \{z \in D : \omega(z, E, D) > 0\}.$$

To compute $\omega(z, E, D)$ we will use some more results from [3].

Lemma 3. *Let $D \subset \mathbb{C}^N$ be a domain, $E \subset D$ and let $A \subset D \setminus E$ be closed and pluripolar. Then for $z \in D \setminus A$ we have $\omega(z, E, D) = \omega(z, E, D \setminus A)$.*

Proposition 4. *Let $D \subset \mathbb{C}^N$ and $G \subset \mathbb{C}^M$ be domains and let $h : D \to G$ be holomorphic. Then for $z \in D$ and $E \subset G$*

$$\omega(z, h^{-1}(E), D) \leq \omega(h(z), E, G). \tag{2.2}$$

Moreover, equality holds in (2.2) if the following additional requirements are met: The map h is a covering map; E admits a simply connected open neighborhood V such that $h^{-1}(V)$ is a disjoint union of connected open V_j; $z \in D$ has the property

$$\lim_{j\to\infty} \omega(z, \cup_{k=j}^{\infty} V_k, D) = 0 \tag{2.3}$$

Lemma 5. *Let D be a relatively compact subdomain of a domain $G \subset \mathbb{C}^N$. Suppose that $E \subset D$ is compact, $V \subset G \setminus E$ is a domain and $z \in V \cap D$. Put $K = \overline{\partial V \cap D}$. If $\omega(z, E, D) = a$ then $\omega(w, E, G) \geq a$ for some $w \in K$.*

3. Hulls of graphs at essential singularities

In this section we prove the main result. We start with a lemma which supplements Proposition 4 and is a variation of Lemma 3.2 in [2].

Lemma 6. *Let $D \subset \mathbb{C}^N$ and $G \subset \mathbb{C}^M$ be domains and let $h : D \to G$ be a finite branched holomorphic covering. Then for $z \in D$ and $E \subset G$*

$$\omega(z, h^{-1}(E), D) = \omega(h(z), E, G). \tag{3.1}$$

Proof. Let $z \in D$. In view of Proposition 4, we only have to prove the inequality $\omega(z, h^{-1}(E), D) \geq \omega(h(z), E, G)$. Let $\epsilon > 0$ and let u be PSH on D, $u \leq -\chi_{h^{-1}(E)}$ and $u(z) \geq -\omega(z, h^{-1}(E), D) - \epsilon$. Define the function $\tilde{u}$ on G by

$$\tilde{u}(w) = \max_{z\in h^{-1}(w)} u(z).$$

Obviously, $\tilde{u}$ is PSH outside the branch locus B of h. The set B is pluripolar and $\tilde{u}$ is bounded from above, therefore the function u^* defined by

$$u^*(w) = \limsup_{\substack{v\to w\\ v\notin B}} \tilde{u}(v)$$

is a PSH extension of $\tilde{u}|_{G\setminus B}$ to G. Clearly $\tilde{u} = u^*$ on all of G. Moreover $\tilde{u} \leq -\chi_E$. Hence

$$\omega(h(z), E, G) \leq -\tilde{u}(h(z)) \leq -u(z) \leq \omega(z, h^{-1}(E), D) + \epsilon.$$

It follows that $\omega(z, h^{-1}(E), D) \geq \omega(h(z), E, G)$ and we are done. □

In order to keep close to [3], where a covering map in the first coordinate is used, we formulate our theorem with w as independent variable. The symbol M will denote a positive number or ∞. Let φ denote a branched cover of the Riemann sphere of the form

$$\varphi(z) = \alpha z + \frac{\beta}{z},$$

where $\alpha, \beta \in \mathbb{C}$ are not both equal to 0.

Theorem 7. *Suppose that f is holomorphic on $\{|w| < M\} \subset \mathbb{C}$, $f(0) \neq 0$ and let n be a positive integer. Let $E = \{(z, w) \in \mathbb{C}^2 : z = \varphi(e^{f(w)/w^n}), 0 < |w| < M\}$ and let Ω be the domain $\{(z, w) \in \mathbb{C}^2 : |w| < M\}$. Then $E^*_\Omega = E$.*

Proof. In view of of Theorem 1, it suffices to show that $(E \cap D)^-_D = E \cap D$ for every bidisk D compactly contained in Ω. We may assume that $M > 1$ and that D equals $kU \times U$, where U is the unit disk in $\mathbb{C}$ and $k \geq 1$. Observing that $E \cap D$ is never empty, let $\Delta = \{z : |z - a| \leq b\} \subset U$, $(b < |a|)$, be contained in the image of U under the map $w \mapsto \varphi(e^{f(w)/w^n})$. Write $E^- = (E \cap D)^-_D$. Let η be so small that the set

$$F = \{(z, w) : z = \varphi(e^{f(w)/w^n}), z \in \Delta, \eta \leq |w| \leq 1 - \eta\}$$

is compact and contains a relatively open subset of the analytic variety E. Then we have $F_D^- = E^-$. In view of Proposition 2, the proof is complete once we have shown that

$$\omega((z,w),F,D) > 0 \Longleftrightarrow (z,w) \in E \cap D.$$

Now the proof divides into two cases, namely $\alpha \cdot \beta = 0$ or $\alpha \cdot \beta \neq 0$. In the first case we may assume $\alpha = 1$, $\beta = 0$ and proceed as follows.

First let $(z,w) \in D \setminus E$, $z \neq 0$. Let $A = D \cap \{z = 0\}$. Then by Lemma 3

$$\omega((z,w),F,D) = \omega((z,w),F,D \setminus A). \tag{3.2}$$

Let $H = \{\zeta : \Re\zeta < 0\} \subset \mathbb{C}$, $G = H \times U$ and $h : G \to D \setminus A$, $(\zeta, w) \mapsto (ke^\zeta, w)$. The map h is a holomorphic covering map. Let $T = \{z : |z-a| < b+\epsilon\} \subset U \setminus \{0\}$, so that $|\arg z - \arg a| < \pi/2$ on T. The set $V = T \times U$ is a simply connected neighborhood of F. We have

$$h^{-1}(V) = \cup_{j=-\infty}^{\infty}(T_j' \times U),$$

where $T_j' \subset \{\zeta : \Re\zeta < 0,\ (2j-1/2)\pi < \Im\zeta - \arg a < (2j+1/2)\pi\}$. Then for the (ordinary) harmonic measure we have

$$\lim_{j\to\infty} \omega(\zeta, \bigcup_{|l|\geq j} T_l', H) = 0,$$

hence by Proposition 4 applied to the map $(\zeta,w) \mapsto \zeta$, also

$$\lim_{j\to\infty} \omega((\zeta,w), (\bigcup_{|l|\geq j} T_l') \times U, G) = 0.$$

Again by Proposition 4 and (3.2)

$$\omega((\zeta,w),h^{-1}(F),G) = \omega((ke^\zeta,w),F,D). \tag{3.3}$$

Now $h^{-1}(E) = \cup_{j\in\mathbb{Z}} E_j$ with

$$E_j = \{(\zeta,w) \in G : \ \zeta + \ln k + 2\pi i j = f(w)/w^n\}.$$

For each j the function $u_j(\zeta,w) = \ln|(\zeta + \ln k + 2\pi i j)w^n - f(w)| - \ln|\zeta - 1|$ is bounded on G and equals $-\infty$ precisely on E_j. It follows that

$$\omega((\zeta,w),E_j,G) = 0, \tag{3.4}$$

if $(\zeta,w) \in G \setminus E_j$. By considering $u(\zeta,w) = \sum_j 2^{-|j|}u_j(\zeta,w)$, we conclude that for $(\zeta,w) \in G \setminus h^{-1}(E)$

$$\omega((\zeta,w),h^{-1}(E),G) = 0,$$

hence by (3.3), if $(z,w) \in D \setminus (E \cup A)$, then

$$\omega((z,w),F,D) = 0. \tag{3.5}$$

Next, let $(z,w) \in E \cap D$, $z \neq 0$. Let $(\zeta,w) \in E_0$ be such that $h(\zeta,w) = (z,w)$. From (3.4) we obtain

$$\omega((\zeta,w),h^{-1}(F),G) = \omega((\zeta,w),F_0,G),$$

where $F_0 = \{(\zeta,w) \in G : \ \zeta w^n - f(w) = 0, \zeta \in \Delta_0\}$ and

$$\Delta_0 = \{\zeta \in h^{-1}(\Delta), -\pi/2 < \Im\zeta - \arg a < \pi/2\}.$$

Now $\omega(\zeta, \Delta_0, H)$ is positive, but tends to 0 as $\Re\zeta \to -\infty$. From Proposition 4 we see again that $\omega((\zeta, w), F_0, G) \to 0$ as $\Re\zeta \to \infty$, and therefore keeping in mind (3.5)

$$\omega((z,w), F, D) \to 0, \qquad \text{as } |z| \to 0. \tag{3.6}$$

Thusfar we have proved that for *every* polydisk D compactly contained in Ω, $\omega((z,w), F, D) \to 0$ as $|z| \to 0$. Now we apply Lemma 5 to deal with $z = 0$. Let $\omega((0,w), F, D) = c \geq 0$. Let D' be a polydisk with $D \subset D' \subset \Omega$ and let $V = \{(z,w) \in D' : \ |z| < r\}$, where r is so small that $\omega((z,w), F, D') \leq c/2$ if $|z| = r$. From Lemma 5 we see that there is a point P in $\partial V \cap D'$ with $\omega(P, F, D') \geq c$. It follows that $c = 0$. The conclusion is that $\omega((z,w), F, D) > 0$ if and only if $(z,w) \in E$ and the proof of the first case is finished.

Next we assume that $\alpha \cdot \beta \neq 0$. We consider the map $h_1 : (\zeta, w) \mapsto (\varphi(\zeta), w)$ which defines a 2-sheeted branched covering $h_1^{-1}(D) \to D$. Let $(z,w) \in D$ and $z = \varphi(\zeta)$ for some ζ. By Lemma 6 we have

$$\omega((z,w), F, D) = \omega((\zeta, w), h_1^{-1}(F), h_1^{-1}(D)).$$

Now $h_1^{-1}(F) = F_1 \cup F_2$, where

$$F_1 = \{(\zeta, w) : \zeta = e^{f(w)/w^n}, \varphi(\zeta) \in \Delta\}$$

and

$$F_2 = \{(\zeta, w) : \zeta = e^{\frac{-b}{a} f(w)/w^n}, \varphi(\zeta) \in \Delta\}.$$

Let $l < M$. From the first part of the proof it follows that $\omega((\zeta, w), F_1, kU \times lU) = 0$ if and only if $\zeta \neq e^{f(w)/w^n}$, hence also $\omega((\zeta, w), F_1, h_1^{-1}(D)) = 0$ if and only if $\zeta \neq e^{f(w)/w^n}$. Similarly, $\omega((\zeta, w), F_2, h_1^{-1}(D)) = 0$ if and only if $\zeta \neq e^{\frac{-b}{a} f(w)/w^n}$. Using (2.1) we obtain

$$\omega((z,w), F, D) \leq \omega((\zeta, w), F_1, h_1^{-1}(D)) + \omega((\zeta, w), F_2, h_1^{-1}(D)) = 0$$

if and only if $z \neq \varphi(e^{f(w)/w^n})$, i.e $(z,w) \notin E \cap D$. This completes the proof. $\square$

Theorem 7 implies immediately that the graph

$$\{(x,y) \in \mathbb{C}^2 : \ y = \varphi(e^{f(x)/x^n}), 0 < x < M\}$$

is plurithin at the origin. Typical graphs that can be handled by the theorem are those of $y = e^{-1/x}$ and $y = \sin \frac{1}{x}$.

Note added in proof.

I recently became aware of the following result of Zeriahi, [6], Prop. 2.1: Let E be a pluripolar subset of a pseudoconvex domain Ω in $\mathbb{C}^n$. If $E^*_\Omega = E^*$ and E is a G_δ as well as an F_σ, then E is complete pluripolar in Ω.

It is easy to see that the graphs E that we consider in Theorem 7 are G_δ and F_σ. Theorem 7 gives $E = E^*_\Omega$, therefore E is complete pluripolar in Ω, that is, there exists a plurisubharmonic function u on Ω such that

$$E = \{p \in \Omega : \ u(p) = -\infty\}.$$

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