From Deterministic ODEs to Dynamic Structural Causal Models

Paul K. Rubenstein*
Department of Engineering
University of Cambridge
pkr23@cam.ac.uk

Stephan Bongers
Informatics Institute
University of Amsterdam
S.R.Bongers@uva.nl

Joris M. Mooij
Informatics Institute
University of Amsterdam
j.m.mooij@uva.nl

Bernhard Schölkopf
MPI-IS
Tübingen
bs@tue.mpg.de

August 30, 2016

Abstract

We show how, under certain conditions, the asymptotic behaviour of an Ordinary Differential Equation under non-constant interventions can be modelled using Dynamic Structural Causal Models. In contrast to earlier work, we study not only the effect of interventions on equilibrium states; rather, we model asymptotic behaviour that is *dynamic* under interventions that vary in time, and include as a special case the study of static equilibria.

1 Introduction

Ordinary Differential Equations (ODEs) provide a universal language to describe deterministic physical systems in the world via equations that determine how variables change in time as a function of other variables. They are *causal* in the sense that at least in principle they allow us to reason about interventions: any physical intervention in a system — e.g., moving an object by applying a force — can be modelled using modified differential equations by, for instance, including suitable forcing terms. In practice, of course, this may be arbitrarily difficult.

Statistical dependences in the world are generally the outcome of time evolution governed by differential equations. For instance, to produce a dataset of labelled handwritten digits, we might instruct a writer to produce images of digits belonging to specified classes. This instantiates a complex dynamical process involving neural networks in the brain as well as muscles. At the end, we are left with statistical dependences that machine learning methods can exploit. However, the causal structure that was present in the differential equations is lost.

Structural Causal Models (SCMs, also known as Structural Equation Models) are another language capable of describing causal relations and interventions and have been widely applied in the social

*Also affiliated with the Max Planck Institute for Intelligent Systems, Tübingen sciences, economics, genetics and neuroscience [1; 2]. One of the successes of SCMs over other causal frameworks has been their ability to express cyclic causal models [3; 4; 5; 6; 7; 8].

We view SCMs as an intermediate level of description, which ideally retains the benefits of a data-driven statistical approach while still allowing a limited set of causal statements about the effect of interventions. While we have a good understanding of how an SCM can imply a statistical model, much less is known about how a differential equation model — our most fundamental level of modelling — can imply an SCM in the first place. Recent work has begun to address this question by showing how, under strong assumptions, SCMs can be derived from an underlying discrete time difference equation or continuous time ODE [6; 7; 9; 10; 11]. With the exception of [6], each of these methods assume that the dynamical system comes to a static equilibrium, with the derived SCM describing how this equilibrium changes under intervention.

If the equilibrium assumption is reasonable for a particular system under study, the SCM framework can be useful. Although the derived SCM then lacks information about the (possibly rich) transient dynamics of the system, if the system equilibrates quickly then the description of the system as an SCM may be a more natural and compact representation of the causal structure of interest. By making assumptions on the dynamical system and the interventions being made, the SCM effectively allows us to reason about a 'higher level' qualitative description of the dynamics — in this case, the equilibrium states.

There are, however, two major limitations that stem from the equilibrium assumption. First, for many dynamical systems the assumption that the system settles to equilibrium, either in its observational state or under intervention, may be unnatural. Second, this framework is only capable of modelling interventions in which a subset of variables are clamped to fixed values (*constant* interventions). Even for rather simple physical systems such as a forced damped simple harmonic oscillator, these assumptions are violated.

The work presented in this paper is a step towards broadening the above by extending it to encompass time-dependent dynamics and interventions. In Section 2, we introduce notation to describe ODEs. In Section 3, we describe how to apply the notion of an intervention on an ODE to the dynamic case. In Section 4, we define regularity conditions on the asymptotic behaviour of an ODE under a set of interventions. In Section 5, we present our main result: subject to conditions on the dynamical system and interventions being modelled, an SCM can be derived that allows one to reason about how the asymptotic dynamics change under interventions on variables in the system. We refer to this as a *Dynamic SCM* to distinguish from the static equilibrium case for the purpose of exposition, but note that this is conceptually the same as an SCM on a fundamental level. In doing so, we draw inspiration from the approach of [11].

2 Ordinary Differential Equations

Let $\mathcal{I} = \{1, \dots, D\}$ be a set of variable labels. Consider time-indexed variables $X_i(t) \in \mathcal{R}_i$ for $i \in \mathcal{I}$, where $\mathcal{R}_i \subseteq \mathbb{R}^{d_i}$ and $t \in \mathbb{R}_{\geq 0}$. For $I \subseteq \mathcal{I}$, we write $\mathbf{X}_I(t) \in \prod_{i \in I} \mathcal{R}_i$ for the tuple of variables $(X_i(t))_{i \in I}$. By an ODE $\mathcal{D}$, we mean a collection of D coupled ordinary differential equations with initial conditions $\mathbf{X}_0^{(k)}$:

$$\mathcal{D}: \left\{ \; f_i(X_i, \mathbf{X}_{\mathtt{pa}(i)})(t) = 0, \quad X_i^{(k)}(0) = (\mathbf{X}_0^{(k)})_i \quad 0 \leq k \leq n_i - 1, \quad i \in \mathcal{I} \right.$$

where the ith equation determines the evolution of the variable X_i in terms of $\mathbf{X}_{\mathtt{pa}(i)}$ and X_i itself, and where n_i is the order of the highest derivative $X_i^{(k)}$ of X_i that appears in equation i.

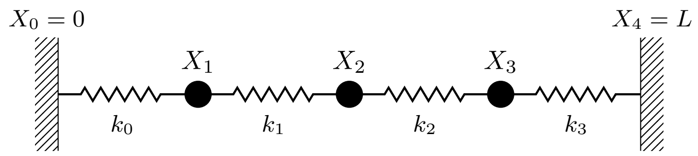


Figure 1: The mass-spring system of Example 1 with $D = 3$

f_i is a functional that can include time-derivatives of its arguments. In addition to defining the evolution of X_i in terms of $\mathbf{X}_{\mathtt{pa}(i)}$, we can also interpret it as a constraint: $f_i(X_i, \mathbf{X}_{\mathtt{pa}(i)})(t) = 0 \,\forall t \in \mathbb{R}_{\geq 0}$ if and only if X_i and $\mathbf{X}_{\mathtt{pa}(i)}$ obey the physical laws of $\mathcal{D}$.

One possible way to write down an ODE is to canonically decompose it into a collection of first order differential equations, such as is done in [11]. We choose to present our ODEs as "one equation per variable" rather than splitting up the equations due to complications that would otherwise occur when considering time-dependent interventions (Sec. 3.3)

Example 1 ([11])**.** *Consider a one-dimensional system of D particles of mass m_i ($i = 1, \ldots, D$) with positions X_i coupled by springs with natural lengths l_i and spring constants k_i, where the ith spring connects the ith and $(i+1)$th masses and the outermost springs have fixed ends (see Figure 1). Assume further that the ith mass undergoes linear damping with coefficient b_i.*

Denoting by $\dot{X}_i$ and $\ddot{X}_i$ the first and second time derivatives of X_i respectively, the equation of motion for the ith variable is given by:

$$m_i\ddot{X}_i(t) = k_i[X_{i+1}(t) - X_i(t) - l_i] - k_{i-1}[X_i(t) - X_{i-1}(t) - l_{i-1}] - b_i\dot{X}_i(t)$$

where we take $X_0 = 0$ and $X_D = L$ to be the fixed positions of the end springs. For the case that $D = 3$, we can write the system of equations as:

$$\mathcal{D} : \begin{cases} m_1\ddot{X}_1(t) + b_1\dot{X}_1(t) + (k_1 + k_0)X_1(t) - k_1X_2(t) - k_0l_0 + k_1l_1 & = 0 \\ m_2\ddot{X}_2(t) + b_2\dot{X}_2(t) + (k_2 + k_1)X_2(t) - k_2X_3(t) - k_1X_1(t) - k_2l_1 + k_2l_2 & = 0 \\ m_3\ddot{X}_3(t) + b_3\dot{X}_3(t) + (k_3 + k_2)X_3(t) - k_3L - k_2X_2(t) - k_2l_2 + k_3l_3 & = 0 \\ X_i^{(k)}(0) = (\mathbf{X}_0^{(k)})_i \quad k \in \{0, 1\},\ i \in \{1, 2, 3\} \end{cases}$$

We can represent the dependence structure between variables implied by the functions f_i graphically, in which variables are nodes and arrows point $X_j \longrightarrow X_i$ if $j \in \mathtt{pa}(i)$. Self loops $X_i \longrightarrow X_i$ exist if $X_i^{(k)}$ appears in the expression of f_i for more than one value of k. This is illustrated for the system described in Example 1 in Figure 2a.

3 Interventions

3.1 Time-dependent perfect interventions

Usually in the causality literature, by a *perfect intervention* it is meant that a variable is clamped to take a specific given value. The natural analogue of this in the time-dependent case is a perfect intervention that forces a variable to take a particular *trajectory*. That is, given a subset $I \subseteq \mathcal{I}$ and a function $\boldsymbol{\zeta}_I : \mathbb{R}_{\geq 0} \longrightarrow \prod_{i \in I} \mathcal{R}_i$, we can intervene on the subset of variables $\mathbf{X}_I$ by forcing $\mathbf{X}_I(t) = \boldsymbol{\zeta}_I(t) \,\forall t \in \mathbb{R}_{\geq 0}$. Using Pearl's do-calculus notation [1] and for brevity omitting the t, we write $\mathtt{do}(\mathbf{X}_I = \boldsymbol{\zeta}_I)$ for this intervention. Such interventions are more general objects than those of the equilibrium or time-independent case, but in the specific case that we restrict ourselves to constant trajectories the two notions coincide.

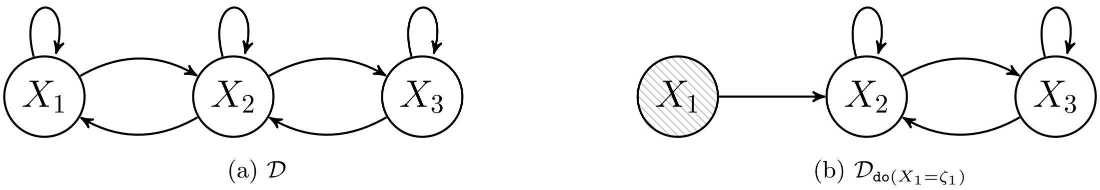


Figure 2: Graphical models representing the causal structure of the mass-spring system of Example 1 for: (a) the observational system; and (b) after the intervention on variable X_1 described in Example 2. As a result of the intervention, X_1 is not causally influenced by any variable, while the causal mechanisms of X_2 and X_3 remain unchanged.

3.2 Sets of interventions

The Dynamic SCMs that we will derive will describe the asymptotic dynamics of the ODE and how they change under different interventions. If we want to model 'all possible interventions', then the resulting asymptotic dynamics that can occur are arbitrarily complicated. The idea is to fix a simpler set of interventions and derive an SCM that models only these interventions, resulting in a model that is simpler than the original ODE but still allows us to reason about interventions we are interested in. In the examples in this paper, we restrict ourselves to periodic or quasi-periodic interventions, but the results hold for more general sets of interventions that satisfy the stability definitions presented later.

We need to define some notation to express the sets of interventions we will model. Since interventions correspond to forcing variables to take some trajectory, we describe notation for defining sets of trajectories: For $I \subseteq \mathcal{I}$, let $\mathtt{Dyn}_I$ be a set of trajectories in $\prod_{i \in I} \mathcal{R}_i$. Let $\mathtt{Dyn} = \cup_{I \in \mathcal{P}(\mathcal{I})} \mathtt{Dyn}_I$. Thus, an element $\boldsymbol{\zeta}_I \in \mathtt{Dyn}_I$ is a function $\mathbb{R}_{\geq 0} \longrightarrow \prod_{i \in I} \mathcal{R}_i$, and $\mathtt{Dyn}$ consists of such functions for different $I \subseteq \mathcal{I}$.

The following observation is an important point for later on: **elements of $\mathtt{Dyn}_{\mathcal{I}}$ define trajectories for the whole system, and will be used later on to define the asymptotic dynamics of the system**. For technical reasons, we will make use of the following definition later on. This should be interpreted as saying that interventions on each variable can be made independently and combined in any way, and it is thus related to notions that have been discussed in the literature under various headings [1] (e.g., modularity, autonomy, and invariance).

Definition 1. *A set of trajectories* $\mathtt{Dyn}$ *is* ***modular*** *if, for any* $\{i_1, \ldots, i_n\} = I \subseteq \mathcal{I}$,

$$\boldsymbol{\zeta}_I \in \mathtt{Dyn} \iff \zeta_{i_k} \in \mathtt{Dyn} \quad \forall k \in \{1, \ldots, n\}$$

Not all sets of trajectories $\mathtt{Dyn}$ are modular; later on we will assume that the sets of trajectories we are considering *are* for the purposes of constructing the Dynamic SCMs.

3.3 Describing interventions in ODEs

We can realise a perfect intervention by replacing the equations of the intervened variables with new equations that fix them to take the specified trajectories:

$$\mathcal{D}_{\mathtt{do}(\mathbf{X}_I = \boldsymbol{\zeta}_I)} : \begin{cases} f_i(X_i, \mathbf{X}_{\mathtt{pa}(i)})(t) & = 0, \quad X_i^{(k)}(0) = (\mathbf{X}_0^{(k)})_i \quad 0 \leq k \leq n_i - 1, \qquad i \in \mathcal{I} \setminus I \\ X_i(t) - \zeta_i(t) & = 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\;\; i \in I \end{cases}$$

This procedure is analogous to the notion of intervention in an SCM. Physically, this corresponds to decoupling the intervened variables from their usual causal mechanism by forcing them to take a particular value, while leaving the non-intervened variables' causal mechanisms unaffected.

Perfect interventions will not generally be realisable in the real world. In practice, an intervention on a variable would correspond to altering the differential equation governing its evolution by adding extra forcing terms; perfect interventions could be realised by adding forcing terms that push the variable towards its target value at each instant in time, and considering the limit as these forcing terms become infinitely strong so as to dominate the usual causal mechanism determining the evolution of the variable.

Example 2. *Consider the mass-spring system described in Example 1. If we were to intervene on the system to force the mass X_1 to undergo simple harmonic motion, we could express this as a change to the system of differential equations as:*

$$\mathcal{D}_{\mathtt{do}(X_1(t)=l_1+A\cos(\omega t))}:$$
$$\begin{cases} X_1(t) - l_1 - A\cos(\omega t) & = 0 \\ m_2\ddot{X}_2(t) + b_2\dot{X}_2(t) + (k_2+k_1)X_2(t) - k_2X_3(t) - k_1X_1(t) - k_2l_1 + k_2l_2 & = 0 \\ m_3\ddot{X}_3(t) + b_3\dot{X}_3(t) + (k_3+k_2)X_3(t) - k_3L - k_2X_2(t) - k_2l_2 + k_3l_3 & = 0 \\ X_i^{(k)}(0) = (\mathbf{X}_0^{(k)})_i \quad k\in\{0,1\},\ i\in\{2,3\} \end{cases}$$

This induces a change to the graphical description of the causal relationships between the variables. We break any incoming arrows to any intervened variables, including self loops, as the intervened variables are no longer causally influenced by any other variables in the system. See Figure 2b for the result of the intervention in Example 2.

4 Dynamic Stability

A crucial assumption of [11] was that the systems considered were *stable* in the sense that they would converge to unique stable equilibria after arbitrary (constant) interventions. This made them amenable to study by considering the $t \longrightarrow \infty$ limit in which any complex but transient dynamical behaviour would have decayed. The SCMs derived would allow one to reason about the asymptotic equilibrium states of the systems after interventions. Since we want to consider non-constant asymptotic dynamics, this is not a notion of stability that is fit for our purposes.

Instead, we define our stability with reference to a set of trajectories. We will use $\mathtt{Dyn}_{\mathcal{I}}$ for this purpose. Recall that elements of $\mathtt{Dyn}_{\mathcal{I}}$ are trajectories for all variables in the system. To be totally explicit, we can think of an element $\boldsymbol{\eta}\in\mathtt{Dyn}_{\mathcal{I}}$ as a function

$$\begin{aligned} \boldsymbol{\eta}: \quad & \mathbb{R}_{\geq 0} \longrightarrow \mathcal{R}_{\mathcal{I}} \\ & t \mapsto (\eta_1(t), \eta_2(t), \ldots, \eta_D(t)) \end{aligned}$$

where $\eta_i(t)\in\mathcal{R}_i$ is the state of the ith variable X_i at time t. Note that $\mathtt{Dyn}_{\mathcal{I}}$ is not a single fixed set, independent of the situation we are considering. We can choose $\mathtt{Dyn}_{\mathcal{I}}$ depending on the ODE $\mathcal{D}$ under consideration, and the interventions that we may wish to make on it.

Informally, stability in this paper means that the asymptotic dynamics of the dynamical system converge to an element of $\mathtt{Dyn}_{\mathcal{I}}$, independent of initial condition. If $\mathtt{Dyn}_{\mathcal{I}}$ is in some sense simple, we can simply characterise the asymptotic dynamics of the system under study. In the particular case that $\mathtt{Dyn}_{\mathcal{I}}$ consists of constant trajectories, the following definitions of stability can be seen to be closely related to those of [11].

Definition 2. *The ODE $\mathcal{D}$ is **dynamically stable with reference to** $\texttt{Dyn}_{\mathcal{I}}$ if there exists a unique $\boldsymbol{\eta}_{\emptyset} \in \texttt{Dyn}_{\mathcal{I}}$ such that $\mathbf{X}_{\mathcal{I}}(t) = \boldsymbol{\eta}_{\emptyset}(t)\ \forall t$ is a solution to $\mathcal{D}$ and that for all initial conditions $\mathbf{X}_0^{(k)}$, $\mathbf{X}_{\mathcal{I}}(t) \longrightarrow \boldsymbol{\eta}_{\emptyset}(t)$ as $t \longrightarrow \infty$.*

We use a subscript $\emptyset$ to emphasise that $\boldsymbol{\eta}_{\emptyset}$ describes the asymptotic dynamics of $\mathcal{D}$ without any intervention. Observe that $\texttt{Dyn}_{\mathcal{I}}$ could consist of the single element $\{\boldsymbol{\eta}_{\emptyset}\}$ in this case. The requirement that this hold for all initial conditions can be relaxed to hold for almost all initial conditions, but means that the proofs later on require some more technical details. For the purpose of exposition, we stick to this simpler case.

Example 3. *Consider a single mass on a spring that is undergoing simple periodic forcing and is underdamped. Such a system could be expressed as a single (parent-less) variable with ODE description:*

$$\mathcal{D} : \Big\{ m\ddot{X}_1(t) + b\dot{X}_1(t) + k(X_1(t) - l) = F\cos(\omega t + \phi), \quad X_1^{(k)}(0) = (X_0^{(k)}) \quad k \in \{0, 1\}.$$

The solution to this differential equation is

$$X_1(t) = r(t) + l + A\cos(\omega t + \phi') \tag{$*$}$$

where $r(t)$ decays exponentially quickly (and is dependent on the initial conditions) and A and ϕ' depend on the parameters of the initial equation (but not on the initial conditions).

Therefore such a system would be dynamically stable with reference to (for example)

$$\texttt{Dyn}_{\mathcal{I}} = \{l + A\cos(\omega t + \phi') : A \in \mathbb{R},\ \phi' \in [0, 2\pi]\}.$$

We straightforwardly extend this definition to intervened systems:

Definition 3. *Let $I \subseteq \mathcal{I}$ and let $\boldsymbol{\zeta}_I$ be a trajectory in $\prod_{i \in I} \mathcal{R}_i$. The intervened ODE $\mathcal{D}_{\texttt{do}(\mathbf{X}_I = \boldsymbol{\zeta}_I)}$ is **dynamically stable with reference to** $\texttt{Dyn}_{\mathcal{I}}$ if there exists a unique $\boldsymbol{\eta}_{\boldsymbol{\zeta}_I} \in \texttt{Dyn}_{\mathcal{I}}$ such that $\mathbf{X}_{\mathcal{I}}(t) = \boldsymbol{\eta}_{\boldsymbol{\zeta}_I}(t)\ \forall t$ is a solution to $\mathcal{D}_{\texttt{do}(\mathbf{X}_I = \boldsymbol{\zeta}_I)}$ and that for all initial conditions, $\mathbf{X}_{\mathcal{I}}(t) \longrightarrow \boldsymbol{\eta}_{\boldsymbol{\zeta}_I}(t)$ as $t \longrightarrow \infty$.*

Remark 1. We use a subscript $\boldsymbol{\zeta}_I$ to emphasise that $\boldsymbol{\eta}_{\boldsymbol{\zeta}_I}$ describes the asymptotic dynamics of $\mathcal{D}$ after performing the intervention $\texttt{do}(\mathbf{X}_I = \boldsymbol{\zeta}_I)$. Observe that $\texttt{Dyn}_{\mathcal{I}}$ could consist only of the single element $\{\boldsymbol{\eta}_{\boldsymbol{\zeta}_I}\}$ and the above definition would be satisfied. But then the original ODE wouldn't be dynamically stable with reference to $\texttt{Dyn}_{\mathcal{I}}$, nor would other intervened versions of $\mathcal{D}$. This motivates the following definition, extending dynamic stability to sets of intervened systems.

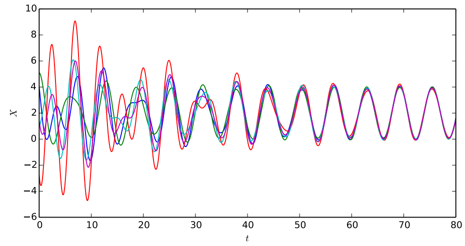
Definition 4. *Let $\texttt{Traj}$ be a set of trajectories. We say that the pair $(\mathcal{D}, \texttt{Traj})$ is **dynamically stable with reference to** $\texttt{Dyn}_{\mathcal{I}}$ if for any $\boldsymbol{\zeta}_I \in \texttt{Traj}$, $\mathcal{D}_{\texttt{do}(\mathbf{X}_I = \boldsymbol{\zeta}_I)}$ is dynamically stable with reference to $\texttt{Dyn}_{\mathcal{I}}$, and the non-intervened ODE $\mathcal{D}$ is dynamically stable with reference to $\texttt{Dyn}_{\mathcal{I}}$.*

Example 3 (continued)**.** *Suppose we are interested in modelling the effect of changing the forcing term, either in amplitude, phase or frequency. We introduce a second variable X_2 to model the forcing term:*

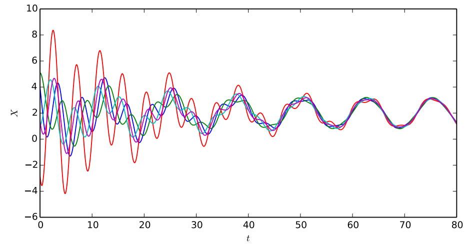
$$\mathcal{D} : \begin{cases} f_1(X_1, X_2)(t) & = m\ddot{X}_1(t) + b\dot{X}_1(t) + k(X_1(t) - l) - X_2(t) & = 0 \\ f_2(X_2)(t) & = X_2(t) - F_0\cos(\omega_0 t + \phi_0) & = 0 \end{cases}$$

If we want to change the forcing term that we apply to the mass, we can interpret this as performing an intervention on X_2. We could represent this using the notation we have developed as

$$\texttt{Dyn}_2 = \{\zeta_2(t) = F\cos(\omega t + \phi) :\ F, \omega \in \mathbb{R},\ \phi \in [0, 2\pi]\}.$$



(a)



(b)

Figure 3: Simulations from the forced simple harmonic oscillator in Example 3 showing the evolution of X_2 with different initial conditions for different forcing terms (interventions on X_1). The parameters used were $m = 1, k = 1, l = 2, F = 2, b = 0.1$, with (a) $\omega = 3$ and (b) $\omega = 2$. Dynamic stability means that asymptotic dynamics are independent of initial conditions, and the purpose of the DSCM is to quantify how the asymptotic dynamics change under intervention.

For any intervention $\zeta_2 \in \texttt{Dyn}_2$*, the dynamics of* X_1 *in* $\mathcal{D}_{\texttt{do}(X_2=\zeta_2)}$ *will be of the form* $(*)$*. Therefore* $(\mathcal{D}, \texttt{Dyn}_2)$ *will be dynamically stable with reference to*

$$\texttt{Dyn}_{\mathcal{I}} = \{\boldsymbol{\zeta}(t) = (l + F\cos(\omega t + \phi), F'\cos(\omega t + \phi')) : \ F, F', \omega \in \mathbb{R},\ \phi, \phi' \in [0, 2\pi]\}.$$

The independence of initial conditions for Example 3 is illustrated in Figure 3.

Note that if $(\mathcal{D}, \texttt{Traj})$ is dynamically stable with reference to $\texttt{Dyn}_{\mathcal{I}}$, and $\texttt{Dyn}'_{\mathcal{I}} \supseteq \texttt{Dyn}_{\mathcal{I}}$ is such that $\forall \boldsymbol{\zeta} \neq \boldsymbol{\zeta}' \in \texttt{Dyn}'_{\mathcal{I}}$, $\boldsymbol{\zeta}(t) \nrightarrow \boldsymbol{\zeta}'(t)$ as $t \longrightarrow \infty$,[1] then $(\mathcal{D}, \texttt{Traj})$ is dynamically stable with reference to $\texttt{Dyn}'_{\mathcal{I}}$.

5 Dynamic Structural Causal Models

A deterministic SCM $\mathcal{M}$ is a collection of Structural Equations, the ith of which defines the value of variable X_i in terms of its parents. We extend this to the case that our variables do not take fixed values but rather represent entire trajectories.

Definition 5. *A deterministic Dynamic Structural Causal Model (DSCM) on the time-indexed variables* $\mathbf{X}_{\mathcal{I}}$ *taking value in a modular set of trajectories* $\texttt{Dyn}$ *is a collection of* structural equations

$$\mathcal{M} : \{ \ X_i = F_i(\mathbf{X}_{\texttt{pa}(i)}) \quad i \in \mathcal{I}$$

where $\texttt{pa}(i) \subseteq \mathcal{I} \setminus \{i\}$ *and each* F_i *is a map* $\texttt{Dyn}_{\texttt{pa}(i)} \longrightarrow \texttt{Dyn}_i$*.*

The point of this paper is to show that, subject to restrictions on $\mathcal{D}$ and $\texttt{Dyn}$, we can derive a DSCM that allows us to reason about the effect on the asymptotic dynamics as a result of making interventions using trajectories in $\texttt{Dyn}$.

'Traditional' deterministic SCMs and DSCMs are identical except for the fact that the variables in an SCM usually take value in a discrete set or a finite dimensional space such as $\mathbb{R}^n$, whilst variables in a DSCM are trajectories in such sets (and as such are infinite dimensional).

In an ODE, the equations f_i determine the causal relationship between the variable $X_i(t)$ and its parents $\mathbf{X}_{\texttt{pa}(i)}(t)$ *at each instant* in time. In contrast, we think of the function F_i of the DSCM

[1] That is, $\texttt{Dyn}_{\mathcal{I}} \subseteq \texttt{Dyn}'_{\mathcal{I}}$ and $\texttt{Dyn}'_{\mathcal{I}}$ still satisfies the uniqueness conditions in the definition of dynamic stability.

as a causal mechanism that determines the entire trajectory of X_i in terms of the variables $\mathbf{X}_{\mathtt{pa}(i)}$, summarising the instantaneous causal effects over all time. In the case that $\mathtt{Dyn}$ consists of constant trajectories (and thus the instantaneous causal effects are constant over time), a DSCM reduces to the definition of a normal deterministic SCM.

The rest of this section is laid out as follows. In Sec. 5.1 we define what it means to make an intervention in a DSCM. In Sec. 5.2 we show how, subject to certain conditions, a DSCM can be derived from a pair $(\mathcal{D}, \mathtt{Dyn})$. The procedure for doing this relies on intervening on all but one variable at a time. In Sec. 5.3, Theorem 2 states that the DSCM thus derived is capable of modelling the effect of intervening on arbitrary subsets of variables, even though it was constructed by considering the case that we consider interventions on exactly $D-1$ variables. Theorem 3 in Sec. 5.4 proves that the notions of intervention in ODE and the derived DSCM coincide. Collectively, these theorems tell us that we can derive a DSCM that allows us to reason about the effects of interventions on the asymptotic dynamics of the ODE. Proofs of these theorems are provided in Section A of the Supplementary Materials.

5.1 Interventions in a DSCM

Interventions in (D)SCMs are realized by replacing the structural equations of the intervened variables. Given $\boldsymbol{\zeta}_I \in \mathtt{Dyn}_I$ for some $I \subseteq \mathcal{I}$, the intervened DSCM $\mathcal{M}_{\mathtt{do}(\mathbf{X}_I = \boldsymbol{\zeta}_I)}$ can be written:

$$\mathcal{M}_{\mathtt{do}(\mathbf{X}_I = \boldsymbol{\zeta}_I)} : \begin{cases} X_i & = F_i(\mathbf{X}_{\mathtt{pa}(i)}) \quad i \in \mathcal{I} \setminus I \\ X_i & = \zeta_i \qquad\qquad\quad i \in I \end{cases}$$

The causal mechanisms determining the non-intervened variables are unaffected, so their structural equations remain the same. The intervened variables are decoupled from their usual causal mechanisms and are forced to take the specified trajectory.

5.2 Deriving DSCMs from ODEs

We require the following consistency property between the asymptotic dynamics of the ODE and the set of interventions.

Definition 6 (Structural dynamic stability)**.** *Let* $\mathtt{Dyn}$ *be modular. The pair* $(\mathcal{D}, \mathtt{Dyn})$ *is* ***structurally dynamically stable*** *if* $(\mathcal{D}, \mathtt{Dyn}_{\mathcal{I}\setminus\{i\}})$ *is dynamically stable with reference to* $\mathtt{Dyn}_{\mathcal{I}}$ *for all i.*

This means that for any trajectory $\boldsymbol{\zeta}_{\mathcal{I}\setminus\{i\}} \in \mathtt{Dyn}_{\mathcal{I}\setminus\{i\}}$, the asymptotic dynamics of the intervened ODE $\mathcal{D}_{\mathtt{do}(\mathbf{X}_{\mathcal{I}\setminus\{i\}} = \boldsymbol{\zeta}_{\mathcal{I}\setminus\{i\}})}$ are expressible uniquely as an element of $\mathtt{Dyn}_{\mathcal{I}}$. Since $\mathtt{Dyn}$ is modular, the asymptotic dynamics of the non-intervened variable can be realised as the trajectory $\boldsymbol{\zeta}_i \in \mathtt{Dyn}_i$, and thus $\mathtt{Dyn}$ is rich enough to allow us to make an intervention which forces the non-intervened variable to take this trajectory. This is a crucial property that allows the construction of the structural equations. In the particular case that $\mathtt{Dyn}$ consists of all constant trajectories, structural dynamic stability means that after any intervention on all-but-one-variable, the non-intervened variable settle to a unique equilibrium. In the language of [11], this would imply that the ODE is *structurally stable*.

It should be noted that $(\mathcal{D}, \mathtt{Dyn})$ being structurally dynamically stable is a strong assumption in general. If $\mathtt{Dyn}$ is too small,[2] then it may be possible to find a larger set $\mathtt{Dyn}' \supset \mathtt{Dyn}$ such that $(\mathcal{D}, \mathtt{Dyn}')$ *is* structurally dynamically stable. The procedure described in this section describes how to derive a DSCM capable of modelling all interventions in $\mathtt{Dyn}'$, which can thus be used to model interventions in $\mathtt{Dyn}$.

[2] For example, if $\mathtt{Dyn}$ is not modular or represents interventions on only a subset of the variables.

Henceforth, we use the notation $I_i = \mathcal{I} \setminus \{i\}$ for brevity. Suppose that $(\mathcal{D}, \mathtt{Dyn})$ is structurally dynamically stable. We can **derive structural equations** $F_i : \mathtt{Dyn}_{\mathtt{pa}(i)} \longrightarrow \mathtt{Dyn}_i$ to describe the asymptotic dynamics of children variables as functions of their parents as follows. Pick $i \in \mathcal{I}$. The variable X_i has parents $\mathbf{X}_{\mathtt{pa}(i)}$. Since $\mathtt{Dyn}$ is modular, for any configuration of parent dynamics $\boldsymbol{\eta}_{\mathtt{pa}(i)} \in \mathtt{Dyn}_{\mathtt{pa}(i)}$ there exists $\boldsymbol{\zeta}_{I_i} \in \mathtt{Dyn}_{I_i}$ such that $(\boldsymbol{\zeta}_{I_i})_{\mathtt{pa}(i)} = \boldsymbol{\eta}_{\mathtt{pa}(i)}$.

By structural dynamic stability, the system $\mathcal{D}_{\mathtt{do}(\mathbf{X}_{I_i} = \boldsymbol{\zeta}_{I_i})}$ has asymptotic dynamics specified by a unique element $\boldsymbol{\eta} \in \mathtt{Dyn}_{\mathcal{I}}$, which in turn defines a unique element $\eta_i \in \mathtt{Dyn}_i$ specifying the asymptotic dynamics of variable X_i since $\mathtt{Dyn}$ is modular.

Theorem 1. *Suppose that $(\mathcal{D}, \mathtt{Dyn})$ is structurally dynamically stable. Then the functions*

$$F_i : \mathtt{Dyn}_{\mathtt{pa}(i)} \longrightarrow \mathtt{Dyn}_i$$
$$\boldsymbol{\eta}_{\mathtt{pa}(i)} \longrightarrow \eta_i$$

constructed as above are well-defined.

Given the structurally dynamically stable tuple $(\mathcal{D}, \mathtt{Dyn})$ we define the derived DSCM:

$$\mathcal{M}_{\mathcal{D}} : \{ \;\; X_i = F_i(\mathbf{X}_{\mathtt{pa}(i)}) \quad i \in \mathcal{I}$$

where the $F_i : \mathtt{Dyn}_{\mathtt{pa}(i)} \longrightarrow \mathtt{Dyn}_i$ are defined as above. Note that structural dynamic stability was a crucial property that ensured $F_i(\mathtt{Dyn}_{\mathtt{pa}(i)}) \subseteq \mathtt{Dyn}_i$. If $(\mathcal{D}, \mathtt{Dyn})$ is not structurally dynamically stable, we cannot build structural equations in this way.

5.3 Solutions of a DSCM

Theorem 1 states that we can construct a DSCM by the described procedure. We constructed each equation by intervening on $D - 1$ variables at a time. The result of this section states that the DSCM can be used to correctly model interventions on arbitrary subsets of variables. We say that $\boldsymbol{\eta}_{\mathcal{I}} \in \mathtt{Dyn}_{\mathcal{I}}$ is a *solution* of $\mathcal{M}$ if $\eta_i = F_i(\boldsymbol{\eta}_{\mathtt{pa}(i)}) \; \forall i \in \mathcal{I}$.

Theorem 2. *Suppose that $(\mathcal{D}, \mathtt{Dyn})$ is structurally dynamically stable. Let $I \subseteq \mathcal{I}$, and let $\boldsymbol{\zeta}_I \in \mathtt{Dyn}_I$. Then $\mathcal{D}_{do(X_I = \boldsymbol{\zeta}_I)}$ is dynamically stable if and only if the intervened SCM $\mathcal{M}_{(\mathcal{D}_{\mathtt{do}(\mathbf{X}_I = \boldsymbol{\zeta}_I)})}$ has a unique solution. If there is a unique solution, it coincides with the element of $\mathtt{Dyn}_{\mathcal{I}}$ describing the asymptotic dynamics of $\mathcal{D}_{\mathtt{do}(\mathbf{X}_I = \boldsymbol{\zeta}_I)}$.*

Remark 2. We could also take $I = \emptyset$, in which case the above theorem applies to just $\mathcal{D}$.

5.4 Causal reasoning is preserved

We have defined ways to model interventions in both ODEs and DSCMs. The following theorem proves that these notions of intervention coincide, and hence that DSCMs provide a calculus to reason about the asymptotic behaviour of the ODE under interventions in $\mathtt{Dyn}$. A consequence of this is that the diagram in Figure 4 commutes.

Theorem 3. *Suppose that $(\mathcal{D}, \mathtt{Dyn})$ is structurally dynamically stable. Let $I \subseteq \mathcal{I}$ and let $\boldsymbol{\zeta}_I \in \mathtt{Dyn}_I$. Then $\mathcal{M}_{(\mathcal{D}_{\mathtt{do}(\mathbf{X}_I = \boldsymbol{\zeta}_I)})} = (\mathcal{M}_{\mathcal{D}})_{\mathtt{do}(\mathbf{X}_I = \boldsymbol{\zeta}_I)}$.*

To summarise, Theorems 1 - 3 collectively state that if $(\mathcal{D}, \mathtt{Dyn})$ is dynamically structurally stable then it is possible to derive a DSCM that allows us to reason about the asymptotic dynamics of the ODE under any possible intervention in $\mathtt{Dyn}$. A worked example showing how to derive a DSCM for the mass-spring system of Example 1 is included in the Supplementary Material.

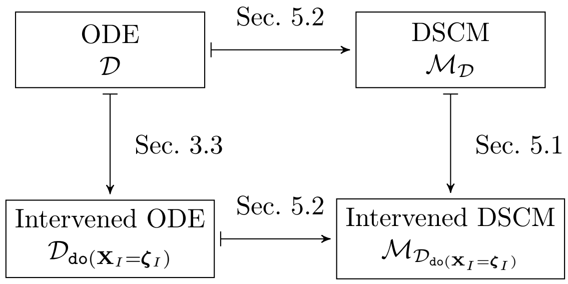


Figure 4: Left-to-right arrows: Theorems 1 and 2 together state that if $(\mathcal{D}, \mathtt{Dyn})$ is structurally dynamically stable then we can construct a DSCM to describe the asymptotic behaviour of $\mathcal{D}$ under different interventions in the set $\mathtt{Dyn}$. Top-to-bottom arrows: Both ODEs and DSCMs are equipped with notions of intervention. Theorem 3 says that these two notions of intervention coincide, and thus the diagram commutes.

6 Discussion

The main contribution of this paper is to show that the SCM framework can be applied to reason about time-dependent interventions on an ODE in a dynamic setting. In particular, we showed that if an ODE is sufficiently well-behaved under a set of interventions, a DSCM can be derived that captures how the asymptotic dynamics change under these interventions. This is in contrast to previous approaches to connecting the language of ODEs with the SCM framework, which used SCMs to describe the stable equilibria of the ODE and how they change under intervention.

There are two possible directions in which to extend this work. The first is to relax the assumption that the asymptotic dynamics are *independent of initial conditions*. This rules out, for example, simple models of neural dynamics such as the FitzHugh-Namugo model which exhibits a limit cycle in the observational system [12]. Indeed, if one were to start two systems at two different points along the limit cycle, they would remain forever out of phase, violating independence of initial condition. Intuitively, systems exhibiting limit cycles might still have dynamics that are sufficiently simple to characterise in an SCM-like framework. The second extension is to move away from deterministic systems and consider Stochastic Differential Equations. This framework could be used to take into account model uncertainty, but also to include systems that may be inherently stochastic.

Acknowledgments

Stephan Bongers was supported by NWO, the Netherlands Organization for Scientific Research (VIDI grant 639.072.410). This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement n° 639466).

References

[1] J. Pearl. *Causality: Models, Reasoning, and Inference.* Cambridge University Press, New York, NY, 2nd edition, 2009.

[2] Kenneth A Bollen. *Structural equations with latent variables.* John Wiley & Sons, 2014.

[3] Marek J Druzdzel and Herbert A Simon. Causality in bayesian belief networks. In *Proceedings of the Ninth international conference on Uncertainty in artificial intelligence*, pages 3–11. Morgan Kaufmann Publishers Inc., 1993.

[4] Antti Hyttinen, Frederick Eberhardt, and Patrik O Hoyer. Learning linear cyclic causal models with latent variables. *The Journal of Machine Learning Research*, 13(1):3387–3439, 2012.

[5] Joris M Mooij, Dominik Janzing, Tom Heskes, and Bernhard Schölkopf. On causal discovery with cyclic additive noise models. In *Advances in neural information processing systems*, pages 639–647, 2011.

[6] Mark Voortman, Denver Dash, and Marek J Druzdzel. Learning why things change: the difference-based causality learner. *Proceedings of the 26th Conference on Uncertainty in Artificial Intelligence (UAI 2010)*, 2010.

[7] Gustavo Lacerda, Peter L Spirtes, Joseph Ramsey, and Patrik O Hoyer. Discovering cyclic causal models by independent components analysis. *In Proceedings of the Twenty-Fourth Conference on Uncertainty in Artificial Intelligence (UAI2008)*, 2008.

[8] Peter Spirtes. Directed cyclic graphical representations of feedback models. In *Proceedings of the Eleventh conference on Uncertainty in artificial intelligence*, pages 491–498. Morgan Kaufmann Publishers Inc., 1995.

[9] Yumi Iwasaki and Herbert A Simon. Causality and model abstraction. *Artificial intelligence*, 67 (1):143–194, 1994.

[10] Denver Dash. Restructuring dynamic causal systems in equilibrium. In *AISTATS*. Citeseer, 2005.

[11] J. Mooij, D. Janzing, and B. Schölkopf. From ordinary differential equations to structural causal models: the deterministic case. In A. Nicholson and P. Smyth, editors, *Proceedings of the Twenty-Ninth Conference Annual Conference on Uncertainty in Artificial Intelligence*, pages 440–448, Corvallis, OR, 2013. AUAI Press. URL `http://www.is.tuebingen.mpg.de/fileadmin/user_upload/files/publications/2013/MooijJS2013-uai.pdf`.

[12] Richard FitzHugh. *Mathematical models of excitation and propagation in nerve*. McGraw-Hill Book Co., 1966.

Supplementary Material

A Proofs

A.1 Proof of Theorem 1

Proof. We need to show that if $\boldsymbol{\zeta}_{I_i}$ and $\boldsymbol{\zeta}'_{I_i}$ are such that $(\boldsymbol{\zeta}_{I_i})_{\mathtt{pa}(i)} = (\boldsymbol{\zeta}'_{I_i})_{\mathtt{pa}(i)} = \boldsymbol{\eta}_{\mathtt{pa}(i)}$, then $\eta_i = \eta'_i$. To see that this is the case, observe that the system of equations for $\mathcal{D}_{\mathtt{do}(\mathbf{X}_{I_i}=\boldsymbol{\zeta}_{I_i})}$ is given by:

$$\mathcal{D}_{\mathtt{do}(\mathbf{X}_{I_i}=\boldsymbol{\zeta}_{I_i})} : \begin{cases} X_j(t) = \zeta_j(t) & j \in \mathcal{I} \setminus (\mathtt{pa}(i) \cup \{i\}) \\ X_j(t) = \eta_j(t) & j \in \mathtt{pa}(i) \\ f_i(X_i, \mathbf{X}_{\mathtt{pa}(i)})(t) = 0 & \end{cases}$$

The equations for $\mathcal{D}_{\mathtt{do}(\mathbf{X}_{I_i}=\boldsymbol{\zeta}'_{I_i})}$ are similar, except with $X_j(t) = \zeta'_j(t)$ for $j \in \mathcal{I} \setminus (\mathtt{pa}(i) \cup \{i\})$

In both cases, the equations for all variables except X_i are solved already. The equation for X_i in both cases reduces to the same thing by substituting in the values of the parents, namely

$$f_i(X_i, \boldsymbol{\eta}_{\mathtt{pa}(i)})(t) = 0$$

The solution to this equation in $\mathtt{Dyn}_i$ must be unique, else the dynamic stability of the intervened systems $\mathcal{D}_{\mathtt{do}(\mathbf{X}_{I_i}=\boldsymbol{\zeta}_{I_i})}$ and $\mathcal{D}_{\mathtt{do}(\mathbf{X}_{I_i}=\boldsymbol{\zeta}'_{I_i})}$ would not hold, contradicting the dynamic structural stability of $(\mathcal{D}, \mathtt{Dyn})$. It follows that $\eta_i = \eta'_i$. □

A.2 Proof of Theorem 2

Proof. By construction of the SCM, $\boldsymbol{\eta} \in \mathtt{Dyn}_{\mathcal{I}}$ is a solution of $\mathcal{M}_{(\mathcal{D}_{\mathtt{do}(\mathbf{X}_I=\boldsymbol{\zeta}_I)})}$ if and only if:

- For $i \in \mathcal{I} \setminus I$, $X_i(t) = \eta_i(t)\ \forall t$ is a solution to the differential equation $f_i(X_i, \boldsymbol{\eta}_{\mathtt{pa}(i)})(t) = 0$
- For $i \in I$, $\eta_i(t) = \zeta_i(t)$ for all t.

which is true if and only if $\mathbf{X} = \boldsymbol{\eta}$ is a solution to $\mathcal{D}_{\mathtt{do}(\mathbf{X}_I=\boldsymbol{\zeta}_I)}$ in $\mathtt{Dyn}_{\mathcal{I}}$.

Thus, by definition of dynamic stability, $\mathcal{D}_{\mathtt{do}(\mathbf{X}_I=\boldsymbol{\zeta}_I)}$ is dynamically stable with asymptotic dynamics describable by $\boldsymbol{\eta} \in \mathtt{Dyn}$ if and only if $\mathbf{X} = \boldsymbol{\eta}$ uniquely solves $\mathcal{M}_{(\mathcal{D}_{\mathtt{do}(\mathbf{X}_I=\boldsymbol{\zeta}_I)})}$. □

A.3 Proof of Theorem 3

Proof. We need to show that the structural equations of $\mathcal{M}_{(\mathcal{D}_{\mathtt{do}(\mathbf{X}_I=\boldsymbol{\zeta}_I)})}$ and $(\mathcal{M}_{\mathcal{D}})_{\mathtt{do}(\mathbf{X}_I=\boldsymbol{\zeta}_I)}$ are equal. Observe that the equations for $\mathcal{D}_{\mathtt{do}(\mathbf{X}_I=\boldsymbol{\zeta}_I)}$ are given by:

$$\mathcal{D}_{\mathtt{do}(\mathbf{X}_I=\boldsymbol{\zeta}_I)} : \begin{cases} X_i = \zeta_i, & i \in I \\ f_i(X_i, \mathbf{X}_{\mathtt{pa}(i)}) = 0, & i \in \mathcal{I} \setminus I \end{cases}$$

Therefore, when we do the procedure to derive the structural equations for $\mathcal{D}_{\mathtt{do}(\mathbf{X}_I=\boldsymbol{\zeta}_I)}$, we see that:

- if $i \in I$, the ith structural equation will simply be $X_i = \zeta_i$ since intervening on I_i does not affect variable X_i.
- if $i \in \mathcal{I} \setminus I$, the ith structural equation will be the same as for $\mathcal{M}_{\mathcal{D}}$, since the dependence of X_i on the other variables is unchanged.

i.e. the structural equations for $\mathcal{M}_{(\mathcal{D}_{\mathtt{do}(\mathbf{X}_I=\boldsymbol{\zeta}_I)})}$ are given by:

$$\mathcal{M}_{(\mathcal{D}_{\mathtt{do}(\mathbf{X}_I=\boldsymbol{\zeta}_I)})} : \begin{cases} X_i = \zeta_i, & i \in I \\ X_i = F_i(\mathbf{X}_{\mathtt{pa}(i)}), & i \in \mathcal{I} \setminus I \end{cases}$$

and therefore $\mathcal{M}_{(\mathcal{D}_{\mathtt{do}(\mathbf{X}_I=\boldsymbol{\zeta}_I)})} = (\mathcal{M}_{\mathcal{D}})_{\mathtt{do}(\mathbf{X}_I=\boldsymbol{\zeta}_I)}$ □

B Deriving the DSCM for the mass-spring system

Consider the mass-spring system of Example 1, but with $D \geq 3$ an arbitrary integer.

We repeat the setup. We have D masses attached together on springs. The location of the ith mass at time t is $X_i(t)$, and its mass is m_i. For notational ease, we denote by $X_0 = 0$ and $X_{D+1} = L$ the locations of where the ends of the springs attached to the edge masses meet the walls to which they are affixed. X_0 and X_{D+1} are constant. The natural length and spring constant of the spring connecting masses i and $i+1$ are l_i and k_i respectively. The ith mass undergoes linear damping with coefficient b_i, where b_i is small to ensure that the system is underdamped.

The equation of motion for the ith mass ($1 \leq i \leq D$) is given by:

$$m_i\ddot{X}_i(t) = k_i[X_{i+1}(t) - X_i(t) - l_i] - k_{i-1}[X_i(t) - X_{i-1}(t) - l_{i-1}] - b_i\dot{X}_i(t)$$

so, defining

$$f_i(X_i, X_{i-1}, X_{i+1})(t) = m_i\ddot{X}_i(t) - k_i[X_{i+1}(t) - X_i(t) - l_i] + k_{i-1}[X_i(t) - X_{i-1}(t) - l_{i-1}] + b_i\dot{X}_i(t)$$

we can write the system of equations $\mathcal{D}$ for our mass-spring system as

$$\mathcal{D} : \left\{ \; f_i(X_i, X_{i-1}, X_{i+1})(t) = 0 \qquad i \in \mathcal{I} \right.$$

In the rest of this section we will explicitly calculate the structural equations for the DSCM derived from $\mathcal{D}$ with two different sets of interventions. First, we will derive the structural equations for the case that `Dyn` consists of all constant trajectories, corresponding to constant interventions that fix variables to constant values for all time. This shows the correspondence between this paper and [11]. Next, we will derive the structural equations for the case that `Dyn` consists of interventions corresponding to sums of periodic forcing terms.

B.1 Mass-spring with constant interventions

In order to derive the structural equations we only need to consider, for each variable, the influence of its parents on it. (Formally, this is because of Theorem 1)

Considering variable i. If we intervene to fix its parents to have locations $X_{i-1}(t) = \eta_{i-1}$ and $X_{i+1}(t) = \eta_{i+1}$ for all t, then the equation of motion for variable i is given by

$$m_i\ddot{X}_i(t) + b_i\dot{X}_i(t) + (k_i + k_{i-1})X_i(t) = k_i[\eta_{i+1} - l_i] + k_{i-1}[\eta_{i-1} + l_{i-1}]$$

There may be some complicated transient dynamics that depend on the initial conditions $X_i(0)$ and $\dot{X}_i(0)$ but provided that $b_i > 0$, we know that the $X_i(t)$ will converge to a constant and therefore the asymptotic solution to this equation can be found by setting $\ddot{X}_i$ and $\dot{X}_i$ to zero. Note that in general, we could explicitly find the solution to this differential equation (and indeed, in the next example we will) but for now there is a shortcut to deriving the structural equations[3]

The asymptotic solution is therefore:

$$X_i = \frac{k_i[\eta_{i+1} - l_i] + k_{i-1}[\eta_{i-1} + l_{i-1}]}{k_i + k_{i-1}}$$

[3]This is analogous to the approach taken in [11] in which they first define the Labelled Equilibrium Equations and from these derive the SCM.

Therefore the ith structural equation is:

$$F_i(X_{i-1}, X_{i+1}) = \frac{k_i[X_{i+1} - l_i] + k_{i-1}[X_{i-1} + l_{i-1}]}{k_i + k_{i-1}}$$

Hence the SCM for $(\mathcal{D}, \mathtt{Dyn}_c)$ is

$$\mathcal{M}_{\mathcal{D}} : \left\{ \; X_i = \tfrac{k_i[X_{i+1}-l_i]+k_{i-1}[X_{i-1}+l_{i-1}]}{k_i+k_{i-1}} \qquad i \in \mathcal{I} \right.$$

And we can use this model to reason about the effect of constant interventions on the asymptotic dynamics of the system.

B.2 Sums of periodic interventions

Suppose now we want to be able to make interventions of the form:

$$\mathtt{do}\left(X_i(t) = A\cos(\omega t + \phi)\right) \tag{†}$$

Such interventions cannot be described by the DSCM derived in Section B.1. In this section we will explicitly derive a DSCM capable of reasoning about the effects of such interventions. It will also illustrate why we need dynamic structural stability.

By Theorem 1, to derive the structural equation for each variable we only need to consider the effect on the child of intervening on the parents according to interventions of the form †.

Consider the following linear differential equation:

$$m\ddot{X}(t) + b\dot{X}(t) + kX(t) = g(t) \tag{1}$$

In general, the solution to this equation will consist of two parts - the *homogeneous* solution and the *particular* solution. The homogeneous solution is one of a family of solutions to the equation:

$$m\ddot{X}(t) + b\dot{X}(t) + kX(t) = 0 \tag{2}$$

and this family of solutions is parametrised by the initial conditions. If there $b > 0$ then all of the homogeneous solutions decay to zero as $t \longrightarrow \infty$. The particular solution is any solution to the original equation with arbitrary initial conditions. The particular solution captures the asymptotic dynamics due to the forcing term g.

Equation 1 is a linear differential equation. This means that if $X = X_1$ is a particular solution for $g = g_1$ and $X = X_2$ is a particular solution for $g = g_2$, then $X = X_1 + X_2$ is a particular solution for $g = g_1 + g_2$.

In order to derive the structural equations, the final ingredient we need is an explicit representation for a particular solution to (1) in the case that $g(t) = A\cos(\omega t + \phi)$. We state the solution for the case that the system is underdamped - this is a 'standard result' and can be verified by checking that the following satisfies 1:

$$X(t) = A'\cos(\omega t + \phi')$$

where

$$A' = \frac{A}{\sqrt{[k - m\omega^2]^2 + bm\omega^2}} \qquad\qquad \phi' = \phi - \arctan[\frac{b\omega}{k - m\omega^2}] \tag{3}$$

Therefore if we go back to our original equation of motion for variable X_i:

$$m_i \ddot{X}_i(t) + b_i \dot{X}_i(t) + (k_i + k_{i-1}) X_i(t) = k_i[X_{i+1}(t) - l_i] + k_{i-1}[X_{i-1}(t) + l_{i-1}]$$

Then if we make the intervention

$$\texttt{do}(X_{i-1}(t) = A_{i-1}\cos(\omega_{i-1}t + \phi_{i-1}), X_{i+1}(t) = A_{i+1}\cos(\omega_{i+1}t + \phi_{i+1}))$$

we see that we can write the RHS of the above equation as the sum of the three terms

$$\begin{aligned} g_1(t) &= k_{i-1}l_{i-1} - k_i l_i \\ g_2(t) &= k_{i-1}A_{i-1}\cos(\omega_{i-1}t + \phi_{i-1}) \\ g_3(t) &= k_i A_{i+1}\cos(\omega_{i+1}t + \phi_{i+1}) \end{aligned}$$

Using the fact that linear differential equation have superposable solutions and Equation 3 we can write down the resulting asymptotic dynamics of X_i:

$$\begin{aligned} X_i(t) = &\frac{k_{i-1}l_{i-1} - k_i l_i}{k_i + k_{i-1}} \\ &+ \frac{k_{i-1}A_{i-1}}{\sqrt{[k_i + k_{i-1} - m_i\omega_{i-1}^2]^2 + b_i m_i \omega_{i-1}^2}} \cos(\omega_{i-1}t + \phi_{i-1} - \arctan[\frac{b_i\omega_{i-1}}{k_i + k_{i-1} - m_i\omega_{i-1}^2}]) \\ &+ \frac{k_i A_{i+1}}{\sqrt{[k_i + k_{i-1} - m_i\omega_{i+1}^2]^2 + b_i m_i \omega_{i+1}^2}} \cos(\omega_{i+1}t + \phi_{i+1} - \arctan[\frac{b_i\omega_{i+1}}{k_i + k_{i-1} - m_i\omega_{i+1}^2}]) \end{aligned}$$

However, note that if we were using `Dyn` consisting of interventions of the form of equation †, then we have just shown that the mass-spring system would not be structurally dynamically stable with respect to this `Dyn`, since we need two periodic terms and a constant term to describe the motion of a child under legal interventions of the parents.

This illustrates the fact that we may sometimes be only interested in a particular set of interventions that may not itself satisfy structural dynamic stability, and that in this case we must consider a larger set of interventions that *does*.

In this case, we can consider the modular set of trajectories generated by trajectories of the following form for each variable:

$$X(t) = \sum_{j=1}^{\infty} A_j \cos(\omega_j t + \phi_j)$$

where $\sum_{j=1}^{\infty} |A_j| < \infty$ (so that the series is absolutely convergent and thus does not depend on the ordering of the terms in the sum). Call this set $\texttt{Dyn}_{qp}$ (quasi-periodic).

By equation 3, we can write down the structural equations:

$$\begin{aligned} &F_i\left(\sum_{j=1}^{\infty} A_j^{i-1}\cos(\omega_j^{i-1}t + \phi_j^{i-1}), \sum_{j=1}^{\infty} A_j^{i+1}\cos(\omega_j^{i+1}t + \phi_j^{i+1})\right) \\ &= \sum_{j=1}^{\infty} \frac{k_{i-1}A_j^{i-1}}{\sqrt{[k_i + k_{i-1} - m_i{\omega_j^{i-1}}^2]^2 + b_i m_i {\omega_j^{i-1}}^2}} \cos\left(\omega_j^{i-1}t + \phi_j^{i-1} - \arctan\left[\frac{b_i\omega_j^{i-1}}{k_i + k_{i-1} - m_i{\omega_j^{i-1}}^2}\right]\right) \\ &+ \sum_{j=1}^{\infty} \frac{k_i A_j^{i+1}}{\sqrt{[k_i + k_{i+1} - m_i{\omega_j^{i+1}}^2]^2 + b_i m_i {\omega_j^{i+1}}^2}} \cos\left(\omega_j^{i+1}t + \phi_j^{i+1} - \arctan\left[\frac{b_i\omega_j^{i+1}}{k_i + k_{i+1} - m_i{\omega_j^{i+1}}^2}\right]\right) \end{aligned}$$

Since this is also a member of $\texttt{Dyn}_{qp}$, the mass-spring system is dynamically structurally stable with respect to $\texttt{Dyn}_{qp}$ and so the equations F_i define the structural causal model for asymptotic dynamics.