GRAPHS OF HOLOMORPHIC FUNCTIONS WITH ISOLATED SINGULARITIES ARE COMPLETE PLURIPOLAR

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ABSTRACT. Let f be a function on a domain D in $\mathbb{C}$ which is holomorphic except for isolated singularities. In this paper we show that $E = \{(z, w) \in D \times \mathbb{C} : \ w = f(z)\}$ is complete pluripolar in $D \times \mathbb{C}$.

1. Introduction

In classical potential theory one encounters the notions of polar set and complete polar set. A set $E \subset \mathbb{R}^n$ is called polar if there exists a subharmonic u on a neighborhood of E such that $E \subset \{x : \ u(x) = -\infty\}$; E is called complete polar if one actually has $E = \{x : \ u(x) = -\infty\}$. (The function identically equal to $-\infty$ is not considered to be subharmonic.) It is well-known that we may take u to be defined on all of $\mathbb{R}^n$ and also that E is complete polar if and only if E is polar and a G_δ, cf., [4].

In pluripotential theory the situation is more complicated. A set E in a domain $D \subset \mathbb{C}^n$ is called *pluripolar in* D if there exists a plurisubharmonic function u on D such that $E \subset \{z : \ u(z) = -\infty\}$; E is called *complete pluripolar in* D if for some plurisubharmonic function u on D we have $E = \{z : \ u(z) = -\infty\}$. While Josefson's theorem, cf., [3], asserts that E being pluripolar in D implies that E is pluripolar in $\mathbb{C}^n$, the corresponding assertion is false in the complete pluripolar setting. Also, a pluripolar G_δ need not be complete: The open unit disk Δ in the complex line $z_2 = 0$ in $\mathbb{C}^2$ is a G_δ, but not complete in $\mathbb{C}^2$. In fact, every plurisubharmonic function on $\mathbb{C}^2$ that equals $-\infty$ on Δ must equal $-\infty$ on the line $z_2 = 0$. Thus it is reasonable to introduce the *pluripolar hull* of a pluripolar set $E \subset D$ as

$$E^*_D = \{z \in D : \ u|_E = -\infty \Rightarrow u(z) = -\infty \text{ for all } u \in PSH(D)\},$$

where $PSH(D)$ denotes the set of all plurisubharmonic functions on D. We also have use for the *negative pluripolar hull*

$$E^-_D = \{z \in D : \ u|_E = -\infty \Rightarrow u(z) = -\infty \text{ for all } u \in PSH(D), u \leq 0\}.$$

If E is complete pluripolar in D then apparently E is a G_δ and $E^*_D = E$. A partial converse is Zeriahi's theorem, cf., [10]:

Theorem 1. *Let E be a pluripolar subset of a pseudoconvex domain D in $\mathbb{C}^n$. If $E^*_D = E$ and E is a G_δ as well as an F_σ, then E is complete pluripolar in D.*

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Completeness and hulls of various pluripolar sets have been studied in [8], [6], [5] and [9].

In [8] Sadullaev posed the following questions. Consider the sets

$$E_1 = \{(x, y) \in \mathbb{C}^2 : \; y = x^\alpha, \, x \in (0, 1)\}$$

with α irrational, and

$$E_2 = \{(x, y) \in \mathbb{C}^2 : \; y = e^{-1/x}, \, x \in (0, 1)\}.$$

Does there exist a plurisubharmonic function h_j on a neighborhood V of $\overline{E_j}$ such that

$$h_j(0) > \limsup_{\substack{x \to 0 \\ x \in E_j}} h_j(x).$$

See also Bedford's survey [1]. In [5] Levenberg and Poletsky gave a positive answer for E_1, in [9] the author gave a positive answer for E_2. In fact, in both cases it was shown that the pluripolar hulls $E^*_{\mathbb{C}^2}$ are equal to the graphs of the maximal analytic extension of $y = x^\alpha$, respectively $y = e^{-1/x}$.

In the present paper we generalize the results of [9] as follows.

Theorem 2. *Suppose that D is a domain in $\mathbb{C}$ and A a sequence of points in D without density point in D. Let f be holomorphic on $D \setminus A$. Let E denote the graph of f in $(D \setminus A) \times \mathbb{C}$. Then E is complete pluripolar in $D \times \mathbb{C}$.*

Necessary preliminaries are dealt with in section 2. After some preparations we give in section 3 the proof of the theorem. Surprisingly, it is more elementary than the special cases that were treated in [9].

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2. Preliminaries

2.1. **Pluripolar Hulls.** We will need some facts about the pluripolar hulls E^*_D and E^-_D, which were defined in the introduction. Of course, $E^*_D \subset E^-_D$. Moreover, these hulls are related as follows, see [5],

Theorem 3. *Let D be pseudoconvex in $\mathbb{C}^N$ and $E \subset D$ pluripolar. Suppose $D = \cup_j D_j$, where D_j form an increasing sequence of relatively compact pseudoconvex subdomains of D. Then*

$$E^*_D = \bigcup_j (E \cap D_j)^-_{D_j}.$$

Moreover, if D is hyperconvex, that is, D admits a bounded plurisubharmonic exhaustion function, then

$$E^-_D = \bigcup_j (E \cap D_j)^-_{D_j}.$$

2.2. **Pluriharmonic measure.** The notion of pluriharmonic measure was introduced in [8], [2]. Proofs of the properties that we state below may be found in [5]. Let E be a subset of a domain $D \subset \mathbb{C}^n$. The *pluriharmonic measure* at $z \in D$ of E relative to D is the number

$$\omega(z, E, D) = -\sup\{u(z) : \ u \in PSH(D) \text{ and } u \leq -\chi_E\}. \tag{2.1}$$

Here χ_E is the characteristic function of E on D.

Notice that we do not regularize the sup in (2.1). If we did, we would get 0 if E were pluripolar, which is the case we will consider, and all information were lost. For $n = 1$, E compact in D and $z \in D \setminus E$ this notion boils down to the usual concept of harmonic measure at z of the boundary of E in the domain $D \setminus E$.

Pluriharmonic measure has the following properties:

$$\omega(z, E, D) = \inf_{\substack{V \text{open} \\ E \subset V \subset D}} \omega(z, V, D)$$

and

$$\omega(z, E_1 \cup E_2, D) \leq \omega(z, E_1, D) + \omega(z, E_2, D). \tag{2.2}$$

The relation of pluriharmonic measure with pluripolar hulls is given by

Proposition 4. *Let D be a hyperconvex domain in $\mathbb{C}^N$ and let $E \subset D$ be pluripolar. Then*

$$E_D^- = \{z \in D : \omega(z, E, D) > 0\}.$$

Now let E be the graph of some holomorphic function f on a domain G. Let $M > 0$ and let $B = \{w : |w| < M\}$. Next, let K be a closed disk in G such that $f(K) \subset B$ and let E_K denote the graph of f over K. Then

$$\omega(z, E \cap (G \times B), G \times B) = 0 \iff \omega(z, E_K, G \times B) = 0, \tag{2.3}$$

because $E \cap (G \times B)$ and E_K have the same pluripolar hull.

3. Proof of theorem 2

3.1. **Construction of a plurisubharmonic function.** Our isolated singularities will of course be poles or essential singularities. The principal part at such a singularity will be considered as an infinite Laurent series, which may consist of only finitely many non-zero terms.

Proposition 5. *Let f be holomorphic on a bounded domain D in $\mathbb{C}$ except for finitely many isolated singularities at $a_1, \dots, a_n \in D$. Let E denote the graph of f in $D \times B$, where B is a disk about the origin. Then there is a negative continuous plurisubharmonic function on $D \times B$ which equals $-\infty$ precisely at $E \bigcup \cup_{i=1}^n \{z = a_i\}$. In particular $E \bigcup \cup_{i=1}^n \{z = a_i\}$ is complete pluripolar in $D \times B$.*

Proof. We write down the Mittag-Leffler decomposition of f:

$$f = f_0 + f_1 + \dots + f_n,$$

where f_0 is holomorphic on D while for $j = 1, \dots, n$, the function f_j is holomorphic on $\mathbb{C} \setminus \{a_j\}$, vanishes at ∞ and has an isolated singularity at a_j. The functions f_j have the expansion

$$f_j = \sum_{m=1}^{\infty} c_{jm}(z - a_j)^{-m},$$

with

$$|c_{jm}|^{1/m} \to 0, \quad \text{as } m \to \infty. \tag{3.1}$$

After introducing new coordinates

$$w' = w - f_0(z), \quad z' = z,$$

we may assume that $f_0 = 0$.

Let

$$P_N(z,w) = (w - \sum_{j=1}^{n} \sum_{m=1}^{N} c_{jm}(z-a_j)^{-m}) \prod_{j=1}^{n} (z-a_j)^N$$

and let

$$h_N(z,w) = \frac{1}{N} \log |P_N(z,w)|.$$

Then h_N is a plurisubharmonic function on $D \times \mathbb{C}$.

We make some estimates of h_N. Let $\varepsilon_m = \max_{k>m} |c_{jm}|^{1/m}$. Then ε_m decreases to 0 according to (3.1).

Fix $\delta > 0$ and let K be the compact set consisting of those $z \in D$ with distance to the boundary of $D \setminus \{a_1, \ldots, a_n\}$ at least δ. Let M be the diameter of D. On $(K \times B) \cap E$ we have for $\varepsilon_N < \delta/2$

$$\begin{aligned} h_N(z, f(z)) &= \frac{1}{N} \log |f(z) - \sum_{j=1}^{n} \sum_{m=1}^{N} c_{jm}(z-a_j)^{-m}| + \sum_{j=1}^{n} \log |z - a_j| \\ &= \frac{1}{N} \log |\sum_{j=1}^{n} \sum_{m=N+1}^{\infty} c_{jm}(z-a_j)^{-m}| + \sum_{j=1}^{n} \log |z-a_j| \\ &\leq \frac{1}{N} \log (\sum_{j=1}^{n} \sum_{m=N+1}^{\infty} \varepsilon_N^m |z-a_j|^{-m}) + \sum_{j=1}^{n} \log |z-a_j| \\ &\leq (1 + \frac{1}{N}) \log \varepsilon_N + \log 2n - (1+\frac{1}{N}) \log \delta + n \log M \\ &\leq \log \varepsilon_N + C, \end{aligned} \tag{3.2}$$

where C depends on δ, M and n, but not on ε_N. Next let A be positive. For $z \in K$ and $|w - f(z)| > A > 0$ we have

$$\begin{aligned} h_N(z,w) &= \frac{1}{N} \log |w - \sum_{j=1}^{n} \sum_{m=1}^{N} c_{jm}(z-a_j)^{-m}| + \sum_{j=1}^{n} \log |z-a_j| \\ &\geq \frac{1}{N} \log \left| |w - f(z)| - \sum_{j=1}^{n} \sum_{m=N+1}^{\infty} \varepsilon_N^m |z-a_j|^{-m} \right| + n \log |\delta| \\ &\geq \frac{1}{N} \log (A - 2n(\varepsilon_N/\delta)^{N+1}) + n \log |\delta| \geq -C', \end{aligned} \tag{3.3}$$

for some positive constant C', if N is sufficiently large.

The final estimate is that for large enough C_0 and C'', the inequality

$$h_N(z,w) < \frac{C_0}{N} \log(|w|+1) + C_0 \leq C'' \tag{3.4}$$

holds on $D \times B$.

Now we introduce the negative plurisubharmonic functions

$$u_N = \max(h_N - C'', \log \varepsilon_N).$$

Choose a sequence of positive integers N_i and a sequence of positive numbers d_i with the following property: $\sum d_i$ converges, but $\sum d_i \log \varepsilon_{N_i}$ diverges to $-\infty$. This is possible since $\varepsilon_N \downarrow 0$. We form the series

$$u(z,w) = \sum_{i=1}^{\infty} d_i u_{N_i}(z,w) \tag{3.5}$$

This is on $D \times B$ a limit of a decreasing sequence of plurisubharmonic functions. On $E \cap (K \times B)$ we use (3.2) and find that $u = -\infty$ on $E \cap (K \times B)$ hence also on E. By (3.3) we obtain that $u > -\infty$ if $w \neq f(z)$. The convergence properties of (3.5) are independent of δ; we conclude that u represents a negative plurisubharmonic function on $D \times B$ wich satisfies

$$E \subset \{(z,w) : \ u(z,w) = -\infty\} \subset E \cup \bigcup_{1 \le j \le n} \{z = a_j\}.$$

Moreover, the convergence of (3.5) is uniformly on compact sets in the complement of $E \cup \bigcup_{1 \le j \le n} \{z = a_j\}$. Hence u is continuous in this complement. Finally, the function

$$u(z,w) + \sum_{i=1}^{n} \log |z - a_i| - n \log M$$

satisfies all our conditions. □

3.2. **Estimates for harmonic measure.** The next one-variable proposition will allow us to estimate pluriharmonic measure in the proof of the main theorem. We will denote classical harmonic measure by ω_0

Proposition 6. *Let G be a Dirichlet domain in $\mathbb{C}$, let K be a closed disk contained in G and let a be a point in the boundary of G. Assume that there is an arc $\gamma : [0,1] \to \mathbb{C}$ contained in the complement of G with $\gamma(0) = a$. Then for every $\varepsilon > 0$ there exists a $\delta > 0$ and a negative subharmonic function h on $G_\delta = G \cup \{z : |z - a| < \delta\}$ such that $h|_K = -1$ and $h(a) \ge -\varepsilon$.*

Proof. Applying a Möbius transformation if necessary, we can assume that $\infty = \gamma(1)$. For every $0 < \delta < 1$ we let γ_δ be the component of $\gamma \setminus B(0,\delta)$ that contains ∞. Abusing notation, let $\gamma_\delta(\delta)$ denote the other endpoint of γ_δ, so that $|\gamma_\delta(\delta)| = \delta$.

Let $\Omega_0 = \mathbb{C}^* \setminus \gamma_\delta(\delta)$. Choose an analytic branch f_δ of $\log(z - \gamma_\delta(\delta))$ on Ω_0. Let B_δ be the disk $\{z : \ |z - \gamma_\delta| < 1\}$ and C_δ its boundary. The image of $G_\delta \cap B_\delta$ under f_δ is contained in the half plane $H = \{\Re w < 0\}$ and X, the intersection of its boundary with the imaginary axis has length $< 2\pi$. Set $u(z) = -\omega_0(z, X, H)$. Then $u \circ f_\delta$ is a negative harmonic function on $G_\delta \cap B_\delta$ that equals -1 on $G_\delta \cap C_\delta$. Moreover, $u(f_\delta(0)) \ge 2/\log \delta$. Now let $h(z) = -\omega_0(z, K, G_\delta)$. Then $h(z) > -1$ on $G_\delta \cap C_\delta$, $h(z) = 0$ on $\partial G_\delta \cap B_\delta$. Hence $h(z) \ge u(z)$ on $\partial(G_\delta \cap B_\delta)$. Therefore $h(0) \ge u(0) \ge 2/\log \delta$, so that $h(0) > -\varepsilon$ if δ is sufficiently small. Also, h is negative subharmonic on G_δ and $h = -1$ on K. □

3.3. **Proof of Theorem 2.**

Proof. Let $D'' \subset D'$ be subdomains of D which are compactly contained in each other. Assume in addition that D' is a Dirichlet domain. Let K be a (small) closed disk in D' that doesn't meet A and denote by E_K the graph of f over K. Let M be the maximum of $|f|$ on K and let $B'' \subset B'$ be disks about the origin with different radii bigger than M. Let $\Omega' = D' \times B'$, $\Omega'' = D'' \times B''$. We put $Z = E \cup \{(z, w) : \ z \in A\}$.

By proposition 5 there exists a negative continuous plurisubharmonic function u on Ω' such that u equals $-\infty$ precisely on $\Omega' \cap Z$. It follows that

$$\omega((z, w), E_K, \Omega'') = 0 \quad \text{for } (z, w) \in \Omega'' \setminus Z.$$

We next estimate $\omega((a, w), E_K, \Omega'')$ for $a \in A \cap D''$. Let $\varepsilon > 0$ and let G denote the projection of $E \cap \Omega'$ on the first coordinate. If a is a pole of f, then a is an interior point of the complement of G. The function h defined by -1 on G and $-\varepsilon$ on a small neighborhood of a is harmonic on $G \cup B(a, \delta)$ for sufficiently small δ. If a is an essential singularity, then a is a boundary point of G. Since there exists a curve ending in a along which f tends to ∞, it is clear that the conditions of proposition 6 are met. We thus find a small $\delta > 0$ and a negative subharmonic function h on $G \cup B(a, \delta)$ with $h(a) > -\varepsilon$ and $h|_K = -1$.

Now we view the function h as a plurisubharmonic function on $(G \cup B(a, \delta)) \times B'$. It is plurisubharmonic in a neighborhood of $\{(z, w) \in \Omega' : \ u(z, w) = -\infty\}$. There exists an $\eta > 0$ such that this neighborhood contains $X = \{(z, w) \in \Omega'' : \operatorname{dist}((z, w), Z) = \eta\}$. Now the supremum of h on X is negative and u is continuous. Thus for sufficiently small, positive λ we find that $\lambda u > h$ on X.

The function h can be extended to a negative plurisubharmonic function on Ω'' as follows. Set

$$\tilde{h}(z, w) = \begin{cases} \max\{(h(z, w), \lambda u(z, w)\} & \text{if } \operatorname{dist}((z, w), Z) < \eta; \\ \lambda u(z, w) & \text{otherwise.} \end{cases}$$

The function $\tilde{h}$ competes for the supremum in the definition of $\omega((a, w), E_K, \Omega'')$. We conclude that $\omega((a, w), E_K, \Omega'') = 0$.

It now follows from proposition 4 that $(E_K)^-_{\Omega''} = E \cap \Omega''$. Application of theorem 3 shows that $(E_K)^*_{D \times \mathbb{C}} = E$. Finally, observing that E is a G_δ as well as an F_σ, we infer from theorem 1 that E is complete pluripolar in $D \times \mathbb{C}$. $\square$

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