Causal Calculus in the Presence of Cycles, Latent Confounders and Selection Bias

Patrick Forrê
Informatics Institute
University of Amsterdam
The Netherlands
p.d.forre@uva.nl

Joris M. Mooij
Informatics Institute
University of Amsterdam
The Netherlands
j.m.mooij@uva.nl

Abstract

We prove the main rules of causal calculus (also called do-calculus) for i/o structural causal models (ioSCMs), a generalization of a recently proposed general class of non-/linear structural causal models that allow for cycles, latent confounders and arbitrary probability distributions. We also generalize adjustment criteria and formulas from the acyclic setting to the general one (i.e. ioSCMs). Such criteria then allow to estimate (conditional) causal effects from observational data that was (partially) gathered under selection bias and cycles. This generalizes the backdoor criterion, the selection-backdoor criterion and extensions of these to arbitrary ioSCMs. Together, our results thus enable causal reasoning in the presence of cycles, latent confounders and selection bias. Finally, we extend the ID algorithm for the identification of causal effects to ioSCMs.

1 INTRODUCTION

Statistical models are governed by the rules of probability (e.g. sum and product rule), which link joint distributions with the corresponding (conditional) marginal ones. Causal models follow additional rules, which relate the observational distributions with the interventional ones. In contrast to the rules of probability theory, which directly follow from their axioms, the rules of causal calculus need to be proven, when based on the definition of structural causal models (SCMs). As SCMs will among other things depend on the underlying graphical structure (e.g. with or without cycles or bidirected edges, etc.), the used function classes (e.g. linear or non-linear, etc.) and the allowed probability distributions (e.g. discrete, continuous, singular or mixtures, etc.) the respective endeavour is not immediate.

Such a framework of causal calculus contains rules about when one can 1.) insert/delete observations, 2.) exchange action/observation, 3.) insert/delete actions; and about when and how to recover from interventions and/or selection bias (backdoor and selection-backdoor criterion), etc. (see [1, 4, 5, 14, 21–24, 26, 27, 32–35]). While these rules have been extensively studied for acyclic causal models, e.g. (semi-)Markovian models, which are attached to directed acyclic graphs (DAGs) or acyclic directed mixed graphs (ADMGs) (see [1, 4, 5, 14, 21–24, 26, 27, 32–35]), the case of causal models with cycles stayed in the dark.

To deal with cycles and latent confounders at the same time in this paper we will introduce the class of input/output structural causal models (ioSCMs), a “conditional” version of the recently proposed class of modular structural causal models (mSCMs) (see [10, 11]) to also include “input” nodes that can play the role of parameter/context/action/intervention nodes. ioSCMs have several desirable properties: They allow for arbitrary probability distributions, non-/linear functional relations, latent confounders and cycles. They can also model non/probabilistic external and probabilistic internal nodes in one framework. The cycles are modelled in a least restrictive way such that the class of ioSCMs still becomes closed under arbitrary marginalizations and interventions. All causal models that are based on acyclic graphs like DAGs, ADMGs or mDAGs (see [9, 28]) can be interpreted as special acyclic ioSCMs. Besides feedback over time ioSCMs can also express instantaneous and equilibrated feedback under the made model assumptions (e.g. the ODEs in [2, 18]). All models where the non-trivial cycles are “contractive” (negative feedback loops, see [11]) are ioSCMs without further assumptions. Thus ioSCMs generalize all these classes of causal models in one framework, which goes beyond the acyclic setting and also allows for conditional...
versions of those (e.g. CADMGs), expressed via external non-/probabilistic “input” nodes. Also the generalized directed global Markov property for mSCMs (see [10, 11]) generalizes to ioSCMs, i.e. ioSCMs entail the conditional independence relations that follow from the σ-separation criterion in the underlying graph, where σ-separation generalizes the usual δ-separation (also called m- or m′-separation, see [9, 20, 24, 28, 38]) from acyclic graphs to directed mixed graphs (DMGs) (and even HEDGes [10] and σ-CGs [11]) with or without cycles in a non-naïve way.

This paper now aims at proving the mentioned main rules of causal calculus for ioSCMs and derive adjustment criteria with corresponding adjustment formulas like generalized (selection-)backdoor adjustments. We also provide an extension of the ID algorithm for the identification of causal effects to the general setting, which reduces to the usual one in the acyclic case.

The paper is structured as follows: We will first give the precise definition of ioSCMs closely mirroring mSCMs from [10, 11]. We will then review σ-separation and generalize its criterion from mSCMs (see [10, 11]) to ioSCMs. As a preparation for the causal calculus, which relates observational and interventional distributions, we will then show how one can extend a given ioSCM to one that also incorporates additional interventional variables indicating the regime of interventions on the observed nodes. We will then show how the rules of causal calculus directly follow from applying the σ-separation criterion to such an extended ioSCM. We then derive the mentioned general adjustment criteria with corresponding adjustment formulas. Finally, we introduce the right definitions for ioSCMs to extend the ID algorithm for the identification of causal effects to the general setting.

2 INPUT/OUTPUT STRUCTURAL CAUSAL MODELS

In this section we will define input/output structural causal models (ioSCMs), which can be seen as a “conditional” version of modular structural causal models (mSCMs) defined in [10, 11]. We will then construct marginalized ioSCMs and intervened ioSCMs. To allow for cycles we first need to introduce the notion of loop of a graph and its strongly connected components.

Definition 2.1 (Loops). Let G = (V, E) be a directed graph (with or without cycles).

1. A set of nodes S ⊆ V is called a loop of G if for every two nodes v1, v2 ∈ S there are two directed walks v1 −→ · · · −→ v2 and v2 −→ · · · −→ v1 in G such that all the intermediate nodes are also in S (if any). The sets S = {v} are also considered as loops (independent of v −→ v ∈ E or not).
2. The set of loops of G is written as L(G).
3. The strongly connected component of v in G is defined to be: ScG(v) := AncG(v) ∩ DescG(v).
4. The set of strongly connected components is S(G).

Remark 2.2. Let G = (V, E) be a directed graph.

1. We always have v ∈ ScG(v) and ScG(v) ∈ L(G).
2. If G is acyclic then: L(G) = { {v} | v ∈ V }.

In the following all spaces are meant to be equipped with σ-algebras and all maps to be measurable. Whenever (regular) conditional distributions occur we implicitly assume standard measurable spaces (to ensure existence).

Definition 2.3 (Input/Output Structural Causal Model). An input/output (i/o) structural causal model (ioSCM) by definition consists of:

1. a set of nodes V+ = V ∪ U ∪ J, where elements of V correspond to output/observed variables, elements of U to probabilistic latent variables and elements of J to input/intervention variables.
2. an observation/latent/action space Xv for every v ∈ V+, Xv := ∏x∈V+ Xv.
3. a product probability measure P = ∏u∈U Pu on the latent space XU := ∏u∈U XU.
4. a directed graph structure G+ = (V+, E+) with the properties:
   (a) V = ChG+(U ∪ J),
   (b) PaG+(U ∪ J) = ∅,
   where ChG+ and PaG+ stand for children and parents in G+, resp.
5. a system of causal mechanisms g = (gs)S∈L(G+):
   gS : {Xv}v∈PaG+(S)\S → {Xv}v∈S
   that satisfy the following global compatibility conditions: For every nested pair of loops S′ ⊆ S ⊆ V of G+ and every element xP=PaG+(S)∪S Xv we have the implication:
   gS(xP=PaG+(S∪S)\S) = xS
   xS′
   where xP=PaG+(S∪S)\S and xS′ denote the corresponding components of xP=PaG+(S∪U).
The ioSCM will be denoted by $M = (G^+, X, P_U, g)$. A modular structural causal model (mSCM) is an ioSCM without input nodes, i.e. $J = \emptyset$.

**Definition 2.4** (Modular structural causal model, see [10]). A modular structural causal model (mSCM) is an ioSCM without input nodes, i.e. $J = \emptyset$.

**Remark 2.5** (Composition of ioSCMs). Consider two ioSCMs $M_1$, $M_2$ and an identification of subsets $I_1 \subseteq V_1^+$ with $I_2 \subseteq J_2$ and maps $g_{i_2} : X_{i_1} \rightarrow X_{i_2}$, for $i_1$ corresponding to $i_2$, e.g. $g_{i_2} = id$ if possible. We can now “glue” them together to get a new ioSCM $M_3$ given by $V_3 := V_1 \cup V_2 \cup J_2$, $U_3 := U_1 \cup U_2$, $J_3 := J_1 \cup J_2 \setminus I_2$ and $G_3^+ := G_1^+ \cup G_2^+$, where we add the the edges $i_1 \rightarrow i_2$, and the mechanisms $g_{i_2}$ and $P_{U_3} := P_{U_1} \otimes P_{U_2}$.

**Example 2.6** (Constructing mSCMs from ioSCMs). Given an ioSCM $M = (G^+, X, P_U, g)$ with graph $G^+ = (V \cup U \cup J, E^+)$ we can construct a well-defined mSCM by specifying a product distribution $P : J, E \rightarrow \mathbb{R}$ with

\[ P(\mathbb{1}_{S^+} \circ \mathbb{1}_{\bar{S}}) \quad : \quad \text{for subsets } S \subseteq X \]

where $S^+ := \text{Sc}G^+(v) := \text{AnC}G^+(v) \cap \text{DesC}G^+(v)$ and where the second index $v$ refers to the $v$-component of $g_S$. The induction is taken over any topological order of the strongly connected components of $G^+$, which always exists (see [10]).

By the compatibility condition for $g$ we then have that for every $S \subseteq E(G^+)$ with $S \subseteq V$ the following equality holds:

\[ X_S^{do(x_j)} = g_S(X_{Pa^G(A)}^{do(x_j)} \mid S), \]

where we put $X_A := \prod_{v \in A} X_v$ and $X_A := (X_v)_{v \in A}$ for subsets $A$.

**Definition 2.7**. Let $M = (G^+, X, P_U, g)$ be an ioSCM with $G^+ = (V \cup U \cup J, E^+)$.

1. The latent variables are given by $(X_u)_{u \in U} \sim \mathcal{P}_U$.
2. For $j \in J$ we put $X_j^{do(x_j)} := x_j$, the constant variable given by $j$-component of $X_j$.
3. The observed variables $(X_v^{do(x_j)})_{v \in V}$ are inductively defined by:

\[ X_v^{do(x_j)} := g_{S,v}(X_{Pa^G(A)}^{do(x_j)} \mid S \subseteq Pa^G(S) \setminus S), \]

where $S := \text{Sc}G^+(v) := \text{AnC}G^+(v) \cap \text{DesC}G^+(v)$ and where the second index $v$ refers to the $v$-component of $g$. The induction is taken over any topological order of the strongly connected components of $G^+$, which always exists (see [10]).

4. By the compatibility condition for $g$ we then have that for every $S \subseteq E(G^+)$ with $S \subseteq V$ the following equality holds:

\[ X_S^{do(x_j)} = g_S(X_{Pa^G(A)}^{do(x_j)} \mid S \subseteq Pa^G(S) \setminus S), \]

where we put $X_A := \prod_{v \in A} X_v$ and $X_A := (X_v)_{v \in A}$ for subsets $A$.

**Definition 2.8**. Let $M = (G^+, X, P_U, g)$ be an ioSCM with $G^+ = (V \cup U \cup J, E^+)$.

1. Define the graph $G^+_{do(W)}$ by removing all the edges $v \rightarrow w$ for all nodes $w \in W$ and $v \in Pa^G(w)$.
2. Put $V_{do(W)} := V \setminus W$ and $J_{do(W)} := J \setminus W$.
3. Remove the functions $g_S$ for loops $S$ with $S \cap W \neq \emptyset$.

The remaining functions then are clearly globally compatible and we get a well-defined ioSCM $M_{do(W)}$.
3 CONDITIONAL INDEPENDENCE

Here we generalize conditional independence for structured families of distributions. The main application will be the distributions \((P_V(X_V \mid \text{do}(X_J = x_J))_{x_J \in X_J})\) coming from ioSCMs, but the following definition might be of more general importance.

**Definition 3.1** (Conditional independence). Let \(X_V := \prod_{v \in V} X_v\) and \(X_J := \prod_{j \in J} X_j\) be product spaces and \(P := (P_V(X_V | x_J))_{x_J \in X_J}\) a family of distributions on \(X_V\) (measurable) parametrized by \(X_J\). For subsets \(A, B, C \subseteq V \cup J\) we write:

\[
X_A \perp \perp X_B | X_C
\]

if and only if for every product distribution \(P_J = \bigotimes_{j \in J} P_j\) on \(X_J\) we have:

\[
X_A \perp \perp X_B | X_C, \quad \text{i.e.}: \quad P_{V \cup J}(X_A | X_B, X_C) = P_{V \cup J}(X_A | X_C) \quad P_{V \cup J}-a.s.
\]

where \(P_{V \cup J}(X_A | X_B, X_C) := P_V(X_V | X_J) \otimes P_J(X_J)\) is the distribution given by \(X_J \sim P_J\) and then \(X_V \sim P_V\).

**Remark 3.2.**

1. The definition \[3.7\] assumes that the input variables \(J\) are considered independent, in contrast to \[3.29\], where all \(J\) are implicitly assumed to be jointly confounded. We discuss this further in Supplementary Material.

2. In contrast with \[3.29\] definition \[3.7\] can accommodate any variable from \(V\) or \(J\) at any spot of the conditional independence statement.

3. \(\perp \perp\) satisfies the separoid axioms (see \[6.7.13.25\] or see rules 1-5 in Lem.\[4.5\] or \(\perp \perp\)) as these rules are preserved under conjunction.

4 \(\sigma\)-SEPARATION

In this section we will define \(\sigma\)-separation on directed mixed graphs (DMG) and present the generalized directed global Markov property stating that every ioSCM will entail the conditional independencies that come from \(\sigma\)-separation in its induced DMG. We will again closely follow the work in \[11\].

**Definition 4.1** (Directed mixed graph (DMG)). A directed mixed graph (DMG) \(G\) consists of a set of nodes \(V\) together with a set of directed edges (\(\rightarrow\)) and bidirected edges (\(\leftrightarrow\)). In case \(G\) contains no directed cycles it is called an acyclic directed mixed graph (ADMG).

\[^{3}\]\text{We require that for every measurable} \(F \subseteq X_V\) \text{the map} \(X_J \rightarrow [0, 1]\) \text{given by} \(x_J \rightarrow P_V(X_V \in F | x_J)\) \text{is measurable. Such families of distributions are also called channels or (stochastic) Markov (transition) kernels (see} \[16\].\)

**Definition 4.2** (\(\sigma\)-Open walk in a DMG). Let \(G\) be a DMG with set of nodes \(V\) and \(C \subseteq V\) a subset. Consider a walk \(\pi\) in \(G\) with \(n \geq 1\) nodes:

\[
v_1 \equiv \cdots \equiv v_n\]

The walk will be called \(C\)-\(\sigma\)-open if:

1. the endnodes \(v_1, v_n \notin C\), and
2. every triple of adjacent nodes in \(\pi\) that is of the form:
   - (a) collider: \(v_{i-1} \equiv v_i \equiv v_{i+1}\), satisfies \(v_i \in C\),
   - (b) left chain: \(v_{i-1} \rightarrow v_i \equiv v_{i+1}\), satisfies \(v_i \notin C\) or \(v_i \in C \cap Sc^G(v_{i-1})\),
   - (c) right chain: \(v_{i-1} \equiv v_i \rightarrow v_{i+1}\), satisfies \(v_i \notin C\) or \(v_i \in C \cap Sc^G(v_{i+1})\),
   - (d) fork: \(v_{i-1} \rightarrow v_i \equiv v_{i+1}\), satisfies \(v_i \notin C\) or \(v_i \in C \cap Sc^G(v_{i-1}) \cap Sc^G(v_{i+1})\).

Similar to d-separation we define \(\sigma\)-separation in a DMG.

**Definition 4.3** (\(\sigma\)-Separation in a DMG). Let \(G\) be a DMG with set of nodes \(V\). Let \(A, B, C \subseteq V\) be subsets.

1. We say that \(A\) and \(B\) are \(\sigma\)-connected by \(C\) or not \(\sigma\)-separated by \(C\) if there exists a walk \(\pi\) (with \(n \geq 1\) nodes) in \(G\) with one endnode in \(A\) and one endnode in \(B\) that is \(C\)-\(\sigma\)-open. In symbols this statement will be written as follows:

\[
A \sigma_{G} B | C.
\]

2. Otherwise, we will say that \(A\) and \(B\) are \(\sigma\)-separated by \(C\) and write:

\[
A \not\sigma_{G} B | C.
\]

**Remark 4.4.**

1. In any DMG we will always have that \(\sigma\)-separation implies d-separation, since every \(C\)-d-open walk is also \(C\)-\(\sigma\)-open because \(\{v\} \subseteq Sc^G(v)\).

2. If a DMG \(G\) is acyclic, i.e. an ADMG, then \(\sigma\)-separation coincides with d-separation (also called \(m\)-or \(m^*\)-separation in this context).

It was shown in \[10\] that \(\sigma\)-separation satisfies the graphoid/separdoid axioms (see \[6.7.13.25\]):

**Lemma 4.5** (Graphoid and separoid axioms). Let \(G\) be a DMG with set of nodes \(V\) and \(A, B, C, D \subseteq V\) subsets. Then we have the following rules for \(\sigma\)-separation in \(G\) (with \(\perp \perp\) standing for \(\perp \perp\)):
where (see [10, 11]): marginalization

It was also shown that Theorem 4.6 (Lany sets $\{u, \sigma\}$) that this will give us an induced directed mixed graph $M(G)$ of an ioSCM. An alternative version with confounded input $G = (X, \mathcal{X}, \mathcal{P}, \mathcal{P}_d)$ is defined as follows:

1. **Redundancy:** $A \perp B \mid A$ always holds.
2. **Symmetry:** $A \perp B \mid D \implies B \perp A \mid D$.
3. **Decomposition:** $A \perp B \cup C \mid D \implies A \perp B \mid D$.
4. **Weak Union:** $A \perp B \cup C \mid D \implies A \perp B \mid C \cup D$.
5. **Contraction:** $(A \perp B \mid C \cup D) \wedge (A \perp C \mid D) \implies A \perp B \cup C \mid D$.
6. **Intersection:** $(A \perp B \mid C \cup D) \wedge (A \perp C \mid B \cup D) \implies A \perp B \cup C \mid D$, whenever $A, B, C, D$ are pairwise disjoint.
7. **Composition:** $(A \perp B \mid D) \wedge (A \perp C \mid D) \implies A \perp B \cup C \mid D$.

It was also shown that $\sigma$-separation is stable under marginalization (see [10,11]):

**Theorem 4.6** ($\sigma$-Separation under marginalization, see [10,11]). Let $G$ be a DMG with set of nodes $V$. Then for any sets $A, B, C \subseteq V$ and $L \subseteq V \setminus (A \cup B \cup C)$ we have the equivalence:

$$A \perp_G B \mid C \iff A \perp_{G \setminus L} B \mid C,$$

where $G \setminus L$ is the DMG that arises from $G$ by marginalizing out the variables from $L$.  

## 5 A GLOBAL MARKOV PROPERTY

The most important ingredient for our results is a generalized directed global Markov property that relates the graphical structure of any ioSCM $M$ to the conditional independencies of the observed random variables via a $\sigma$-separation criterion. Since we have no access to the latent nodes $u \in U$ of an ioSCM with graph $G^+$ we need to marginalize them out (see Supplementary Material [3]). This will give us an induced directed mixed graph (DMG) $G$.

**Definition 5.1** (Induced DMG of an ioSCM). Let $M = (G^+, \mathcal{X}, \mathcal{P}_U, \mathcal{P}_d)$ be an ioSCM with $G^+ = (V \cup U \cup J, E^+)$. The induced directed mixed graph (DMG) $G$ of $M$ is defined as follows:

1. $G$ contains all nodes from $V \cup J$.
2. $G$ contains all the directed edges of $G^+$ whose endnodes are both in $V \cup J$.
3. $G$ contains the bidirected edge $v \leftrightarrow w$ with $v, w \in V$ if and only if $v \neq w$ and there exists a $u \in U$ with $v, w \in \mathrm{Ch}^{G^+}(u)$, i.e. $v$ and $w$ have a common latent confounder.

The following generalized directed global Markov property directly generalizes from mSCMs (see [10,11]) to ioSCMs. An alternative version with confounded input is given in [3,5].

**Theorem 5.2** ($\sigma$-Separation criterion). Let $M$ be an ioSCM with induced DMG $G$. Then for all subsets $A, B, C \subseteq V \cup J$ we have the implication:

$$A \perp_G B \mid C \implies X_A \perp \sigma_G X_B \mid X_C.$$

In words, if $A$ and $B$ are $\sigma$-separated by $C$ in $G$ then the corresponding variables $X_A$ and $X_B$ are conditionally independent given $X_C$ under $\mathcal{P}$, i.e. under the joint distribution $\mathbb{P}_U(X_V \mid \text{do}(X_J)) \otimes \mathbb{P}_J(X_J)$ for any product distribution $\mathbb{P}_J = \otimes_{j \in J} \mathbb{P}_j$.

**Proof.** As mentioned, after specifying the product distribution $\mathbb{P}_J$ the ioSCM $M$ constitutes a well-defined mSCM with the same induced DMG $G$. So the $\sigma$-separation criterion for ioSCMs directly follows from the mSCM-version proven in [10,11].

**Remark 5.3.** Note that, since $\sigma$-separation is stable under marginalization (see [10,11]), also the $\sigma$-separation criterion is stable under marginalization.

**Remark 5.4** (Causal calculus for mechanism change). The $\sigma$-separation criterion 5.2 can be viewed as the causal calculus for mechanism change (also sometimes called “soft” interventions, see [8,17,19,24]). As an example consider $A, B \subseteq V, I \subseteq J$. Then the graphical separation $A \perp_G I \mid B \cup (J \setminus I)$ implies that the conditional probability $\mathbb{P}_U(X_A \mid X_B, \text{do}(X_J))$ is independent of the actual input variables in $I$.

## 6 THE EXTENDED IOSCM

In this section we want to consider (perfect) interventions onto the observed nodes and improve upon the general rules mentioned in 5.4. For an elegant treatment of this we need to gather for a given ioSCM $M$ all interventional ioSCMs $M_{\text{do}(w)}$, where $W$ runs through all subsets of observed variables, and glue them all together into one big extended ioSCM $M$. To consider all interventions at once we will need to introduce additional intervention variables $I_v$ to the graph $G^+$, $v \in V$, which indicate which interventional mechanisms to use. Such techniques were already used in the acyclic case in [21,22,24]. The definition will be made in such a way that $M$ will still be a well-defined ioSCM. So all the results for ioSCMs will apply to $M$, most importantly the $\sigma$-separation criterion (Thm. 5.2).

**Definition 6.1.** Let $M = (G^+, \mathcal{X}, \mathcal{P}_U, \mathcal{P}_d)$ be an ioSCM with $G^+ = (V \cup U \cup J, E^+)$. The extended ioSCM $M = (\hat{G}^+, \hat{\mathcal{X}}, \mathcal{P}_U, \hat{\mathcal{P}}_d)$ will be defined as follows:

1. For every $v \in V$ define the interventional domain $I_v := \mathcal{X}_V \cup \{\hat{\sigma}_v\}$, where $\hat{\sigma}_v$ is a new symbol corresponding to the observational (non-interventional) regime. For a set $A \subseteq V$ we put $\hat{I}_A := \prod_{v \in A} I_v$ and $\hat{\sigma}_A := (\hat{\sigma}_v)_{v \in A}$.
2. Let $\hat{G}^+$ be the graph $G^+$ with the additional intervention nodes $I_{\nu}$ and directed edges $I_{\nu} \rightarrow v$ for every $v \in V$. For a uniform notation we sometimes write $I_j$ instead of $j \in J$. So we have: 
$$\hat{J} := \hat{J} \cup \{\hat{I}_V \mid v \in V\} = \{\hat{I}_V \mid w \in V \cup J\}.$$ 

3. For every $A \subseteq V$ we will define the mechanism:

$$\hat{g}_A : \hat{X}_{Pa^{\hat{G}^+}(A) \setminus A} = \hat{I}_A \times \hat{X}_{Pa^{\hat{G}^+}(A) \setminus A} \rightarrow \hat{X}_A = \hat{\hat{X}}_A.$$ 

First, for $x_A \in \hat{I}_A$ we put $I(x_A) := \{v \in A \mid x_v \neq \varnothing\}$. Consider the subgraph of $G^+$:

$$H(x_A) := (Pa^{\hat{G}^+}(A) \cup A)_{do(I(x_A))}.$$ 

Then define recursively for $v \in A$:

$$\hat{g}_{A,v}(x_A, x_{Pa^{\hat{G}^+}(A) \setminus A}) := \begin{cases} x_v & \text{if } v \in I(x_A), \\
\hat{g}_{S,v}(x_{\hat{P}a^{\hat{G}^+}(x_A) \setminus S}) & \text{if } v \notin I(x_A), \end{cases}$$

where $S := Sc^{H(x_A)}(v)$ is also a loop in $G^+$.

4. These functions then are again globally compatible and $M$ constitutes a well-defined ioSCM.

5. All the distributions in $\hat{M}$ then are given by the general procedure of ioSCMs (see Def. 2.7). We introduce the notation for $C \subseteq V$ and $(x_C, x_J) \in \hat{I}_C \times \hat{I}_J$:

$$\hat{P}_{U}(X_V \mid I_C = x_C, X_J = x_J) := \hat{P}_{U}(X_V \mid do(I_C, I_{V \setminus C}, X_J) = (x_C, \varnothing_{\setminus C}, x_J)).$$

6. The extended DMG $\hat{G}$ of $G^+$ is then the induced DMG of $\hat{G}^+$, i.e. the extended DMG $\hat{G}$ with the additional edges $I_{\nu} \rightarrow v$ for every $v \in V$.

The following result now relates the interventional distributions of the ioSCM $\hat{M}$ with the ones from the extended ioSCM $M$. These relations will be used in the following.

**Proposition 6.2.** Let $M = (G^+, \mathcal{X}, \mathcal{P}_U, g)$ be an ioSCM with $G^+ = (V \cup U \cup J, E^+)$ and $\hat{M}$ the extended ioSCM. Let $A, B, C \subseteq V$ be pairwise disjoint set of nodes and $x_{C \cup J} \in X_{C \cup J}$. Then we have the equations:

$$\begin{align*}
\hat{P}_{U}(X_A \mid X_B, do(X_{C \cup J} = x_{C \cup J})) &= \hat{P}_{U}(X_A \mid X_B, I_C = x_C, X_J = x_J) \\
&= \hat{P}_{U}(X_A \mid X_B, I_C = x_C, X_C = x_C, X_J = x_J).
\end{align*}$$

**Proof.** This follows from $I(x_C, \varnothing_{\setminus C}) = C$. See Supplementary Material [D.1].

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7 THE THREE MAIN RULES OF CAUSAL CALCULUS

**Notation 7.1.** Since everything has been defined in detail in the last section we now want to make use of a simplified and more suggestive notation for better readability.

1. We identify variables $X_A$ with the set of nodes $A$.
2. We omit values $x_V$ and the subscript in $\mathcal{P}_U$. E.g. we write $\mathcal{P}(Y \mid I_T, T, Z, do(W))$ instead of

$$\hat{P}_{U}(X_V \mid I_T = x_T, X_T = x_T, X_Z = x_Z, do(X_W = x_W)).$$

where the latter comes from the extended ioSCM of the intervened ioSCM $M_{do(W)} := M_{do(W)} \cup J$ of $M$.
3. We abbreviate $X_Y \perp \! \! \! \perp Z, do(W)$ as $Y \perp \! \! \! \perp Z, do(W)$.
4. We write $Y \perp \! \! \! \perp Z, do(W)$ to mean $Y \perp \! \! \! \perp Z, do(W)$.

**Theorem 7.2 (The three main rules of causal calculus).** Let $M$ be an ioSCM with set of observed nodes $V$ and input nodes $J$ and induced DMG $G$. Let $X, Y, Z \subseteq V$ and $J \subseteq W \subseteq V \cup J$ be subsets.

1. **Insertion/deletion of observation:**

$$\begin{align*}
Y \perp \! \! \! \perp Z, do(W) & \quad \text{then:} \\
\mathcal{P}(Y \mid X, Z, do(W)) &= \mathcal{P}(Y \mid Z, do(W)).
\end{align*}$$

2. **Action/observation exchange:**

$$\begin{align*}
Y \perp \! \! \! \perp z, do(W) & \quad \text{then:} \\
\mathcal{P}(Y \mid do(X), Z, do(W)) &= \mathcal{P}(Y \mid X, Z, do(W)).
\end{align*}$$

3. **Insertion/deletion of actions:**

$$\begin{align*}
Y \perp \! \! \! \perp z, do(W) & \quad \text{then:} \\
\mathcal{P}(Y \mid do(X), Z, do(W)) &= \mathcal{P}(Y \mid Z, do(W)).
\end{align*}$$

The proofs follow directly from the $\sigma$-separation criterion [5,2] and Prop. 6.2. applied to the extended ioSCM and can be found in Supplementary Material [E.1].

---

8 ADJUSTMENT CRITERIA

**Notation 8.1.** Let $M = (G^+, \mathcal{X}, \mathcal{P}, g)$ be an ioSCM with $G^+ = (V \cup U \cup J, E^+)$.

The following set of nodes/variables will play the described roles:

- $Y$: the outcome variables,
- $X$: the treatment or intervention variables,
- $Z_0$: the core set of adjustment variables,
- $Z_+$: additional adjustment variables,
- $Z := Z_0 \cup Z_+$: all actual adjustment variables,
- $L$: “marginalizable” adjustment variables,
- $C$: context variables,
- $W$: default intervention variables containing $J$,
- $S$: variables inducing selection bias given $S = s$. 
We are interested in finding a “do(X)-free” expression for the (conditional) causal effect \( P(Y | C, do(X), do(W)) \) only using data for \( C, X, Y, Z \) that was gathered under selection bias \( S = s \) and intervention \( do(W) \) and additional unbiased observational data for \( C, Z \) given \( do(W) \). The task can be achieved via the following criterion, which is a generalization of the acyclic case of the selection-backdoor criterion (see \((\text{11})\)), the backdoor criterion (see \((\text{21})[\text{22}][\text{24}]\)) and its extensions (also see \((\text{4})[\text{26}][\text{27}][\text{32}]\)) to general ioSCMs.

**Theorem 8.2 (General adjustment criterion and formula).** Let the setting be like in \((\text{8})[\text{1}]\). Assume that data was collected under selection bias, \( P(V | S = s, do(W)) \) (or under \( P(V | do(W)) \) and \( S = \emptyset \)), and there are unbiased samples from \( P(Z | C, do(W)) \). Further assume that the variables satisfy:

1. \((Z_0, L) \sigma \int G \text{I} X \mid C, do(W), and \)
2. \(Y \sigma \int G (I_X, Z_+ \mid C, X, Z_0, L, do(W), and \)
3. \(Y \sigma \int G S \mid C, X, Z, do(W), and \)
4. \(L \sigma \int G X \mid C, Z, do(W). \)

Then one can estimate the conditional causal effect \( P(Y | C, do(X), do(W)) \) via the adjustment formula:

\[
P(Y | C, do(X), do(W)) = \int P(Y | X, Z, C, S = s, do(W)) \, dP(Z | C, do(W)).
\]

The proof again follows directly from the \( \sigma \)-separation criterion \((\text{5})[\text{2}]\) and Prop. \((\text{6})[\text{2}]\) applied to the extended ioSCM and can be found in the Supplementary Material \((\text{F})[\text{1}]\).

**Remark 8.3.** Note that the adjustment formula in theorem \((\text{8})[\text{2}]\) does not depend on \( L \). This thus allows us to even choose variables for \( L \) that come from an ioSCM \( M' \) that marginalizes to \( M \), e.g. \( L \subseteq U \) or by extending directed edges \( v \rightarrow w \) by \( v \rightarrow \ell \rightarrow w \) with \( \ell \in L \). This technique was used in \((\text{12})[\text{1}]\) to find all adjustment sets in the acyclic case with \( C = S = \emptyset \).

**Corollary 8.4.** Let the notations be like in \((\text{8})[\text{1}] \) and \( W = J = \emptyset \). We have the following special cases, which in the acyclic case will reduce to the ones given by the indicated references:

1. General selection-backdoor (see \((\text{11})\)): \( C = \emptyset \).
2. Selection-backdoor (see \((\text{11})\)): \( C = L = \emptyset \).
3. Extended backdoor (see \((\text{26})[\text{32}]\)): \( C = S = \emptyset \).
4. Backdoor (see \((\text{27})[\text{22}][\text{24}]\)): \( C = S = L = Z_+ = \emptyset \):
   
   (a) \( Z \sigma \int G I_X, and \)
   
   (b) \( Y \sigma \int G I_X \mid X, Z, implies:\n   
   \[
P(Y | do(X)) = \int P(Y | X, Z) \, dP(Z).
   \]

More details can be found in the Supplementary Material \((\text{F})[\text{2}]\). Also a generalization of the criterion for selection without/partial external data of \((\text{4})[\text{3}]\) is given there.

**Remark 8.5.** The conditions in theorems \((\text{2})[\text{2}] \) and corollaries \((\text{8})[\text{2}] \) are in the acyclic setting usually phrased in terms of sub-structures of the graph \( G \) (see \((\text{27})[\text{22}][\text{24}]\)):

1. For rule 3 in Thm. \((\text{2})[\text{2}] \) one usually requires \( Y \perp d X \mid Z \) in the graph \( G_{do(W)} \) that is further mutilated on the set \( X(Z) \), the set of all \( X \)-nodes that are not ancestors of any \( Z \)-node in \( G_{do(W)} \).
2. For the backdoor criterion instead of \( L \perp d G I_X \) we could have written that \( L \) does not contain any descendent of \( X \); and for \( Y \perp d G I_X \mid X, Z \) that \( Z \) blocks all “backdoor paths” from \( X \) to \( Y \).

We presented the results in the formulaic terms of \( \sigma \)-separation because the relations to their use is directly indicated (e.g. in the proofs), it makes the generalization to ioSCMs possible and when reduced to the acyclic case it will be equivalent to the usual description.

### 9 IDENTIFYING CAUSAL EFFECTS

Here we extend the ID algorithm for the identification of causal effects to ioSCMs. The main references are \((\text{12})[\text{14}][\text{15}][\text{24}][\text{29}][\text{34}][\text{37}]\). The task is to decide if a causal effect \( P(Y | do(W)) \) in an ioSCM can be identified from (i.e., expressed in terms of) the observational distributions \( P(V | do(J)) \) and the induced graph \( G \). Note that having more dependence structure (like latent founders, feedback cycles, etc.) will leave us with less identifiable causal effects in general. Due to space limitations, we can only provide here the bare necessities to state the generalized ID algorithm. We assume that the

![Diagram](image-url)
reader is already familiar with the ID algorithm formulated for ADMGs (for example, the treatment in [36]).

We generalize the notion of districts / $C$-components:

**Definition 9.1** (Consolidated districts). Let $G$ be a directed mixed graph (DMG) with set of nodes $V$. Let $v \in V$. The consolidated district $CdG(v)$ of $v$ in $G$ is given by all nodes $w \in V$ for which there exist $k \geq 1$ nodes $(v_1, \ldots, v_k)$ in $G$ such that $v_1 = v$, $v_k = w$ and for $i = 2, \ldots, k$ we have that the bidirected edge $v_{i-1} \leftrightarrow v_i$ is in $G$ or that $v_1 \in ScG(v_i)$. For $B \subseteq V$ we write $CdG(B) := \bigcup_{v \in B} CdG(v)$. Let $CD(G)$ be the set of consolidated districts of $G$.

We also generalize the notion of topological order:

**Definition 9.2** (Apt-order, see [10]). Let $G$ be a DMG with set of nodes $V$. An assemblage pseudo-topological order (apt-order) of $G$ is a total order $< \subseteq V$ with the following two properties:

1. For every $v, w \in V$ we have:
   
   \[ w \in AncG(v) \setminus ScG(v) \implies w < v. \]

2. For every $v_1, v_2, w \in V$ we have:
   
   \[ v_2 \in ScG(v_1) \setminus \{v_1 \leq \cdots \leq v_2\} \implies w \in ScG(v_1). \]

**Remark 9.3.** Let $G$ be a DMG.

1. If $G$ is acyclic then an apt-order $< \subseteq V$ is the same as a topological order (i.e. $w \in PaG(v) \implies w < v$).
2. If $G$ has a topological order then $G$ is acyclic.
3. For any DMG $G$ there always exists an apt-order $<$.\footnote{In contrast to topological orders.}

**Notation 9.4.** Let $G$ be a DMG with set of nodes $V$ and $\prec \subseteq V$ an apt-order on $G$. For elements $v \in V$ and subsets $B \subseteq V$ we put:

1. $\text{Pred}_G^< (v) := \{ w \in V \mid w < v \}$.
2. $\text{Pred}_G^{\subseteq} (v) := \{ w \in V \mid w = v \text{ or } v < w \}$.
3. $\text{Pred}_G^< (B) := \bigcup_{v \in B} \text{Pred}_G^< (v)$.
4. $\text{Pred}_G^{\subseteq} (B) := \text{Pred}_G^{\subseteq} (B) \setminus B$.

**Remark 9.5.** If $B$ is strongly-connected, then $\text{Pred}_G^{\subseteq} (B)$ is ancestral in $G$, i.e. $\text{Anc}_G (\text{Pred}_G^{\subseteq} (B)) = \text{Pred}_G^{\subseteq} (B)$.

The notion of input variables enables the following convenient and intuitive construction:

**Definition 9.6** (Sub-ioSCMs). Let $M = (G^+, \mathcal{X}, \mathbb{P}_U, g)$ be an ioSCM with $G^+ = (V \cup U \cup J, \mathcal{E}^+)$ for $C \subseteq V$ non-empty define the ioSCM $M_C$ as follows:

1. Put $G'_C$ to be the subgraph of $G$ induced by $C \cup PaG^+(C)$.
2. $V_C := C$, $J_C := PaG^+(C) \setminus (C \cup U)$, $U_C := U \cup PaG^+(C)$.

3. Keep all functions $g_S$ with $S \subseteq C$.
4. $\mathbb{P}_U(c) := \bigotimes_{u \in U_C} \mathbb{P}_u$, i.e. the marginal of $\mathbb{P}_U$ and we will use the notation $\mathbb{P}_U$ (or just $\mathbb{P}$) for both.

For $C \subseteq V \cup J$ with $C \cap J \neq \emptyset$ put $M_{C|J} := M_{C\cap J}$.

By the definition of the random variables induced by an ioSCM we immediately get the following basic result:

**Lemma 9.7.** Let $M = (G^+, \mathcal{X}, \mathbb{P}_U, g)$ be an ioSCM with $G^+ = (V \cup U \cup J, \mathcal{E}^+)$ for $C \subseteq V$, we have (indices for emphasis):

\[ \mathbb{P}_M(C|do(PaG^+(C) \setminus C)) = \mathbb{P}_M(C|do(J \cup U)), \]

for any $W \subseteq V \setminus C$ that contains $(PaG^+(C) \cap V) \setminus C$. As a special case: if $A \subseteq G$ is ancestral, i.e., $\text{Anc}_G(A) = A$,

\[ \mathbb{P}_M(A \cap V|do(A \cap J)) = \mathbb{P}_M(A \cap V|do(J \cup W)) \]

for any $W \subseteq V \setminus A \cap V$.

The ID algorithm works by repeatedly applying the previous lemma and the following rules:

**Proposition 9.8.** Let $M = (G^+, \mathcal{X}, \mathbb{P}_U, g)$ be an ioSCM with $G^+ = (V \cup U \cup J, \mathcal{E}^+)$ and $< \subseteq V$ an apt-order for $G^+$.

1. $\mathbb{P}(V|do(J)) = \bigotimes_{S \subseteq V \setminus C} \mathbb{P}(S|do(PaG^+(S) \cap V, do(J)))$.

2. For $S \subseteq V$ a strongly connected component of $G$,

\[ \mathbb{P}_M(S|do(PaG^+(S) \cap V, do(J))) = \mathbb{P}_M(S|do(PaG^+(S) \cap D, do(P))). \]

3. For $D \subseteq V$ a consolidated district of $G$:

\[ \mathbb{P}(D|do(J \cup D)) = \bigotimes_{S \subseteq D \setminus C} \mathbb{P}(S|do(PaG^+(S) \cap V, do(J))). \]

**Proof.** 1. uses the chain rule; 2. is proved in Supplementary Material\footnote{G.2}; 3. is shown by applying 1. and Remark 9.7 to $G[D]$ and then making use of 2.. \hfill \square

**Remark 9.9.** Naively putting the equations of Prop. 9.8 into each other would give us the equation:

\[ \mathbb{P}(V|do(J)) = \bigotimes_{D \subseteq V \setminus C} \mathbb{P}(D|do(J \cup V \setminus D)). \]

Note that the product might not be well-defined as the consolidated districts i.e. are not totally ordered by <
Assume that for every strongly connected component \( P \) of \( V \) with set of observed nodes \( Y \) let 
\[
\bigotimes_{S \in S(G)} \mu_S.
\]
Then the densities \( p(D | do(J \cup V \setminus D)) \) can be multiplied in any order and the integration can be separately done via the \( \mu_S \) in reverse order of \( < \).

We now have all the prerequisites to state the generalized ID algorithm (Algorithm 1) and prove its correctness:

**Theorem 9.10** (Consequence of 9.8, 9.9). Let \( M = (G^+, X, \mathbb{P}_{G^+}, g) \) be an ioSCM with \( G^+ = (V \cup U \cup J, E^+) \) with set of observed nodes \( V \) and input nodes \( J \) and distributions \( \mathbb{P}(V | do(J)) \). Let be an apt-order for \( G^+ \).

Assume that for every strongly connected component \( S \subseteq V \) we have a measure \( \mu_S \) such that \( \mathbb{P}(V | do(J)) \) has a density w.r.t. the product measure 
\[
\bigotimes_{S \in S(G)} \mu_S.
\]
Let \( Y \subseteq V \) and \( W \subseteq J \cup V \) be subsets. If the extended ID algorithm (see Algorithm 7) does not “FAIL” then the causal effect \( \mathbb{P}(Y | do(W)) \) is identifiable, i.e. it can be computed from \( \mathbb{P}(V | do(J)) \) alone, and the expression is obtained by postprocessing the output of the algorithm.

**Remark 9.11.** 1. We make no claim about the completeness of the algorithm here.
2. The algorithm reduces to the usual version in the acyclic case (see 29, 35, 37).
3. The main idea of the generalized ID algorithm is to exploit that the causal effects onto ancestral subsets and consolidated districts are identifiable. The algorithm then alternates these constructions to shrink towards the queried set \( C \) until convergence, i.e. until a set \( A \) is reached that is both the ancestral closure of \( C \) and a consolidated district in itself. If \( C = A \) then the causal effect onto \( C \) is identifiable, otherwise it outputs “FAIL” as no shrinking can be done with these techniques anymore. Also see Supplementary Material [G7].

**10 CONCLUSION**

We proved the three main rules of causal calculus and general adjustment criteria with corresponding formulas to recover from interventions and selection bias for general ioSCMs, which allow for arbitrary probability distributions, non-linear functional relations, latent confounders, external non/probabilistic parameter/assignment/intervention/context/input nodes and cycles. This generalizes all the corresponding results of acyclic causal models (see 14, 24, 25, 26, 27, 28) to general ioSCMs. We also showed how to extend the ID algorithm for the identification of causal effects from the acyclic setting to general ioSCMs. In supplementary material [A] we also show how to do counterfactual reasoning in ioSCMs. Future work might address completeness questions of the ID algorithm (see 14, 24, 25, 33, 34).

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References


SYMPLEMENTARY MATERIAL

A TWIN NETWORKS AND COUNTERFACTUALS

In addition to probablistic and causal reasoning about interventions, ioSCMs allow for counterfactual reasoning. Given an ioSCM \( M \) with graph \( G^+ = (V \cup U \cup J, E^+) \), a set \( W \subseteq V \cup J \) and the corresponding intervened ioSCM \( M_{\text{do}(W)} \) with graph \( G^+_{\text{do}(W)} \) one can construct a (merged) twin ioSCM \( M_{\text{twin}} \) similarly to the acyclic case (see [24], or a single world intervention graph (SWIG, see [30]). This is done by identifying/merging the corresponding nodes, mechanisms and variables from the non-descendants of \( W \), i.e., NonDesc\( G^+ (W) \) and NonDesc\( G^+_{\text{do}(W)} (W) \), which are unchanged by the action \( \text{do}(W) \). Then one has the two different branches Desc\( G^+ (W) \) and Desc\( G^+_{\text{do}(W)} (W) \) in the network. This construction then allows one to formulate counterfactual statements like in the acyclic case (see [24]), but now for general ioSCMs. E.g., one could state the assumption of strong ignorability (see [24,31]) as:

\[
\left( Y_{\text{do}(\emptyset)}, Y_{\text{do}(X)} \right) \overset{\sigma}{\overset{\text{d}}{\to}} G_{\text{twin}}(X | Z),
\]

or the conditional ignorability (see [31,32]) as:

\[
Y_{\text{do}(X)} \overset{\sigma}{\overset{\text{d}}{\to}} G_{\text{twin}}(X | Z).
\]

All the causal reasoning rules derived in this paper can thus also be applied to reason about counterfactuals.

B MARGINALIZATION OF DIRECTED MIXED GRAPHS

For completeness, we provide here the definition of marginalization of directed mixed graph. For more details and the relationship with the marginalization of an mSCM (or as a straightforward generalization, an ioSCM), we refer the reader to [10].

Definition B.1 (Marginalization of DMGs). Let \( G = (V, E, B) \) be a directed mixed graph (DMG) with set of nodes \( V \), directed edges \( E \) and bidirected edges \( B \). Let \( W \subseteq V \) be a subset of nodes. We define the marginalized DMG \( G^W = (V^\prime, E^\prime, B^\prime) \) (“marginalizing out \( W \)”), also called latent projection of \( G \) onto \( V \setminus W \), with set of nodes \( V^\prime := V \setminus W \) with the following rules (for \( v_1, v_2 \in V \setminus W = V^\prime \)):

1. \( v_1 \leftrightarrow v_2 \in E^\prime \) iff there exist \( k \geq 0 \) nodes \( w_1, \ldots, w_k \in W \) such that the directed walk:

\[
v_1 \leftrightarrow w_1 \cdots w_k \leftrightarrow v_2
\]

lies in \( G \) (the corner case \( v_1 \leftrightarrow v_2 \in E \) also applies).

2. \( v_1 \leftrightarrow v_2 \in B^\prime \) iff there exist \( k \geq 0 \) nodes \( w_1, \ldots, w_k \in W \) and an index \( 0 \leq m \leq k \) such that a walk of the form:

\[
v_1 \leftrightarrow w_1 \cdots w_m \leftrightarrow w_{m+1} \cdots w_k \leftrightarrow v_2
\]

lies in \( G \) with \( m \geq 1 \) or a walk of the form:

\[
v_1 \leftrightarrow w_1 \cdots w_m \leftrightarrow w_{m+1} \cdots w_k \leftrightarrow v_2
\]

lies in \( G \) (including the corner cases \( v_1 \leftrightarrow v_2 \in B \) and \( v_1 \leftrightarrow w \leftrightarrow v_2 in G with w \in W \).

C CONDITIONAL INDEPENDENCE AND ITS ALTERNATIVE WITH CONFOUNDED INPUTS

Here we want to give a generalization of [3][29] in the flavor of definition 3.1 The main point is that the approaches of conditional independence for families of distributions/Markov kernels in [3][29] implicitly assume that the input variables \( J \) are jointly confounded. The definition 3.1 of conditional independence, in contrast, assumes (via the product distributions) that the variables \( J \) are jointly independent. The approach in definition 3.1 can be easily adapted to the confounded input setting as follows.

C.1 INPUT CONFOUNDED CONDITIONAL INDEPENDENCE

Definition C.1 (Input confounded conditional independence). Let \( \mathcal{X}_V := \prod_{v \in V} \mathcal{X}_v \) and \( \mathcal{X}_J := \prod_{j \in J} \mathcal{X}_j \) be the product spaces of any measurable spaces and

\[
\mathbb{P}_V(\mathcal{X}_V | \mathcal{X}_J)
\]

a Markov kernel (i.e. a family of distributions on \( \mathcal{X}_V \) measurable\(^3\) parametrized by \( \mathcal{X}_J \)). For subsets \( A, B, C \subseteq V \cup J \) we write:

\[
X_A \overset{\mathbb{P}_V(\mathcal{X}_V | \mathcal{X}_J)}{\perp \! \! \! \perp} X_B | X_C
\]

if and only if for every joint distribution \( \mathbb{P}_J \) on \( \mathcal{X}_J \) we have:

\[
X_A \overset{\mathbb{P}_{V \cup J}}{\perp \! \! \! \perp} X_B | X_C,
\]

which means that for all measurable \( F \subseteq \mathcal{X}_A \) we have:

\[
\mathbb{P}_{V \cup J}(X_A \in F | X_B, X_C) = \mathbb{P}_{V \cup J}(X_A \in F | X_C) \quad \text{\( \mathbb{P}_{V \cup J} \)-a.s.,}
\]

\(^3\)We require that for every measurable \( F \subseteq \mathcal{X}_V \) the map \( \mathcal{X}_J \to [0,1] \) given by \( x_j \mapsto \mathbb{P}_V(X_V \in F | X_J = x_j) \) is measurable.
where $P_{V,U,J}(X_{V,U,J}) := P_V(X_V | X_J) \otimes P_J(X_J)$, the distribution given by $X_J \sim P_J$ and then $X_V \sim P_V(\lambda | X_J)$.

**Lemma C.2.** Let the situation be like in C.1 and assume all spaces $X_v, v \in V$, to be standard measurable spaces. Let $A, B, C$ be pairwise disjoint, $A \cap J = \emptyset$ and $J \subseteq B \cup C$. Then every statement implies the one below:

1. There is a version of $P_V(X_A | X_B, X_C)$ such that for all $x_B, x'_B \in X_B, x_C \in X_C$:
   $$P_V(X_A | X_B = x_B, X_C = x_C)$$
   $$= P_V(X_A | X_B = x'_B, X_C = x_C).$$

2. $X_A \updownarrow_{P_V(X_V | X_J)} X_B | X_C.$

3. $X_A \updownarrow_{P_V(X_V | X_J)} X_B | X_C$ (using definition [3]).

4. $X_A \updownarrow_{P_V(X_V | X_J)} X_B | X_C$ for every $x_J \in X_J$.

If there is a Markov kernel $P(V | X_C)$ that is a version of $P_{V,U,J}(X_A | X_B, X_C)$ for every Dirac delta distribution $P_J = \delta_{x_J}$ (e.g. if $J \subseteq C$) then the last point also implies the first.

**Proof.** 1. $\implies$ 2.: Functional dependence only on $x_C$.
2. $\implies$ 3. $\implies$ 4.: Every product distribution is a joint distribution and every Dirac delta distribution is a product distribution.
1. $\iff$ 4.: Let $N \subseteq X_{B \cup C}$ be the measurable set on which the Markov kernels $P_V(X_A | X_B, X_C)$ and $P(X_A | X_C)$ (considered as functions of $(x_B, x_C)$) differ. For every $x_J \in X_J$ we have by assumption:
   $$X_A \updownarrow_{P_V(X_V | X_J)} X_B | X_C.$$
   This shows that:
   $$P_V(X_A | X_B = x_B, X_C = x_C) = P(X_A | X_B = x_B)$$
   for $(x_B, x_C)$ outside of a $P_V(X_{B \cup C}) \setminus J | X_J = x_J)$-zero set, for which we can take the section $N_{x_J}$ of $N$. This implies that $N$ is a $P_V(X_{B \cup C}) \setminus J | X_J)$-zero set. So $P(X_A | X_C)$ is a version of $P_V(X_A | X_B, X_C)$ and satisfies 1.

**Remark C.3.** 1. The existence of the Markov kernel $P(X_A | X_C)$ under the assumption 4. in lemma C.2 always/only holds up to measurability questions, because for every fixed $P_J$ the regular conditional probability distribution $P_{V,U,J}(X_A | X_B, X_C)$ always exists in standard measurable spaces and agrees with $P_{V,U,J}(X_A | X_C)$ (by the assumption 4.).

**Definition C.4** (Input confused global Markov property) Let $M = (G^+, X, P_U, g)$ be an isoSCM with graph $G^+ = (V \cup U \cup J, E^+)$. The corresponding input confused isoSCM $M_\bullet$ is then constructed from $M$ by the following changes:

2. $J_\bullet := \{ \bullet \}$ with a new node $\bullet$ with space $X_\bullet := X_J.$
3. \( E^+_\bullet := E^+ \cup \{ \bullet \rightarrow j | j \in J \} \).
4. add \( g(J) \), the canonical projection from \( X_\bullet \) onto \( X_j \), to \( g \) for \( j \in J \).

With this setting \( M_\bullet \) is a well-defined ioSCM.
Furthermore, let \( G_\bullet \) be the input confounded induced DMG, i.e. the induced DMG of \( G^+_\bullet \) where \( \bullet \) is marginalized out. In other words, \( G_\bullet \) arises from the induced DMG \( G \) of \( G^+_\bullet \) by just adding \( j_1 \leftrightarrow j_2 \) for all \( j_1, j_2 \in J \), \( j_1 \neq j_2 \), to \( G \).

**Theorem C.5** (Input confounded directed global Markov property). Let \( M \) be an ioSCM with input confounded induced DMG \( G_\bullet \). Then for all subsets \( A, B, C \subseteq V \cup J \) we have the implication:

\[
A \perp_{G_\bullet} B | C \quad \Rightarrow \quad X_A \perp_{P_U(V \mid \text{do}(X_J))} X_B | X_C.
\]

In words, if \( A \) and \( B \) are \( \sigma \)-separated by \( C \) in \( G_\bullet \) then the corresponding variables \( X_A \) and \( X_B \) are conditionally independent given \( X_C \) for any distribution \( P_U(X \mid \text{do}(X_J)) \otimes P_J(X_J) \) for any joint distribution \( P_J \) on \( X_J \).

**Proof.** This directly follows from the \( \sigma \)-separation criterion/global Markov property \#5.2 applied to the input confounded ioSCM \( M_\bullet \) and \( G^+_\bullet \), or, alternatively, from the mSCM-version proven in \#10.11 for each fixed joint distribution \( P_J \) on \( X_J \). Note that \( G_\bullet \) is a marginalization of \( G^+_\bullet \) and \( \sigma \)-separation is stable under marginalization. \( \square \)

**D THE EXTENDED IOSCM - PROOFS**

**Proposition D.1.** Let \( M = (G^+, X, P_U, g) \) be an ioSCM with \( G^+ = (V \cup U \cup J, E^+) \) and \( M \) the extended ioSCM. Let \( A, B, C \subseteq V \) be pairwise disjoint set of nodes and \( x_{C \cup J} \in X_{C \cup J} \). Then we have the equations:

\[
P_U(X_A | X_B, \text{do}(X_{C \cup J} = x_{C \cup J})) = P_U(X_A | X_B, I_C = x_C, X_J = x_J) = P_U(X_A | X_B, I_C = x_C, X_C = x_C, X_J = x_J).
\]

**Proof.** Consider the first equality. For any subset \( D \subseteq V \) the variable \( X^D_{\text{do}(X_{C \cup J} = x_{C \cup J})} \) was recursively defined in \( M_{\text{do}(C)} \) via \( g \) using \( G^+_{\text{do}(C)} \), whereas the variable \( X^D_{\text{do}(I_C \cup V \setminus C, X_J = (x_C, \emptyset \cup V \setminus C, x_J))} \) was recursively defined in \( \hat{M} \) via the same \( g \) but using \( I_C \cup V \setminus C \) and \( G^+_{\text{do}(I_C \cup V \setminus C)} \). Since \( x_C \in X_C \) we have that \( I_C \cup V \setminus C \) and thus \( G^+_{\text{do}(I_C \cup V \setminus C)} \). It directly follows that:

\[
X^D_{\text{do}(X_{C \cup J} = x_{C \cup J})} = X^D_{\text{do}(I_C \cup V \setminus C, X_J = (x_C, \emptyset \cup V \setminus C, x_J))}.
\]

This shows the equality of top and middle line. For the equality between the middle and bottom line note that:

\[
I_C = x_C \quad \Rightarrow \quad X_C = x_C.
\]

**E THE THREE MAIN RULES OF CAUSAL CALCULUS - PROOFS**

**Theorem E.1** (The three main rules of causal calculus). Let \( M \) be an ioSCM with set of observed nodes \( V \) and intervention nodes \( J \) and induced DMG \( G \). Let \( X, Y, Z \subseteq V \) and \( J \subseteq W \subseteq V \cup J \) be subsets.

1. **Insertion/deletion of observation:**
   
   If \( Y \perp_{G} X | Z, \text{do}(W) \) then:
   
   \[ P(Y | X, Z, \text{do}(W)) = P(Y | Z, \text{do}(W)). \]

2. **Action/observation exchange:**
   
   If \( Y \perp_{G} I_X | X, Z, \text{do}(W) \) then:
   
   \[ P(Y | \text{do}(X), Z, \text{do}(W)) = P(Y | X, Z, \text{do}(W)). \]

3. **Insertion/deletion of actions:**
   
   If \( Y \perp_{G} I_X | Z, \text{do}(W) \) then:
   
   \[ P(Y | \text{do}(X), Z, \text{do}(W)) = P(Y | Z, \text{do}(W)). \]

**Proof.** 1. Thm. \#5.2 applied to \( G_{\text{do}(W)} \) gives:

\[
Y \perp_{G} X | Z, \text{do}(W) \quad \Rightarrow \quad Y \perp_{P} X | Z, \text{do}(W).
\]

The latter directly gives the claim:

\[ P(Y | X, Z, \text{do}(W)) = P(Y | Z, \text{do}(W)). \]

2. The \( \sigma \)-separation criterion \#5.2 w.r.t. to \( \hat{G}_{\text{do}(W)} \) gives:

\[
Y \perp_{G} I_X | X, Z, \text{do}(W) \quad \Rightarrow \quad Y \perp_{P} I_X | X, Z, \text{do}(W).
\]

Together with Prp. \#6.2 (applied to \( M_{\text{do}(W)} \)) we have:

\[
Y \perp_{G} I_X | X, Z, \text{do}(W) \quad \Rightarrow \quad P(Y | X, Z, \text{do}(W)) = P(Y | X, Z, \text{do}(W)).
\]

3. As before we have:

\[
Y \perp_{G} I_X | Z, \text{do}(W) \quad \Rightarrow \quad Y \perp_{P} I_X | Z, \text{do}(W).
\]

And again:

\[
P(Y | I_X, Z, \text{do}(W)) = P(Y | Z, \text{do}(W)).
\]

\( \square \)
F ADJUSTMENT CRITERIA

F.1 PROOFS

Theorem F.1 (General adjustment criterion and formula). Let the setting be like in 8.1. Assume that data was collected under selection bias, \( P(V|S=s, \text{do}(W)) \) (or under \( P(V|\text{do}(W)) \)) and \( S = \emptyset \), and there are unbiased samples from \( P(Z|C, \text{do}(W)) \). Further assume that the variables satisfy:

1. \((Z_0, L) \overset{\sigma}{\perp} I_X | C, \text{do}(W), \) and
2. \( Y \overset{\sigma}{\perp} (I_X, Z_+ | C, X, Z_0, L, \text{do}(W), \) and
3. \( Y \overset{\sigma}{\perp} S | C, X, Z, \text{do}(W), \) and
4. \( L \overset{\sigma}{\perp} X | C, Z, \text{do}(W). \)

Then one can estimate the conditional causal effect \( P(Y|C, \text{do}(X), \text{do}(W)) \) via the adjustment formula:

\[
P(Y|C, \text{do}(X), \text{do}(W)) = \int P(Y|X, Z, C, S = s, \text{do}(W)) \, dP(Z|C, \text{do}(W)).
\]

Proof. Since \( C, \text{do}(W) \) occur everywhere as a conditioning set, we will suppress \( C, \text{do}(W) \) in the following everywhere. Then note that the \( \sigma \)-separation criterion implies the corresponding conditional independencies in the following when indicated. The adjustment formula then derives from the following computations:

\[
P(Y|\text{do}(X)) = \int P(Y|Z_0, L, \text{do}(X)) \, dP(Z_0, L|\text{do}(X))
\]

\[
\overset{[0.2]}{=} \int P(Y|I_X, X, Z_0, L) \, dP(Z_0, L|I_X)
\]

\[
Y \perp I_X|X, Z_0, L; (Z_0, L) \perp I_X, \quad \int \int P(Y|X, Z_0, L) \, dP(Z_0, L)
\]

\[
\overset{f \, dP(Z_+|Z_0, L) = 1}{=} \int \int P(Y|X, Z_0, L) \, dP(Z_+, Z_0, L)
\]

\[
Y \perp Z_+|X, Z_0, L, \quad \int \int P(Y|X, Z_+, Z_0, L) \, dP(Z_+, Z_0, L)
\]

\[
z = Z_+ \cup Z_0 \quad \int \int P(Y|X, Z, L) \, dP(Z, L)
\]

\[
L \perp X|Z, \quad \int \int P(Y|L, X, Z) \, dP(L|X, Z) \, dP(Z)
\]

\[
= \int \int P(Y|X, Z, L) \, dP(L|Z) \, dP(Z)
\]

\[
Y \perp S|X, Z \quad \int \int P(Y|X, Z, S) \, dP(Z).
\]

\[
\square
\]

F.2 SPECIAL CASES

Corollary F.2. Let the notations be like in 8.1 and 8.2 and \( W = J = \emptyset \). We have the following special cases, which in the acyclic case will reduce to the ones given by the indicated references:

1. General selection-backdoor (see [4]): \( C = \emptyset, \) and
   - (a) \( (Z_0, L) \overset{\sigma}{\perp} I_X, \) and
   - (b) \( Y \overset{\sigma}{\perp} (I_X, Z_+, S) | X, Z_0, L, \) and
   - (c) \( Y \overset{\sigma}{\perp} S | X, Z, \) and
   - (d) \( L \overset{\sigma}{\perp} X | Z, \) implies:

   \[
P(Y|\text{do}(X)) = \int P(Y|X, Z, S = s) \, dP(Z).
   \]

2. Selection-backdoor (see [1]): \( C = L = \emptyset, \) and
   - (a) \( Z_0 \overset{\sigma}{\perp} I_X, \) and
   - (b) \( Y \overset{\sigma}{\perp} (I_X, Z_+, S) | X, Z_0, L, \) and
   - (c) \( Y \overset{\sigma}{\perp} S | X, Z, \) and
   - (d) \( L \overset{\sigma}{\perp} X | Z, \) implies:

   \[
P(Y|\text{do}(X)) = \int P(Y|X, Z, S = s) \, dP(Z).
   \]

3. Extended backdoor (see [20,32]): \( C = S = \emptyset, \) and
   - (a) \( (Z_0, L) \overset{\sigma}{\perp} I_X, \) and
   - (b) \( Y \overset{\sigma}{\perp} (I_X, Z_+, S) | X, Z_0, L, \) and
   - (c) \( L \overset{\sigma}{\perp} X | Z, \) implies:

   \[
P(Y|\text{do}(X)) = \int P(Y|X, Z) \, dP(Z).
   \]

4. Backdoor (see [21,22,24]): \( C = S = L = Z_+ = \emptyset, \) and
   - (a) \( Z \overset{\sigma}{\perp} I_X, \) and
   - (b) \( Y \overset{\sigma}{\perp} I_X | X, Z, \) implies:

   \[
P(Y|\text{do}(X)) = \int P(Y|X, Z) \, dP(Z).
   \]

F.3 MORE ON ADJUSTMENT CRITERIA

The following generalizes the adjustment criterion of type I in [4].

*In the acyclic case it was shown in [32] that when \( L \) is allowed to represent latent variables in a graph \( G' \) that marginalizes to \( G \) then this criterion actually characterizes all adjustment sets for \( G \) and \( P(Y|\text{do}(X)). \)
**Theorem F.3** (General adjustment without external data). Let the setting be like in [5]. Assume that data was collected under selection bias, \( \mathbb{P}(V|S = s) \). Further assume that the variables satisfy:

1. \( Y \perp S \mid \text{do}(X) \),
2. \( Z_0 \perp I_X \mid S \),
3. \( Y \perp Z_+ \mid Z_0, S, \text{do}(X) \),
4. \( Y \perp I_X \mid X, Z, S \).

Then one can estimate the causal effect \( \mathbb{P}(Y|\text{do}(X)) \) via the following adjustment formula from the biased data:

\[
\mathbb{P}(Y|\text{do}(X)) = \int \mathbb{P}(Y|X, Z, S = s) \, d\mathbb{P}(Z|S = s).
\]

**Proof.** First note that the \( \sigma \)-separation criterion Theorem [5.2] implies the corresponding conditional independencies in the following when indicated. We implicitly make use of Proposition [6.2] when needed. The adjustment formula then derives from the following computations:

\[
\begin{align*}
Y \perp S \mid \text{do}(X) & \quad \Rightarrow \quad \mathbb{P}(Y|\text{do}(X)) = \mathbb{P}(Y|S, \text{do}(X)) \quad \text{chain rule} \\
& \quad \Rightarrow \quad \int \mathbb{P}(Y|Z_0, S, \text{do}(X)) \, d\mathbb{P}(Z_0|S, \text{do}(X)) \\
Z_0 \perp I_X \mid S & \quad \Rightarrow \quad \int \mathbb{P}(Y|Z_0, S, \text{do}(X)) \, d\mathbb{P}(Z_0|S) \quad \text{chain rule} \\
& \quad \Rightarrow \quad \int \mathbb{P}(Y|Z_0, S, \text{do}(X)) \, d\mathbb{P}(Z_0|S) \quad \text{by \ref{6.2}} \\
Y \perp Z_+ \mid Z_0, S, \text{do}(X) & \quad \Rightarrow \quad \int \mathbb{P}(Y|Z_+, Z_0, S, \text{do}(X)) \, d\mathbb{P}(Z_+, Z_0) \\
Z = Z_+ \cup Z_0 & \quad \Rightarrow \quad \int \mathbb{P}(Y|Z, S, \text{do}(X)) \, d\mathbb{P}(Z|S) \\
Y \perp I_X \mid X, Z, S & \quad \Rightarrow \quad \int \mathbb{P}(Y|Z, S, X) \, d\mathbb{P}(Z|S) \quad \text{by \ref{6.2}}.
\end{align*}
\]

The following theorem generalizes the adjustment criterion of type III in [5]. For this we have to introduce even more adjustment sets: \( Z_0, Z_0^B, Z_1, Z_1^B, Z_2, Z_3 \) and \( L_0, L_1 \). We write \( Z_0 = (Z_0^A, Z_0^B), Z_1 = (Z_1^A, Z_1^B), \) etc.

**Theorem F.4** (General adjustment with partial external data). Assume that data was collected under selection bias, \( \mathbb{P}(V|S = s) \), but we have unbiased data from \( \mathbb{P}(Z_1^B|S) \). Further assume that the variables satisfy:

1. \( (L_0, Z_0) \perp I_X \),
2. \( Y \perp Z_1 \mid L_0, Z_0, \text{do}(X) \),
3. \( Z_1^A \perp S | Z_1^B \),
4. \( L_0 \perp I_X \mid Z_{\leq 1} \),
5. \( Y \perp S | Z_{\leq 1}, \text{do}(X) \),
6. \( (L_1, Z_2) \perp I_X \mid S, Z_{\leq 1} \),
7. \( Y \perp Z_3 \mid L_1, S, Z_{\leq 2}, \text{do}(X) \),
8. \( L_1 \perp I_X \mid S, Z \),
9. \( Y \perp I_X \mid X, S, Z \).

Then we have the adjustment formula:

\[
\begin{align*}
\mathbb{P}(Y|\text{do}(X)) &= \int \int \mathbb{P}(Y|S = s, Z, X) \, d\mathbb{P}(Z_1^B|S = s, Z_{\leq 1}) \, d\mathbb{P}(Z_{\leq 1}).
\end{align*}
\]

Note that this formula does not depend on \( L_0 \) and \( L_1 \). So \( L_0 \) and \( L_1 \) can be chosen in a graph \( G' \) that marginalizes to \( G \).

**Proof.**
Remark G.1 (More remarks about the ID-algorithm).

1. The extended version of the ID algorithm is equivalent to applying the ID algorithm to the acyclification $G^{+,acy}$ of $G^+$, which here is meant to be the conditional ADMG that arises by adding edges $v \rightarrow w'$ if $v \notin Sc^G(w) \Rightarrow w'$ and $v \rightarrow w \in G^+$, and erasing all edges inside $\text{Sc}^G(w)$, $w \in V$ (see \cite{10}).

2. A consolidated district in $G$ then is the same as a district in $G^{acy}$.

3. Every apt-order of $G$ is a topological order of $G^{acy}$.

4. So identifiability in $G^{acy}$ implies identifiability in $G$.

5. This leads to the rule of thumb that causal effects where both cause and effect nodes are inside one strongly connected component of $G$ are not identifiable from observational data alone, and, that the causal effects of sets of nodes between strongly connected components follow rules similar to the acyclic case.

6. Similarly, the corner cases for the identification of conditional causal effects $\mathbb{P}(Y|R,do(W))$ in $G$ that are not covered by the identification of $\mathbb{P}(Y,R,do(W))$ in $G$ follow from the (acyclic) conditional ID-algorithm from \cite{36} applied to $G^{acy}$ and then translated back to $G$ by the above correspondences.

Lemma G.2. Let $M = (G^+,X,\mathbb{P},U,g)$ be an isoSCM with $G^+ = (V \cup J, E^+)$ and $< \subset$ an apt-order for $G^+$ and $G$ its induced DMG (with nodes $V \cup J$). Let $S \subseteq V$ be a strongly connected component of $G$ and $D \subseteq V$ be any union of consolidated districts in $G$ with $S \subseteq D$ (e.g., $D = \text{Cd}^G(S)$) and $P := \text{Pa}^G(D) \setminus D$. Then we have the equality (indices for emphasis):

$$\mathbb{P}_M(S|\text{Pred}_{G}^<(S) \cap V, do(J)) = \mathbb{P}_{M[D]}(S|\text{Pred}_{G[D]}^<(S) \cap D, do(P)).$$

Proof. First note that since $D$ is a union of strongly connected components and all other variables in $G[D]$ have no parents the total order $<$ is also an apt-order for $G[D]$. It follows that we have the equality of sets of nodes:

$$\text{Pred}_{G[D]}^<(S) \cap D = \text{Pred}_{G}^<(S) \cap D =: D_\prec.$$

Now we introduce the following further abbreviations:

$$D_\succ := D \setminus (S \cup D_\prec),$$

$$P_\prec := \text{Pred}_{G}^<(S) \cap (P \cap V),$$

$$P_\succ := (P \cap V) \setminus \text{Pred}_{G}^<(S),$$

$$P_J := P \cap J,$$

$$J_\prec := \text{Pred}_{G}^<(S) \cap J,$$

$$J_\succ := J \setminus \text{Pred}_{G}^<(S),$$

$$R_\prec := \text{Pred}_{G}^<(S) \cap V \setminus (D \cup P),$$

$$R_\succ := V \setminus (D \cup P \cup \text{Pred}_{G}^<(S)).$$

G IDENTIFYING CAusal EFFECTS

1. The extended version of the ID algorithm is equivalent to applying the ID algorithm to the acyclification $G^{+,acy}$ of $G^+$, which here is meant to be the conditional ADMG that arises by adding edges $v \rightarrow w'$ if $v \notin Sc^G(w) \Rightarrow w'$ and $v \rightarrow w \in G^+$, and erasing all edges inside $\text{Sc}^G(w)$, $w \in V$ (see \cite{10}).
Then we get the relations between the sets of nodes:
\[
V = R_\prec \cup D \cup R_\succ \cup P_\prec \cup P_\succ,
\]
\[
D = D_\prec \cup S \cup D_\succ,
\]
\[
P = P_\prec \cup P_\succ \cup P_j,
\]
\[
\text{Pred}_\prec^G(S) \cap V = D_\prec \cup R_\prec \cup P_\prec,
\]
\[
J = J_\prec \cup J_\succ.
\]
Since \(\text{Pred}_\prec^G(S)\) is ancestral in \(G\) and \(\text{Pred}_\prec^{G(D)}(S)\) is ancestral in \(G(D)\), resp., we can by remark 9.7 arbitrarily intervene on all variables outside of these sets without changing the distributions \(P_M(S|\text{Pred}_\prec^G(S) \cap V, \text{do}(J))\) and \(P_{M(D)}(S|\text{Pred}_\prec^{G(D)}(S) \cap D, \text{do}(P))\), resp. With these remarks and our new notations we have the equalities:
\[
P_M(S|\text{Pred}_\prec^G(S) \cap V, \text{do}(J)) = P_M(S|D_\prec, R_\prec, P_\prec, \text{do}(J)) \tag{J.7}
\]
\[
P_M(S|D_\prec, R_\prec, P_\prec, \text{do}(J, R_\succ, P_\succ, D_\succ));
\]
and:
\[
P_{M(D)}(S|\text{Pred}_\prec^{G(D)}(S) \cap D, \text{do}(P)) = P_{M(D)}(S|D_\prec, \text{do}(P_\prec, P_\succ, P_j)) \tag{J.7}
\]
\[
P_{M(D)}(S|D_\prec, \text{do}(P_\prec, P_\succ, P_j, D_\succ)) \tag{J.7}
\]
\[
P_M(S|D_\prec, \text{do}(P_\prec, P_\succ, J, D_\succ, R_\prec, R_\succ)).
\]
So the equality between those expressions and thus the claim follows by the 2nd rule of causal calculus in Theorem 7.2 with the \(\sigma\)-separation statement:
\[
S \overset{\sigma}{\not\rightarrow}_G I_{R_\prec, P_\prec} \mid D_\prec, R_\prec, P_\prec, \text{do}(J, R_\succ, P_\succ, D_\succ).
\]
To prove the latter note that the intervention \(\text{do}(R_\succ, P_\succ, D_\succ)\) allows us to restrict to the ancestral subgraph \(\text{Pred}_\prec^G(S) \cup J\). Now let \(\pi\) be a path from an indicator variable from \(I_{R_\prec, P_\prec}\) to \(S\) (in \(\text{Pred}_\prec^G(S) \cup J\)). Then the path can only be of the form:
\[
v_1 \cdots v_p \Rightarrow v_d \cdots v_s,
\]
with \(v_i \in I_{R_\prec, P_\prec}, v_p \in P_\prec, v_d \in D, v_s \in S\), as there cannot be any bidirected edge or directed edge in the other direction between \(R_\prec \cup P_\prec\) and \(D\) by the definition of consolidated districts and \(P = \text{Pa}_\prec^G(D) \setminus D\). Since we condition on \(P_\prec\) the path \(\pi\) is \(\sigma\)-blocked.

**Remark G.3.** Another way to deal with the problem that consolidated districts are not topologically ordered in the extended ID-algorithm (see Algorithm 7 and theorem 9.10) as discussed in remark 9.7 is to work with unions of consolidated districts directly instead of working with each single consolidated district at a time (and then having problems multiplying them in a ordered way). The corresponding ID-algorithm then iterates taking the ancestral closure and taking (the unions of) consolidated districts of the queried set until convergence. If the sets agree the causal effect is identifiable and the occurring products can be computed like in proposition 9.8 point 3, with \(D\) now a union of consolidated districts. The soundness then follows again with proposition 9.8 and lemma 6.2 which also work in this case, but the algorithm might more often respond with “FAIL”.