σ-Maximal Ancestral Graphs

Binghua Yao[1] **Joris M. Mooij**[1]

[1]Korteweg-de Vries Institute, University of Amsterdam, Amsterdam, The Netherlands

Abstract

Maximal Ancestral Graphs (MAGs) provide an abstract representation of Directed Acyclic Graphs (DAGs) with latent (selection) variables. These graphical objects encode information about ancestral relations and d-separations of the DAGs they represent. This abstract representation has been used amongst others to prove the soundness and completeness of the FCI algorithm for causal discovery, and to derive a do-calculus for its output. One significant inherent limitation of MAGs is that they rule out the possibility of cyclic causal relationships. In this work, we address that limitation. We introduce and study a class of graphical objects that we coin "σ-Maximal Ancestral Graphs" ("σ-MAGs"). We show how these graphs provide an abstract representation of (possibly cyclic) Directed Graphs (DGs) with latent (selection) variables, analogously to how MAGs represent DAGs. We study the properties of these objects and provide a characterization of their Markov equivalence classes.

1 INTRODUCTION

Maximal Ancestral Graphs (MAGs) were introduced by Richardson and Spirtes [2002] to provide an abstract representation of Directed Acyclic Graphs (DAGs) with latent variables (where the latent variables can either be marginalized out, or conditioned out in case of selection bias). MAGs encode information about ancestral relations and d-separations of the DAGs they represent. Key results in the theory of MAGs are various characterizations of their Markov equivalence classes [Spirtes and Richardson, 1996, Ali et al., 2005, Zhao et al., 2005, Ali et al., 2009]. MAGs provide an appropriate level of abstraction for various complex causal reasoning tasks. For example, MAGs have been used to prove the soundness and completeness of the FCI algorithm for causal discovery [Spirtes et al., 1995, Ali et al., 2005, Zhang, 2008b], and to derive a causal do-calculus for its output [Zhang, 2008a].

One inherent limitation of MAGs is that they rule out the possibility of cyclic causal relationships. Both feedback loops and the merging of variables may lead to directed cycles in the causal graph, and this requires the consideration of directed graphs (DGs) rather than directed acyclic graphs (DAGs). A first attempt to generalize MAGs to cyclic models are the Maximal Almost Ancestral Graphs (MAAGs) introduced by Strobl [2015]. While MAAGs do represent ancestral relations of DGs with latent (selection) variables, they fail to generalize the other key feature of MAGs: MAAGs do *not* capture the d-separation properties of the DGs that they represent. Another approach was taken by Claassen and Mooij [2023]. Already in 1997, Richardson characterized Markov equivalence for DGs (without latent (selection) variables), providing necessary and sufficient conditions for when two DGs entail the same d-separations. Building on this, Claassen and Mooij [2023] proposed Cyclic Maximal Ancestral Graphs (CMAGs) as MAG-like representations of DGs without latent (selection) variables. While CMAGs contain enough information to characterize the Markov equivalence of the DGs they represent, it is not clear at present whether the d-separation properties can be read off easily from the CMAG. Furthermore, it is not clear how CMAGs can be generalized to allow for latent (selection) variables. In conclusion, these proposals provide only partial solutions at best.

Another limitation of existing approaches is that they focus on representing d-separations. While the d-separation Markov property holds for DAG models (e.g., acylic structural causal models and causal Bayesian networks [Lauritzen et al., 1990]) and specific DG models (e.g., linear structural equation models [Spirtes, 1994, Koster, 1996]), it fails to hold for DG models in general [Spirtes, 1995, Neal, 2000].

Recently, a more general theory of cyclic structural causal models (SCMs) has been developed [Bongers et al., 2021]. The subclass of *simple* SCMs was identified as having particularly convenient mathematical properties, while providing a substantive extension of the class of acyclic SCMs. In particular, the d-separation Markov property for acyclic SCMs extends to a more generally applicable σ-separation Markov property for simple SCMs [Forré and Mooij, 2017, Bongers et al., 2021]. In general, a σ-separation in the DG of a simple SCM implies the corresponding d-separation, but not vice versa. The d-separation Markov property, which may imply additional conditional independence relations, only holds for certain subclasses of simple SCMs, in particular for acyclic SCMs, linear SCMs, and SCMs with discrete variables [Forré and Mooij, 2017, Bongers et al., 2021].

In this context, a natural question is whether the notion of MAGs can be extended to represent the σ-separation properties of (possibly cyclic) DGs with latent (selection) variables. In this work, we answer this question affirmatively and in full generality.

Our contributions are as follows. We introduce and study a class of graphical objects that we coin "σ-Maximal Ancestral Graphs" ("σ-MAGs"). We show how these graphs provide an abstract representation of (possibly cyclic) directed graphs with latent (selection) variables, similarly to how MAGs represent DAGs with latent (selection) variables. By drawing inspiration from [Spirtes and Richardson, 1996], we build up the theory of σ-MAGs culminating in a characterization of their Markov equivalence classes.

In this paper, we define how a σ-MAG represents a Directed Mixed Graph (DMG), rather than a DG with latent variables. The reason is that marginalizing over latent variables in DGs is already well-established: DGs can be marginalized into DMGs by a graphical procedure known as "marginalization" or "latent projection" [Bongers et al., 2021]. To avoid redundancy and to streamline our exposition, we focus directly on the abstraction from DMGs to σ-MAGs, which highlights our main contribution: capturing the effects of conditioning on latent selection variables in cyclic settings.

2 PRELIMINARIES

In this work, we propose a new class of mixed graphs, referred to as *σ-Maximal Ancestral Graphs* (σ-MAGs), which provide a representation for sets of Directed Mixed Graphs (DMGs). We begin by establishing some terminologies.

σ-MAGs have multiple edge types. In addition to the three edge types for DMGs ($\rightarrow, \leftarrow, \leftrightarrow$), there is an undirected edge ($-$). Each edge $a \ast\!\!-\!\!\ast\, b$ has two *edge marks*, one at each node, with each edge mark being either a *tail* or an *arrowhead*. We use the "$*$" symbol to denote any of these two edge marks. Thus, the notation $a \ast\!\!-\!\!\ast\, b$ represents all four possible edge types between a and b. Edges of the form $a \leftarrow\!\!\ast\, b$ and $b \ast\!\!\rightarrow a$ are called *into* a. Edges of the form $a -\!\!\ast\, b$ and $b \ast\!\!- a$ are called *out of* a. In order to define σ-MAGs, we extend the common definitions in DMGs to accommodate the new set of edge types.

Definition 1. *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a mixed graph with nodes $\mathcal{V}$ and edges $\mathcal{E}$ of the types $\{\rightarrow, \leftarrow, \leftrightarrow, -\}$. Let $a, b, c, v_0, v_n \in \mathcal{V}$.*

1. *If there is an edge $a \ast\!\!-\!\!\ast\, b$ between a and b, we say that a and b are* adjacent *in $\mathcal{H}$.*
2. *A triple of distinct nodes (a, b, c) in $\mathcal{H}$ is called* unshielded *if b is adjacent to both a and c in $\mathcal{H}$, but a is not adjacent to c in $\mathcal{H}$.*
3. *A* walk *between v_0 and v_n in $\mathcal{H}$ is a finite alternating sequence of nodes and edges:*[1]
$$\langle v_0, e_1, v_1, \ldots, v_{n-1}, e_n, v_n \rangle,$$
for some $n \geq 0$, such that for every $i = 1, \ldots, n$, we have $e_i = v_{i-1} \ast\!\!-\!\!\ast\, v_i \in \mathcal{E}$.
4. *A walk is called a* path *if no node appears more than once on the walk.*
5. *A* directed walk (path) *from v_0 to v_n in $\mathcal{H}$ is a walk (path) of the form:*
$$v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_{n-1} \rightarrow v_n,$$
for some $n \geq 0$.
6. *An* anterior walk (path) *from v_0 to v_n in $\mathcal{H}$ is a walk (path) of the form:*
$$v_0 -\!\!\ast\, v_1 -\!\!\ast \cdots -\!\!\ast\, v_{n-1} -\!\!\ast\, v_n,$$
for some $n \geq 0$.
7. *A* directed cycle *is a directed walk $a \rightarrow \cdots \rightarrow b$, concatenated with the directed edge $b \rightarrow a$.*
8. *An* almost directed cycle *is a directed walk $a \rightarrow \cdots \rightarrow b$, concatenated with the bidirected edge $b \leftrightarrow a$.*
9. *If there is a directed walk from a to b in $\mathcal{H}$, we say that a is an ancestor of b in $\mathcal{H}$, and we write $a \in Anc_{\mathcal{H}}(b)$. For a set $A \subseteq \mathcal{V}$, we define $Anc_{\mathcal{H}}(A) = \bigcup_{v \in A} Anc_{\mathcal{H}}(v)$.*
10. *A triple of consecutive nodes $v_{k-1} \ast\!\!-\!\!\ast\, v_k \ast\!\!-\!\!\ast\, v_{k+1}$ on a walk in $\mathcal{H}$ is called a* collider *if it is of the form $v_{k-1} \ast\!\!\rightarrow v_k \leftarrow\!\!\ast\, v_{k+1}$.*
11. *A triple of consecutive nodes $v_{k-1} \ast\!\!-\!\!\ast\, v_k \ast\!\!-\!\!\ast\, v_{k+1}$ on a walk in $\mathcal{H}$ is called a* non-collider *if it is of the form $v_{k-1} \ast\!\!-\!\!\ast\, v_k -\!\!\ast\, v_{k+1}$ or $v_{k-1} \ast\!\!- v_k \ast\!\!-\!\!\ast\, v_{k+1}$. Furthermore, we also refer to the endpoints v_0 and v_n of a walk between v_0 and v_n in $\mathcal{H}$ as* non-colliders.

[1] Without extra explanations, we consider a walk (path) between v_0 and v_n in the form of $v_0 \ast\!\!-\!\!\ast\, v_1 \ast\!\!-\!\!\ast \cdots \ast\!\!-\!\!\ast\, v_{n-1} \ast\!\!-\!\!\ast\, v_n$.

12. *Let $v_0, v_n \in \mathcal{V}$ and $Z \subseteq \mathcal{V}$. If π is a path between v_0 and v_n in $\mathcal{H}$, then the* Collider Distance Sum to Z *of π is given by*

$$\sum_{i=0}^{n} l_{\mathcal{H}}(v_i, Z) \cdot 1_{\{v_i \text{ is a collider on } \pi\}},$$

where $l_{\mathcal{H}}(v_i, Z)$ denotes the length of the shortest directed path from v_i to Z in $\mathcal{H}$.

3 DEFINITION & CHARACTERIZATION

In this section, we introduce the class of σ-MAGs, which encodes certain ancestral relationships and σ-connections of DMGs. Leveraging this property, we will later introduce the corresponding separation criterion for σ-MAGs and characterize their Markov equivalence.

To facilitate the definition of σ-MAGs, we first define the notion of inducing paths in mixed graphs in analogy to inducing paths in DAGs [Verma and Pearl, 1990]. This differs from the related notion of σ-inducing path in DMGs (Definition 22).

Definition 2 (Inducing Walk (Path))**.** *In a mixed graph $\mathcal{H}$ with nodes $\mathcal{V}$, and edges $\mathcal{E}$ of the types $\{\rightarrow, \leftarrow, \leftrightarrow, -\}$, a walk (path) π in $\mathcal{H}$ between v_0 and v_n is called an* inducing walk (path) *if every node on π, except for the endpoints, is a collider and belongs to $Anc_{\mathcal{H}}(\{v_0, v_n\})$.*

Whereas in MAGs, undirected edges appear in case of conditioning (for example when modeling selection bias), in σ-MAGs they can also encode the existence of directed cycles in the DMGs that the σ-MAG represents. One can sometimes distinguish the two by looking at the connectivities of the undirected edges in the σ-MAG, which motivates the following definition.

Definition 3 (Complete and Incomplete Neighborhoods)**.** *In a mixed graph $\mathcal{H}$ with nodes $\mathcal{V}$, and edges $\mathcal{E}$ of the types $\{\rightarrow, \leftarrow, \leftrightarrow, -\}$, the* neighborhood *of a node a is the set of nodes connected to a by undirected edges, denoted $Nbh_{\mathcal{H}}(a) = \{v \in \mathcal{V} : a - v \in \mathcal{E}\}$.*

We say that the neighborhood of a is complete *if it forms a clique, meaning that for any two nodes $b, c \in Nbh_{\mathcal{H}}(a)$, we have $b - c \in \mathcal{E}$. Otherwise, the neighborhood of a is said to be* incomplete.

As we will demonstrate later, complete neighborhoods in a σ-MAG may correspond to strongly connected components (See Definition 14 for more details) in DMGs, while incomplete neighborhoods must correspond to ancestors of selection variables in DMGs.

We can now state the proposed abstract definition of σ-MAGs:

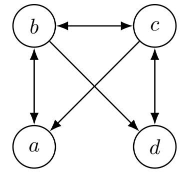


Figure 1: This mixed graph is not a valid σ-MAG, because it has an inducing path $a \leftrightarrow b \leftrightarrow c \leftrightarrow d$ while a and d are non-adjacent.

Definition 4 (σ-MAG)**.** *A mixed graph $\mathcal{H}$ with nodes $\mathcal{V}$ and edges $\mathcal{E}$ of the types $\{\rightarrow, \leftarrow, \leftrightarrow, -\}$ is called a σ-*maximal ancestral graph (σ-MAG) *if all of the following conditions hold:*

1. *Between any two distinct nodes, there is at most one edge, and no node is adjacent to itself.*
2. *$\mathcal{H}$ is* ancestral*: If $\mathcal{H}$ contains an anterior path $a \multimap \cdots \multimap b$ then it does not contain an edge $a \leftrightarrow\!\!* \; b$.*
3. *$\mathcal{H}$ is* σ-maximal*:*
   (a) *$\mathcal{H}$ is* maximal*: There is no inducing path between any two distinct non-adjacent nodes.*
   (b) *$\mathcal{H}$ is* σ-complete*: If $\mathcal{H}$ contains a triple $a \ast\!\!\rightarrow b - c$, then a and c must be adjacent in $\mathcal{H}$. Furthermore, if $\mathcal{H}$ also contains an edge $b - d$, then c and d must be adjacent in $\mathcal{H}$.*

It follows immediately from this definition that MAGs (as defined in [Richardson and Spirtes, 2002]) can be seen as a subclass of σ-MAGs, namely those that have no edges into an undirected edge (the pattern $v_i \ast\!\!\rightarrow v_j - v_k$ cannot occur in a MAG by definition). One can show that the assumptions concerning such patterns also imply certain orientations.

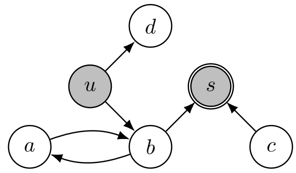
Lemma 1 (Fundamental Property of σ-MAGs)**.** *Let $\mathcal{H}$ be a σ-MAG. If $\mathcal{H}$ contains a triple of the form $a \ast\!\!\rightarrow b - c$, then the edge between a and c is of the same type as the edge between a and b, and the neighborhoods of b and c are both complete.*

Further, it follows that a σ-MAG contains no directed cycles or almost directed cycles.

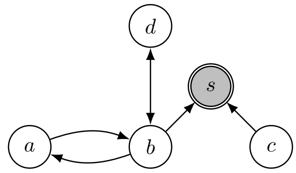
An example of a mixed graph that is not a valid σ-MAG is given in Figure 1. Various examples of valid σ-MAGs are given in Figure 3 (on the left).

The essential connection between σ-MAGs and DMGs is obtained by the following definition, which expresses in which way a σ-MAG "represents" a class of DMGs:

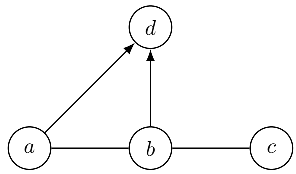
Definition 5 (Representing DMGs by σ-MAGs)**.** *Let $\mathcal{H}$ be a mixed graph with nodes $\mathcal{V}$, and edges $\mathcal{E}$ of the types $\{\rightarrow, \leftarrow, \leftrightarrow, -\}$. Let $\mathcal{G}$ be a DMG with nodes $\mathcal{V}^+ = \mathcal{V} \cup \mathcal{S}$. We say that $\mathcal{H}$ represents $\mathcal{G}$ given $\mathcal{S}$ if all of the following conditions hold:*



(a) DG with latent (selection) variables



(b) DMG after marginalizing out latent variables



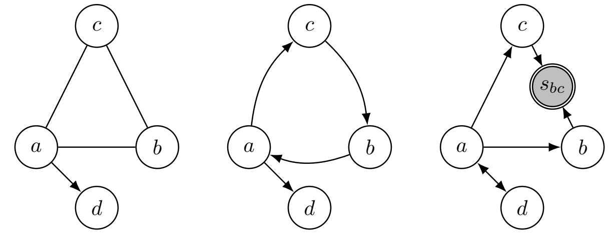
(c) σ-MAG constructed from the DMG

Figure 2: Constructing a σ-MAG from a DG with latent variable u and latent selection variable s.

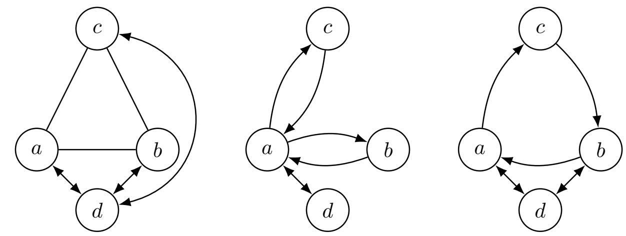
1. *Between any two distinct nodes in $\mathcal{V}$, there is at most one edge, and no edge connects a node to itself in $\mathcal{H}$.*
2. *Two distinct nodes $a, b \in \mathcal{V}$ are adjacent in $\mathcal{H}$ if and only if there exists a σ-inducing path between a and b given $\mathcal{S}$ in $\mathcal{G}$.*
3. *If $a \leftarrow\!\!\ast\; b$ in $\mathcal{H}$, then $a \notin Anc_{\mathcal{G}}(\{b\} \cup \mathcal{S})$.*
4. *If $a \multimap b$ in $\mathcal{H}$, then $a \in Anc_{\mathcal{G}}(\{b\} \cup \mathcal{S})$.*

The above definition yields a systematic procedure for constructing a σ-MAG $\mathcal{H}$ from a given DMG $\mathcal{G}$ that includes selection variables. First, the adjacencies in $\mathcal{H}$ are determined based on the existence of σ-inducing paths given $\mathcal{S}$ between pairs of nodes in $\mathcal{G}$. Next, the orientations of the edges in $\mathcal{H}$ are derived by analyzing the ancestral relationships among the observed nodes in $\mathcal{G}$. Figure 2 illustrates this procedure by showing how to construct a σ-MAG from a Directed Graph (DG) with latent (selection) variables.

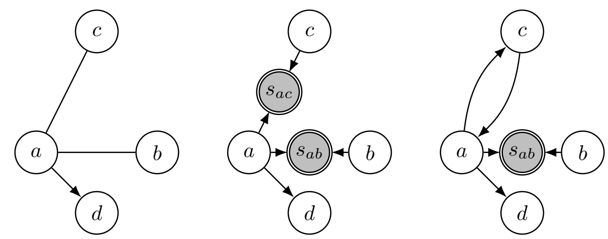
In this example, by marginalizing over latent variables (see Definition 16 for details), we obtain a DMG (Figure 2b) from a DG with latent and selection variables (Figure 2a). Then, according to Definition 5, we construct the corresponding σ-MAG, as shown in Figure 2c. For instance, consider the path $a \to b \leftrightarrow d$ in the DMG, which is σ-inducing given the selection variable $\mathcal{S} = \{s\}$. This implies that a and d are adjacent in the resulting σ-MAG. Moreover, since a is an ancestor of s in the DMG, while d is not an ancestor of either a or s, we can determine the orientation of the edge between a and d in the σ-MAG to be $a \to d$. By repeating this process for all relevant pairs of nodes in the DMG, we can systematically construct the entire σ-MAG.



(a) $\mathcal{H}_1$ (left), $\mathcal{G}_1$ (middle), $\mathcal{G}_2$ (right)



(b) $\mathcal{H}_2$ (left), $\mathcal{G}_3$ (middle), $\mathcal{G}_4$ (right)



(c) $\mathcal{H}_3$ (left), $\mathcal{G}_5$ (middle), $\mathcal{G}_6$ (right)

Figure 3: Examples of σ-MAGs (left) representing DMGs (middle and right). (a) $\mathcal{H}_1$ represents $\mathcal{G}_1$ and represents $\mathcal{G}_2$ given $\mathcal{S} = \{s_{bc}\}$; (b) $\mathcal{H}_2$ represents both $\mathcal{G}_3$ and $\mathcal{G}_4$; (c) $\mathcal{H}_3$ represents $\mathcal{G}_5$ given $\mathcal{S} = \{s_{ac}, s_{ab}\}$ and represents $\mathcal{G}_6$ given $\mathcal{S} = \{s_{ab}\}$.

Some examples illustrating how σ-MAGs represent DMGs are shown in Figure 3. In both $\mathcal{H}_1$ and $\mathcal{H}_2$, the neighborhood of a is complete; the key difference lies in whether there is an arrowhead pointing to a. In particular, $\mathcal{H}_1$ contains the edge $a \to d$, implying that in the DMGs represented by $\mathcal{H}_1$, the set $\{a, b, c\}$ may either form a strongly connected component or simply be ancestors of the set $\mathcal{S}$ (as illustrated in $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively). In contrast, $\mathcal{H}_2$ contains the edge $a \leftrightarrow b$, and Lemma 1 guarantees that $\{a, b, c\}$ must form a strongly connected component, as in $\mathcal{G}_3$ and $\mathcal{G}_4$. Furthermore, if the neighborhood of a is incomplete, as in $\mathcal{H}_3$, then the set $\{a, b, c\}$ must be a subset of $\text{Anc}_{\mathcal{H}_3}(\mathcal{S})$ in the corresponding DMGs represented by it (namely, $\mathcal{G}_5$ and $\mathcal{G}_6$), although a partial strongly connected component may still exist, as seen in $\mathcal{G}_6$.

In the rest of this section we will show that the mixed graphs that represent DMGs are characterized as σ-MAGs. The proofs are provided in the Appendix.

A useful property of σ-MAGs that comes in handy often is:

Lemma 2. *Let $\mathcal{H}$ be a σ-MAG. If $\mathcal{H}$ contains an anterior path that starts with a directed edge:*

$$v_0 \rightarrow v_1 \rightarrow\!\!\ast \cdots \rightarrow\!\!\ast\, v_n$$

for $n \geq 2$, then v_0 belongs to the ancestors of v_n in $\mathcal{H}$.

The following result demonstrates that every σ-MAG can represent at least one DMG, given an additional set of selection nodes.

Lemma 3. *Let $\mathcal{H}$ be a σ-MAG with nodes $\mathcal{V}$. Then, there exists a DMG $\mathcal{G}$ with nodes $\mathcal{V}^+ = \mathcal{V} \cup \mathcal{S}$, such that $\mathcal{H}$ represents $\mathcal{G}$ given $\mathcal{S}$.*

The following elementary properties demonstrate that certain key characteristics of DMGs are encoded in a σ-MAG that represents them.

Lemma 4. *Let $\mathcal{H}$ be a σ-MAG that represents a DMG $\mathcal{G}$ given $\mathcal{S}$. Let a and b be distinct nodes in $\mathcal{H}$.*

1. *$a \in Ant_{\mathcal{H}}(b)$ implies $a \in Anc_{\mathcal{G}}(\{b\} \cup \mathcal{S})$.*
2. *If $\mathcal{H}$ contains the edge $a \ast\!\!\rightarrow b$, then there exists a σ-inducing path given $\mathcal{S}$ in $\mathcal{G}$ between a and b that is into b.*
3. *If $\mathcal{H}$ contains the edge $a \leftrightarrow b$, then there exists a σ-inducing path given $\mathcal{S}$ in $\mathcal{G}$ between a and b that is into both a and b.*

On the other hand, every mixed graph that represents a DMG must be a σ-MAG.

Lemma 5. *Let $\mathcal{H}$ be a mixed graph with nodes $\mathcal{V}$ and edges $\mathcal{E}$ of the types $\{\rightarrow, \leftarrow, \leftrightarrow, -\}$. Let $\mathcal{G}$ be a DMG with nodes $\mathcal{V}^+ = \mathcal{V} \cup \mathcal{S}$. If $\mathcal{H}$ represents $\mathcal{G}$ given $\mathcal{S}$, then $\mathcal{H}$ is a σ-MAG.*

Finally, we obtain the following result characterizing σ-MAGs as the mixed graphs that represent DMGs in the sense of Definition 5.

Theorem 1. *Let $\mathcal{H}$ be a mixed graph with nodes $\mathcal{V}$ and edges $\mathcal{E}$ of the types $\{\rightarrow, \leftarrow, \leftrightarrow, -\}$. The following equivalence holds:*

$$\begin{aligned}&\text{There exists a DMG } \mathcal{G} \text{ with nodes } \mathcal{V}^+ = \mathcal{V} \cup \mathcal{S},\\ &\quad \text{such that } \mathcal{H} \text{ represents } \mathcal{G} \text{ given } \mathcal{S}.\\ &\iff \mathcal{H} \text{ is a } \sigma\text{-MAG}.\end{aligned}$$

4 SEPARATION CRITERION

In this section, we extend the commonly used m-separation criterion for MAGs [Richardson and Spirtes, 2002] to σ-MAGs and demonstrate how it encodes σ-separation (See Definition 20 for more details) in the represented DMGs.

Definition 6 (m-separation for σ-MAGs)**.** *Let $\mathcal{H}$ be a σ-MAG with nodes $\mathcal{V}$. Let $X, Y, Z \subseteq \mathcal{V}$ be subsets of the nodes.*

1. *Consider a walk (path) π in $\mathcal{H}$ with $n \geq 0$ edges:*

   $$v_0 \ast\!\!-\!\!\ast \cdots \ast\!\!-\!\!\ast\, v_n.$$

   *We say that π is Z-m-*blocked *or m-*blocked by *Z if any of the following conditions holds:*

   (a) *there exists a non-collider v_k on π in Z, or*
   (b) *there exists a collider v_k on π that is not in $Anc_{\mathcal{H}}(Z)$, or*
   (c) *there exists a subwalk (subpath) of the form $v_{k-1} \ast\!\!\rightarrow v_k - v_{k+1}$ or $v_{k-1} - v_k \leftarrow\!\!\ast\, v_{k+1}$ on π.*

   *Conversely, we say that π is Z-m-*open *or m-*open given *Z if:*

   (a) *all non-colliders on π are not in Z, and*
   (b) *all colliders on π are in $Anc_{\mathcal{H}}(Z)$, and*
   (c) *it contains no subwalk (subpath) of the form $v_{k-1} \ast\!\!\rightarrow v_k - v_{k+1}$ or $v_{k-1} - v_k \leftarrow\!\!\ast\, v_{k+1}$.*

2. *We say that X* is m-separated from *Y* given *Z if every walk in $\mathcal{H}$ from a node in X to a node in Y is m-blocked by Z. This is denoted as:*

   $$X \underset{\mathcal{H}}{\overset{m}{\perp}} Y \mid Z.$$

   If this property does not hold, we will write

   $$X \underset{\mathcal{H}}{\overset{m}{\not\perp}} Y \mid Z.$$

The following result is frequently employed to simplify proofs and to make the verification of m-separation more tractable in practice.

Proposition 1. *Let $\mathcal{H}$ be a σ-MAG. For $Z \subseteq \mathcal{V}$, and $a, b \in \mathcal{V}$, the following are equivalent:*

1. *there exists a Z-m-open walk between a and b in $\mathcal{H}$;*
2. *there exists a Z-m-open path between a and b in $\mathcal{H}$.*

Lemmas 6 to 8 establish the relation between m-separation in σ-MAGs and σ-separation in DMGs, as stated in Theorem 2.

Lemma 6. *Let $\mathcal{G}$ be a DMG with nodes $\mathcal{V}^+ = \mathcal{V} \cup \mathcal{S}$, and let $\mathcal{H}$ be a σ-MAG that represents $\mathcal{G}$ given $\mathcal{S}$. Let $a, b \in \mathcal{V}$ and $Z \subseteq \mathcal{V}$. If π is a m-open path given Z between a and b in $\mathcal{H}$, then every node on π is in $Anc_{\mathcal{G}}(\{a, b\} \cup Z \cup \mathcal{S})$.*

Lemma 7. *Let $\mathcal{G}$ be a DMG with nodes $\mathcal{V}^+ = \mathcal{V} \cup \mathcal{S}$, and let $\mathcal{H}$ be a σ-MAG that represents $\mathcal{G}$ given $\mathcal{S}$. Let $a, b \in \mathcal{V}$ and $Z \subseteq \mathcal{V}$. If there exists a Z-m-open path between a and b in $\mathcal{H}$, then there exists a $(Z \cup \mathcal{S})$-σ-open path in $\mathcal{G}$ between a and b.*

Lemma 8. *Let $\mathcal{G}$ be a DMG with nodes $\mathcal{V}^+ = \mathcal{V} \cup \mathcal{S}$, and let $\mathcal{H}$ be a σ-MAG that represents $\mathcal{G}$ given $\mathcal{S}$. Let $a, b \in \mathcal{V}$ and $Z \subseteq \mathcal{V}$. If there exists a $(Z \cup \mathcal{S})$-σ-open path between a and b in $\mathcal{G}$, then there exists a Z-m-open path in $\mathcal{H}$ between a and b.*

For proving the latter two lemmata, we drew inspiration from the corresponding proofs for the corresponding statements for MAGs by Spirtes and Richardson [1996, Lemma 17–18]. Combining these yields the following fundamental result.

Theorem 2. *Let $\mathcal{G}$ be a DMG with nodes $\mathcal{V}^+ = \mathcal{V} \cup \mathcal{S}$, and let $\mathcal{H}$ be a σ-MAG that represents $\mathcal{G}$ given $\mathcal{S}$. Let $X, Y, Z \subseteq \mathcal{V}$ be subsets of the nodes. We have the following equivalence:*

$$X \underset{\mathcal{H}}{\overset{m}{\perp}} Y \mid Z \qquad \iff \qquad X \underset{\mathcal{G}}{\overset{\sigma}{\perp}} Y \mid Z \cup \mathcal{S}.$$

The following proposition establishes fundamental properties of m-separation and forms the basis for the concept of inducing paths: specifically, two distinct nodes in a σ-MAG are connected by an inducing path if and only if they cannot be m-separated by any subset of the remaining nodes.

Proposition 2. *Let $\mathcal{H}$ be a σ-MAG with nodes $\mathcal{V}$ and let $a, b \in \mathcal{V}$ be distinct nodes. Then the following are equivalent:*

1. *There is an inducing path in $\mathcal{H}$ between a and b;*
2. *There is an inducing walk in $\mathcal{H}$ between a and b;*
3. *$a \underset{\mathcal{H}}{\overset{m}{\not\perp}} b \mid Z$ for all $Z \subseteq \mathcal{V} \backslash \{a, b\}$;*
4. *$a \underset{\mathcal{H}}{\overset{m}{\not\perp}} b \mid Z$ for $Z = Anc_{\mathcal{H}}(\{a, b\}) \backslash \{a, b\}$.*

The orientations of the outermost edges on an inducing path encode essential information about ancestral or anterior relationships.

Lemma 9. *Let $\mathcal{H}$ be a σ-MAG with nodes $\mathcal{V}$ and let $a, b \in \mathcal{H}$ be distinct. If there exists an inducing path between a and b in $\mathcal{H}$, and all inducing paths in $\mathcal{H}$ between a and b are out of b, then $b \in Ant_{\mathcal{H}}(a)$.*

Lemma 10. *Let $\mathcal{H}$ be a σ-MAG with nodes $\mathcal{V}$ and let $a, b \in \mathcal{V}$ be distinct. If there exists an inducing path between a and b in $\mathcal{H}$ that is into b, and $a \notin Anc_{\mathcal{H}}(b)$, then there exists an inducing path between a and b in $\mathcal{H}$ that is both into a and into b.*

5 MARKOV EQUIVALENCE

With the well-defined notion of m-separation, we now turn our attention to the concept of the *m-Markov Equivalence Class* of σ-MAGs, in which every σ-MAG shares the same m-separation relations.

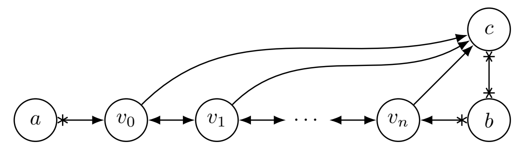


Figure 4: A discriminating path for b between a and c.

Definition 7 (m-Markov Equivalence)**.** *Two σ-MAGs $\mathcal{H}_1, \mathcal{H}_2$ with the same nodes $\mathcal{V}$ are* m-Markov equivalent *if for any three subsets of the nodes $X, Y, Z \subseteq \mathcal{V}$, X is m-separated from Y given Z in $\mathcal{H}_1$ if and only if X is m-separated from Y given Z in $\mathcal{H}_2$.*

One important feature that will play a role in the characterization of m-Markov equivalence involves the concept of *discriminating paths*, which can be seen as a generalization of an unshielded triple.

Definition 8 (Discriminating Path)**.** *A path $\pi = (a, v_0, \ldots, v_n, b, c)$ (with $n \geq 0$) in a σ-MAG $\mathcal{H}$ is a* discriminating path *for b if:*

1. *a is not adjacent to c in $\mathcal{H}$, and*
2. *for $k = 0, \ldots, n$: v_k is a collider on π and a parent of c in $\mathcal{H}$.*

An example of discriminating path is given in Figure 4.

Applying m-separation to discriminating paths leads to the following lemma:

Lemma 11. *In a σ-MAG $\mathcal{H}$ with nodes $\mathcal{V}$, let $\pi = (a, v_0, \ldots, v_n, b, c)$ be a discriminating path for b. Then, the following hold:*

1. *If b is a collider on π, then for any subset of nodes $Z \subseteq \mathcal{V} \backslash \{a, c\}$ such that a and c are m-separated given Z, we have $b \notin Z$.*
2. *If b is a non-collider on π, then for any subset of nodes $Z \subseteq \mathcal{V} \backslash \{a, c\}$ such that a and c are m-separated given Z, we have $b \in Z$.*

Spirtes and Richardson [1996] showed that two MAGs having the "same basic colliders" is a necessary and sufficient condition for "d-Markov equivalence". Here, we aim to verify whether an analogous statement applies to σ-MAGs. The assumption is restated as follows:

Condition 1. *For two σ-MAGs $\mathcal{H}_1, \mathcal{H}_2$ with the same nodes $\mathcal{V}$, the following three conditions hold:*

1. *$\mathcal{H}_1, \mathcal{H}_2$ have the same adjacencies.*
2. *$\mathcal{H}_1, \mathcal{H}_2$ have the same unshielded colliders.*
3. *Let π be a discriminating path for a node v in $\mathcal{H}_1$, and*

let π' be the corresponding path to π in $\mathcal{H}_2$.[2] *If π' is also a discriminating path for v, then v is a collider on π in $\mathcal{H}_1$ if and only if it is a collider on π' in $\mathcal{H}_2$.*

In the rest of this section, we mimic the proof strategy of Spirtes and Richardson [1996] (with small modifications at various places to account for the extension of MAGs to σ-MAGs) to show that this assumption is a necessary and sufficient condition for m-Markov equivalence.

It is straightforward that Condition 1 is a necessary condition to obtain m-Markov equivalence.

Lemma 12. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two σ-MAGs with the same node set $\mathcal{V}$. If $\mathcal{H}_1$ and $\mathcal{H}_2$ are m-Markov equivalent, then they satisfy Condition 1.*

Now, we consider the opposite direction and begin with the definition of *covered nodes*, as follows:

Definition 9 (Covered Node)**.** *In a σ-MAG $\mathcal{H}$, a node v_k on a path π between v_0 and v_n (where $0 < k < n$) is called a* covered node *if its adjacent nodes on π, v_{k-1} and v_{k+1}, are also adjacent in $\mathcal{H}$. Conversely, v_k is called an* uncovered node *if v_{k-1} and v_{k+1} are non-adjacent in $\mathcal{H}$.*

The following proposition establishes an important property: every covered node on a shortest m-open path corresponds to a unique discriminating subpath. This result is built upon Lemma 13 and Lemma 14, which provide the necessary foundation.

Proposition 3. *Let π be a shortest m-open path between v_0 and v_n given Z in a σ-MAG $\mathcal{H}$. If v_j is a covered node on π, then:*

1. *If $v_{j-1} \to v_{j+1}$, there exists a unique index $i < j$ such that the subpath of π between v_i and v_{j+1} is a discriminating path for v_j.*
2. *If $v_{j-1} \leftarrow v_{j+1}$, there exists a unique index $i > j$ such that the subpath of π between v_{j-1} and v_i is a discriminating path for v_j.*

Lemma 13. *Let π be a shortest m-open path between v_0 and v_n given Z in a σ-MAG $\mathcal{H}$. Suppose there exists an edge $v_i *\!\!-\!\!* v_j$ ($i < j$) in $\mathcal{H}$ that is not part of π. Define π' as the path obtained by replacing the subpath between v_i and v_j on π with the edge $v_i *\!\!-\!\!* v_j$. Then, one of the following conditions must hold:*

1. *$v_{i-1} *\!\!\rightarrow v_i \leftarrow\!\!* v_{i+1}$ appears on π and $v_i \to v_j$ exists in $\mathcal{H}$.*
2. *$v_{j-1} *\!\!\rightarrow v_j \leftarrow\!\!* v_{j+1}$ appears on π and $v_i \leftarrow v_j$ exists in $\mathcal{H}$.*

[2] Since $\mathcal{H}_1$ and $\mathcal{H}_2$ have the same nodes and adjacencies, the sequence of nodes in π within $\mathcal{H}_1$ uniquely determines a corresponding path π' in $\mathcal{H}_2$.

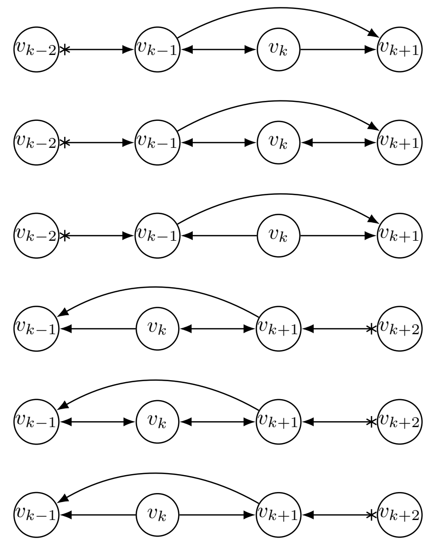


Figure 5: Possible subgraphs for Lemma 14.

Lemma 14. *Let π be a shortest m-open path between v_0 and v_n given Z in a σ-MAG $\mathcal{H}$. Suppose π contains the subpath $v_{k-1} *\!\!\rightarrow\!\!* v_k *\!\!\rightarrow\!\!* v_{k+1}$, where v_{k-1} and v_{k+1} are adjacent in $\mathcal{H}$. Then, $\mathcal{H}$ must contain one of the subgraphs shown in Figure 5.*

Next, from Lemma 15, we can infer Lemmas 16 to 18, which collectively provide a foundation for defining the order of covered nodes.

Lemma 15. *In a σ-MAG $\mathcal{H}$, if π is a shortest m-open path between v_0 and v_n given Z, then no pair of distinct covered nodes v_b and v_j on π can satisfy both of the following conditions: v_b is a covered node on the discriminating subpath of π for v_j, and v_j is a covered node on the discriminating subpath of π for v_b.*

Lemma 16. *In a σ-MAG $\mathcal{H}$, if π is a shortest m-open path between v_0 and v_n given Z, then no triple of distinct covered nodes v_i, v_j, v_k on π can satisfy all of the following conditions: v_i is a covered node on the discriminating subpath of π for v_j, v_j is a covered node on the discriminating subpath of π for v_k, and v_k lies between v_i and v_j on π.*

Lemma 17. *In a σ-MAG $\mathcal{H}$, if π is a shortest m-open path between v_0 and v_n given Z, then no quadruple of distinct covered nodes v_i, v_j, v_k, v_l on π can satisfy all of the following conditions: v_i is a covered node on the discriminating subpath of π for v_j, v_k is a covered node on the discriminating subpath of π for v_l, v_l lies between v_i and v_j on π, and v_j lies between v_l and v_k on π.*

Lemma 18. *In a σ-MAG $\mathcal{H}$, if π is a shortest m-open path between a and b given Z, then no sequence of distinct covered nodes $v_0, v_1, \ldots, v_n$ on π with $n \geq 2$ can satisfy* *the following conditions: for each $i = 0, \ldots, n-1$, the node v_i is a covered node on the discriminating subpath of π for v_{i+1}, and v_n is a covered node on the discriminating subpath of π for v_0.*

Equipped with the above lemmata, we now introduce the *order* of covered nodes on a shortest m-open path.

Definition 10 (Order of covered Node)**.** *In a σ-MAG $\mathcal{H}$, let π be a shortest path between nodes a and b given a conditioning set Z. A covered node v on π is called a* 0-order covered node *if the discriminating subpath of π for v contains no other covered nodes.*

*Furthermore, a covered node v on π is called an n-*order covered node *if the maximum order of any other covered node on the discriminating subpath of π for v is $n-1$.*

Lemma 18 guarantees that this recursive definition is well-founded. Given a shortest m-open path containing covered nodes in a σ-MAG, starting from an arbitrary covered node v_0, if there is no other covered node on the discriminating subpath of π for v_0, then v_0 is a 0-order covered node. If another covered node v_1 exists on this subpath, we proceed by checking whether there is a covered node on the discriminating subpath of π for v_1. Repeating this process, we must eventually reach a covered node that has no other covered nodes on its discriminating subpath, since π has finite length and Lemma 18 ensures that we do not encounter repetitive covered nodes in this process. Similarly, there must exist a covered node such that all covered nodes on its discriminating subpath of π are 0-order. By induction, the definition of n-order covered node is well-defined.

Now we will show that Condition 1 is also a sufficient condition for m-Markov equivalence.

Lemma 19. *If two σ-MAGs $\mathcal{H}_1$ and $\mathcal{H}_2$ with the same nodes $\mathcal{V}$ satisfy Condition 1, and if $\pi = (a, v_0, \ldots, v_n, b, c)$ is a discriminating path for b in $\mathcal{H}_1$, then let π' be the corresponding path to π in $\mathcal{H}_2$. If every node on π', except for the endpoints and b, is a collider, then π' is a discriminating path for b in $\mathcal{H}_2$.*

Equipped with Definition 10, we now show that every node on a given shortest m-open path in one σ-MAG retains the same collider/non-collider status on the corresponding path in another σ-MAG, provided that the two graphs satisfy Condition 1.

Lemma 20. *If two σ-MAGs $\mathcal{H}_1$ and $\mathcal{H}_2$ with the same nodes $\mathcal{V}$ satisfy Condition 1, and if π is a shortest m-open path between v_0 and v_n given Z in $\mathcal{H}_1$, with π' being the corresponding path to π in $\mathcal{H}_2$, then v_k is a collider on π if and only if v_k is a collider on π'.*

The following lemma establishes useful properties for a special case involving covered nodes.

Lemma 21. *If a σ-MAG $\mathcal{H}$ contains a path $a \ast\!\!\rightarrow b \rightarrow c$ and an edge $a \ast\!\!-\!\!\ast\, c$, then:*

1. *The edge between a and c is oriented as $a \ast\!\!\rightarrow c$.*
2. *If $a \ast\!\!-\!\!\ast\, c$ has a different edge mark at a than $a \ast\!\!\rightarrow b$, then the edges are oriented as $a \leftrightarrow b \rightarrow c$ and $a \rightarrow c$.*

By Lemma 20, all non-colliders on a shortest m-open path in one σ-MAG cannot block the corresponding path in another σ-MAG, provided that the two graphs satisfy Condition 1. Lemmas 22 and 23 further show that all colliders do not block the path either, thereby establishing the m-Markov equivalence of the two graphs.

Lemma 22. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two σ-MAGs with the same node set $\mathcal{V}$ that satisfy Condition 1. Suppose π is a shortest m-open path between v_0 and v_n given Z in $\mathcal{H}_1$ and has the smallest collider distance sum to Z.*[3] *If v_i is a collider on π, q is an ancestor of Z in $\mathcal{H}_1$, and there is an edge $v_i \rightarrow q$ on a shortest directed path from v_i to Z in $\mathcal{H}_1$, then the edge $v_i \rightarrow q$ must also be present in $\mathcal{H}_2$.*

Lemma 23. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two σ-MAGs with the same node set $\mathcal{V}$ that satisfy Condition 1. Suppose π is a shortest m-open path between v_0 and v_n given Z in $\mathcal{H}_1$ and has the smallest collider distance sum to Z. If v_k is a collider on π, and $q \in Z$ is the endpoint of a shortest directed path μ from v_k to Z in $\mathcal{H}_1$, then v_k is an ancestor of q in $\mathcal{H}_2$.*

Lemma 24. *If two σ-MAGs $\mathcal{H}_1, \mathcal{H}_2$ with the same node set $\mathcal{V}$ satisfy Condition 1, then they are m-Markov equivalent.*

Consequently, we obtain the following equivalence:

Theorem 3. *Two σ-MAGs $\mathcal{H}_1, \mathcal{H}_2$ with the same nodes $\mathcal{V}$ are m-Markov equivalent if and only if $\mathcal{H}_1, \mathcal{H}_2$ satisfy Condition 1.*

Moreover, by combining it with the equivalence derived in Section 4, we also obtain the σ-Markov equivalence under Condition 1.

Theorem 4. *Let $\mathcal{G}_1, \mathcal{G}_2$ be two DMGs with the same nodes $\mathcal{V}^+ = \mathcal{V} \cup \mathcal{S}$, and let $\mathcal{H}_1, \mathcal{H}_2$ be two σ-MAGs that represents $\mathcal{G}_1, \mathcal{G}_2$ given $\mathcal{S}$ respectively. $\mathcal{G}_1, \mathcal{G}_2$ are σ-Markov equivalent given $\mathcal{S}$ if and only if $\mathcal{H}_1, \mathcal{H}_2$ satisfy Condition 1.*

6 DISCUSSION AND CONCLUSION

It is remarkable that with the same edge types as MAGs, σ-MAGs can account for both cycles and selection bias

[3] That is, there does not exist another m-open path μ between v_0 and v_n given Z in $\mathcal{H}_1$ such that either μ has fewer edges than π, or μ has the same number of edges as π, but a smaller collider distance sum to Z.

(via undirected edges) in such a way that the key properties of MAGs generalize in a predictable way to those of σ-MAGs, even though the undirected edges do not in general distinguish between cycles and selection bias (except in case of the presence of an edge into a clique of undirected edges).

The theory we developed here provides the foundations for possible future work on developing sound and complete extensions of FCI applicable to data generated by simple (possibly cyclic) SCMs in the presence of selection bias. Furthermore, it may prove an important step towards developing a do-calculus for the output of FCI in that general setting.

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Supplementary Material to “σ-Maximal Ancestral Graphs”

Binghua Yao[1] **Joris M. Mooij**[1]

[1]Korteweg-de Vries Institute, University of Amsterdam, Amsterdam, The Netherlands

A PRELIMINARIES

A.1 DIRECTED MIXED GRAPHS (DMGS)

Definition 11 (Directed Mixed Graphs (DMGs))**.** *A* Directed Mixed Graph (DMG) *is a graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{L})$ *with nodes* $\mathcal{V}$ *and two types of edges:* directed edges $\mathcal{E} \subseteq \{(v_i, v_j) : v_i, v_j \in \mathcal{V}, v_i \neq v_j\}$*, and bidirected edges* $\mathcal{L} \subseteq \{(v_i, v_j) : v_i, v_j \in \mathcal{V}, v_i \neq v_j\}$.

We denote a directed edge $(v_i, v_j) \in \mathcal{E}$ as $v_i \to v_j$ or equivalently $v_j \leftarrow v_i$. Self-loops of the form $v_i \to v_i$ are not allowed, but multiple edges (with at most one of each type) between any pair of distinct nodes are permitted. Similarly, we denote a bidirected edge $(v_i, v_j) \in \mathcal{L}$ as $v_i \leftrightarrow v_j$ or equivalently $v_j \leftrightarrow v_i$.

Additionally, we use a star edge mark as a placeholder indicating either an arrowhead or a tail. Specifically,

1. $v_i \ast\!\!\rightarrow v_j$ represents $v_i \to v_j$ or $v_i \leftrightarrow v_j$,
2. $v_i \leftarrow\!\!\ast v_j$ represents $v_i \leftarrow v_j$ or $v_i \leftrightarrow v_j$,
3. $v_i \ast\!\!-\!\!\ast v_j$ represents $v_i \to v_j$, $v_i \leftarrow v_j$, or $v_i \leftrightarrow v_j$.

Definition 12. *Let* $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{L})$ *be a DMG.*

1. *If* $v_i \ast\!\!-\!\!\ast v_j \in \mathcal{E} \cup \mathcal{L}$*, then* v_i *and* v_j *are said to be* adjacent *in* $\mathcal{G}$.
2. *An edge of the form* $v_i \leftarrow\!\!\ast v_j$ *is said to be* into v_i.
3. *An edge of the form* $v_i \to v_j$ *is said to be* out of v_i.

Definition 13 (Walks)**.** *Let* $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{L})$ *be a DMG, and let* $v_0, v_n \in \mathcal{V}$.

1. *A* walk *from* v_0 *to* v_n *in* $\mathcal{G}$ *is a finite alternating sequence of adjacent nodes and edges,*

$$\langle v_0, a_1, v_1, \dots, v_{n-1}, a_n, v_n \rangle,$$

for some $n \geq 0$*, such that for every* $k = 1, \dots, n$*, we have* $a_k = (v_{k-1}, v_k) \in \mathcal{E} \cup \mathcal{L}$*. The* trivial walk *consisting of a single node* v_0 *is also allowed. The walk is called* into v_0 *if* $a_1 = v_0 \leftarrow\!\!\ast v_1$*, and* out of v_0 *if* $a_1 = v_0 \to v_1$*. Similarly, it is called* into v_n *if* $a_n = v_{n-1} \ast\!\!\rightarrow v_n$ *and* out of v_n *if* $a_n = v_{n-1} \leftarrow v_n$.

2. *A* directed walk *from* v_0 *to* v_n *in* $\mathcal{G}$ *is of the form*

$$v_0 \to v_1 \to \cdots \to v_{n-1} \to v_n,$$

for some $n \geq 0$*, where all arrowheads point toward* v_n*, and no arrowheads point backward.*

3. *A walk is called a* path *if no node appears more than once.*

Definition 14 (Family Relationships)**.** *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{L})$ be a DMG, and let $v \in \mathcal{V}$ and $A \subseteq \mathcal{V}$ be a subset of nodes. We define:*

1. *The set of* parents *of $\mathcal{V}$ in $\mathcal{G}$:*
$$Pa_{\mathcal{G}}(v) := \{w \in \mathcal{V} \mid w \to v \in \mathcal{E}\}.$$
The set of parents *of A in $\mathcal{G}$:*
$$Pa_{\mathcal{G}}(A) := \bigcup_{v \in A} Pa_{\mathcal{G}}(v).$$
2. *The set of* children *of $\mathcal{V}$ in $\mathcal{G}$:*
$$Ch_{\mathcal{G}}(v) := \{w \in \mathcal{V} \mid v \to w \in \mathcal{E}\}.$$
The set of children *of A in $\mathcal{G}$:*
$$Ch_{\mathcal{G}}(A) := \bigcup_{v \in A} Ch_{\mathcal{G}}(v).$$
3. *The set of* siblings *of $\mathcal{V}$ in $\mathcal{G}$:*
$$Sib_{\mathcal{G}}(v) := \{w \in \mathcal{V} \mid v \leftrightarrow w \in \mathcal{L}\}.$$
The set of siblings *of A in $\mathcal{G}$:*
$$Sib_{\mathcal{G}}(A) := \bigcup_{v \in A} Sib_{\mathcal{G}}(v).$$
4. *The set of* ancestors *of $\mathcal{V}$ in $\mathcal{G}$:*
$$Anc_{\mathcal{G}}(v) := \{w \in \mathcal{V} \mid \exists \text{ a directed walk } w \to \cdots \to v \text{ in } \mathcal{G}\}.$$
The set of ancestors *of A in $\mathcal{G}$:*
$$Anc_{\mathcal{G}}(A) := \bigcup_{v \in A} Anc_{\mathcal{G}}(v).$$
5. *The set of* descendants *of $\mathcal{V}$ in $\mathcal{G}$:*
$$Desc_{\mathcal{G}}(v) := \{w \in \mathcal{V} \mid \exists \text{ a directed walk } v \to \cdots \to w \text{ in } \mathcal{G}\}.$$
The set of descendants *of A in $\mathcal{G}$:*
$$Desc_{\mathcal{G}}(A) := \bigcup_{v \in A} Desc_{\mathcal{G}}(v).$$
6. *The* strongly connected component *of $\mathcal{V}$ in $\mathcal{G}$:*
$$Sc_{\mathcal{G}}(v) := Anc_{\mathcal{G}}(v) \cap Desc_{\mathcal{G}}(v).$$
The strongly connected component *of A in $\mathcal{G}$:*
$$Sc_{\mathcal{G}}(A) := \bigcup_{v \in A} Sc_{\mathcal{G}}(v).$$

Definition 15 (Directed Graphs (DGs))**.** *A* Directed Graph (DG) *is a DMG that contains no bidirected edges, i.e., $\mathcal{L} = \emptyset$.*

A DMG can be regarded as the result of marginalizing latent variables in a DG, as formalized below:

Definition 16 (Marginalization on DMGs [Bongers et al., 2021])**.** *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{L})$ be a DMG and let $\mathcal{W} \subseteq \mathcal{V}$ be a subset of nodes. The* marginalization *of $\mathcal{G}$ with respect to $\mathcal{W}$ is the DMG*
$$\mathcal{G}^{\mathcal{V} \setminus \mathcal{W}} := \mathcal{G}^{\setminus \mathcal{W}} := (\mathcal{V}^{\setminus \mathcal{W}}, \mathcal{E}^{\setminus \mathcal{W}}, \mathcal{L}^{\setminus \mathcal{W}}),$$
where:

1. *$\mathcal{V}^{\setminus \mathcal{W}} = \mathcal{V} \setminus \mathcal{W}$;*

2. $\mathcal{E}^{\setminus \mathcal{W}}$ *consists of all directed edges* $a \to b$ *with* $a, b \in \mathcal{V} \setminus \mathcal{W}$ *for which there exists a directed walk in* $\mathcal{G}$ *of the form*

$$a \to w_0 \to \cdots \to w_n \to b,$$

where all intermediate nodes $w_0, \ldots, w_n \in \mathcal{W}$ *(if any);*

3. $\mathcal{L}^{\setminus \mathcal{W}}$ *consists of all bidirected edges* $a \leftrightarrow b$ *with* $a, b \in \mathcal{V} \setminus \mathcal{W}$*,* $a \neq b$*, for which there exists a bifurcation in* $\mathcal{G}$ *of the form*

$$a \leftarrow w_0 \leftarrow \cdots \leftarrow w_{k-1} \leftrightarrow\!\!* \, w_k \to \cdots \to w_n \to b,$$

where all intermediate nodes $w_0, \ldots, w_n \in \mathcal{W}$ *(if any).*

A.2 σ-SEPARATION

Definition 17 (Colliders and Non-Colliders)**.** *Let* $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{L})$ *be a DMG, and let* π *be a walk between* v_0 *and* v_n *in* $\mathcal{G}$*. A node* v_k *on* π *is classified as follows:*

1. *A* non-collider *on* π *if there is at most one arrowhead pointing towards* v_k*, i.e., if* v_k *is an endpoint of* π *or of the form*

$$v_{k-1} *\!\!\rightarrow\!\!* \, v_k \to v_{k+1} \quad \textit{or} \quad v_{k-1} \leftarrow v_k *\!\!\leftarrow\!\!* \, v_{k+1}.$$

2. *A* collider *on* π *if there are two arrowheads pointing towards* v_k*, i.e., if* v_k *is of the form*

$$v_{k-1} *\!\!\rightarrow v_k \leftarrow\!\!* \, v_{k+1}.$$

Definition 18 (Blockable and Unblockable Non-Colliders)**.** *Let* $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{L})$ *be a DMG, and let* π *be a walk between* v_0 *and* v_n *in* $\mathcal{G}$*. A non-collider* v_k *on* π *is called an* unblockable non-collider *if it is not an endpoint and has only outgoing edges on* π *leading to nodes within the same strongly connected component of* $\mathcal{G}$*. Specifically,* v_k *follows one of the patterns:*

$$\begin{aligned} v_{k-1} \leftarrow v_k \leftrightarrow\!\!* \, v_{k+1}, &\quad \textit{where } v_{k-1} \in \textit{Sc}_{\mathcal{G}}(v_k), \\ v_{k-1} *\!\!\leftrightarrow v_k \to v_{k+1}, &\quad \textit{where } v_{k+1} \in \textit{Sc}_{\mathcal{G}}(v_k), \\ v_{k-1} \leftarrow v_k \to v_{k+1}, &\quad \textit{where } v_{k-1}, v_{k+1} \in \textit{Sc}_{\mathcal{G}}(v_k). \end{aligned}$$

Otherwise, v_k *is called a* blockable non-collider *on* π*, meaning that either:*

- v_k *is an endpoint, or*
- v_k *has at least one outgoing edge* $v_k \to v_{k\pm1}$ *where* $v_{k\pm1}$ *lies in a different strongly connected component than* v_k*, i.e.,* $v_{k\pm1} \notin \textit{Sc}_{\mathcal{G}}(v_k)$*.*

Definition 19 (σ-blocked Walks (Node Version))**.** *Let* $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{L})$ *be a DMG, and let* $Z \subseteq \mathcal{V}$ *be a subset of nodes. Consider a walk* π *between* v_0 *and* v_n *in* $\mathcal{G}$*. We define:*

1. *The walk* π *is* Z-σ-open *(or* σ-open given Z*) if and only if:*
   (a) *All colliders* v_k *on* π *belong to* $\textit{Anc}_{\mathcal{G}}(Z)$*, and*
   (b) *All blockable non-colliders* v_k *on* π *do not belong to* Z*.*
2. *The walk* π *is* Z-σ-blocked *(or* σ-blocked by Z*) if and only if:*
   (a) *There exists a collider* v_k *on* π *that is not in* $\textit{Anc}_{\mathcal{G}}(Z)$*, or*
   (b) *There exists a blockable non-collider* v_k *on* π *that belongs to* Z*.*

Instead of presenting the well-established concept of d-separation, originally introduced by Pearl [1985], we provide its non-trivial generalization, referred to as σ-separation.

Definition 20 (σ-separation (Node Version) [Forré and Mooij, 2017])**.** *Let* $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{L})$ *be a DMG, and let* $X, Y, Z \subseteq \mathcal{V}$ *be subsets of nodes. We define:*

1. X *is* σ-separated from Y given Z in $\mathcal{G}$, *denoted as*

$$X \overset{\sigma}{\underset{\mathcal{G}}{\perp}} Y \mid Z,$$

if every walk from a node in X *to a node in* Y *is* Z-σ-*blocked.*

2. *If this condition does not hold, we write:*

$$X \overset{\sigma}{\underset{\mathcal{G}}{\not\perp}} Y \mid Z.$$

Definition 21 (σ-separation (Segment Version) [Forré and Mooij, 2017]). *Let* $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{L})$ *be a DMG and* $X, Y, Z \subseteq \mathcal{V}$ *subsets of the nodes.*

1. *Consider a path in* $\mathcal{G}$ *with* $n \geq 1$ *nodes:*

$$v_0 \ast\!\!-\!\!\ast \cdots \ast\!\!-\!\!\ast v_n.$$

Then this path can be uniquely partitioned according to the strongly connected components of $\mathcal{G}$*:*

$$v_{i-1} \ast\!\!-\!\!\ast v_i \ast\!\!-\!\!\ast v_{i+1} \ast\!\!-\!\!\ast \cdots \ast\!\!-\!\!\ast v_{j-1} \ast\!\!-\!\!\ast v_j \ast\!\!-\!\!\ast v_{j+1},$$

with $v_i, \ldots, v_j \in Sc_{\mathcal{G}}(v_i)$ *and* $v_{i-1}, v_{j+1} \notin Sc_{\mathcal{G}}(v_i)$*. Note that* v_{i-1} *or* v_{j+1} *might not appear if* v_i *or* v_j *is an endpoint of the path. We will call the subpath* σ_k *(given by the nodes* $v_i, \ldots, v_j$ *and its corresponding edges) a segment of the path. We abbreviate the left an right endpoint of* σ_k *with* $\sigma_{k,l} = v_i$ *and* $\sigma_{k,r} = v_j$*. The path can then uniquely be written with its segments:*

$$\sigma_1 \ast\!\!-\!\!\ast \cdots \ast\!\!-\!\!\ast \sigma_m.$$

We will call σ_1 *and* σ_m *the end-segments of the path.*

2. *Such a path will be called* Z-σ-blocked *or* σ-blocked by Z *if:*
   (a) *at least one of the endpoints* $v_0 = \sigma_{1,l}$, $v_n = \sigma_{m,r}$ *is in* Z*, or*
   (b) *there is a segment* σ_k *with an outgoing directed edge in the path and its corresponding endpoint lies in* Z*, or*
   (c) *there is a segment* σ_k *with two adjacent edges that form a collider* $\ast\!\!\to \sigma_k \leftarrow\!\!\ast$ *and* $Sc_{\mathcal{G}}(\sigma_k) \cap Anc_{\mathcal{G}}(Z) = \emptyset$*.*

If none of the above holds then the path is called Z-σ-open *or* σ-open given Z.

3. *We say that* X *is* σ*-separated from* Y *given* Z *if every path in* $\mathcal{G}$ *with one endpoint in* X *and one endpoint in* Y *is* σ*-blocked by* Z*. In symbols this will be written as follows:*

$$X \overset{\sigma}{\underset{\mathcal{G}}{\perp}} Y \mid Z.$$

Lemma 25. *Let* $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{L})$ *be a DMG,* $Z \subseteq \mathcal{V}$*, and* π *is a* Z-σ*-open walk between* v_0 *and* v_n *in* $\mathcal{G}$*. Suppose* $v_i \in Sc_{\mathcal{G}}(v_j)$ *for some* $i, j \in \{0, \ldots, n\}$ *with* $i < j$*. If we then replace the subwalk* $v_i \ast\!\!-\!\!\ast \cdots \ast\!\!-\!\!\ast v_j$ *of* π *by*

1. *a shortest directed path* $v_i \to \cdots \to v_j$ *in* $\mathcal{G}$ *if* $j = n$ *or if* $v_j \to v_{j+1}$ *on* π*, or*
2. *a shortest directed path* $v_i \leftarrow \cdots \leftarrow v_j$ *in* $\mathcal{G}$ *otherwise,*

then this new subwalk is entirely within $Sc_{\mathcal{G}}(v_j)$ *and the modified walk* π' *is still* Z-σ*-open.*

Proof. π' cannot become Z-σ-blocked at one of the initial nodes $v_0, \ldots, v_{i-1}$ or at one of the final nodes $v_{j+1}, \ldots, v_n$ on π', since these nodes occur in the same local configuration on π and are not Z-σ-blocked on π by assumption. Furthermore, π' cannot become Z-σ-blocked at one of the nodes strictly between v_i and v_j on π' (if there are any), since these nodes are all non-endnode non-colliders that only point to nodes in the same strongly connected component $\mathrm{Sc}_{\mathcal{G}}(v_j)$. It is also worth noting that π' cannot become Z-σ-blocked at any of its endnodes, which could be v_i or v_j or both, because those are the same in π. So in the following we can w.l.o.g. assume that both v_i and v_j are non-endnodes of π and thus π'.

Case 1: By assumption v_j is in the subwalk $v_{j-1} \ast\!\!-\!\!\ast v_j \to v_{j+1}$ (or the right endnode) on π that is Z-σ-open. Since the same blocking criteria apply to v_j on π' it remains Z-σ-open on π'. If $v_i = v_j$ then also v_i is Z-σ-open on π' (if v_i is the left endnode or not). If $v_i \neq v_j$, then the new directed path $v_i \to \cdots \to v_j$ in π' is Z-σ-open at v_i because all nodes in between lie in the same strongly connected component $\mathrm{Sc}_{\mathcal{G}}(v_i)$ (or v_i is the left endnode anyways).

Case 2: Since case 1 is solved we can assume that we have $j < n$ with $v_j \leftrightarrow\!\!* \, v_{j+1}$ on π. If $v_{i-1} \leftrightarrow\!\!* \, v_i$ on π' (or v_i the left endnode) then this case is analogous to case 1. So we can also assume that we have $i > 0$ and $v_{i-1} *\!\!\to v_i$ on π. So π looks as follows:

$$\pi : \qquad \cdots v_{i-1} *\!\!\to v_i *\!\!-\!\!* \cdots *\!\!-\!\!* v_j \leftarrow\!\!* \, v_{j+1} \cdots .$$

So there must be a smallest number $k \in \{i, \ldots, j\}$ such that a collider appears at v_k on π:

$$\pi : \qquad \cdots v_{i-1} *\!\!\to v_i \to \cdots \to v_k \leftarrow\!\!* \cdots *\!\!-\!\!* v_j \leftarrow\!\!* \, v_{j+1} \cdots .$$

Since π is Z-σ-open we have $v_k \in \mathrm{Anc}_{\mathcal{G}}(Z)$. Since $v_i \in \mathrm{Anc}_{\mathcal{G}}(v_k)$ (otherwise v_k would not be the first collider appearing after v_i) we thus have that also $v_i \in \mathrm{Anc}_{\mathcal{G}}(Z)$. So if we replace the subwalk $v_i *\!\!-\!\!* \cdots *\!\!-\!\!* v_j$ of π by the shortest directed path $v_i \leftarrow \cdots \leftarrow v_j$ in $\mathcal{G}$ we then get for π' the following situation:

$$\pi' : \qquad \cdots v_{i-1} *\!\!\to v_i \leftarrow \cdots \leftarrow v_j \leftarrow\!\!* \, v_{j+1} \cdots ,$$

which is then Z-σ-open at v_i as $v_i \in \mathrm{Anc}_{\mathcal{G}}(Z)$. Note that this holds also when $v_i = v_j$. If $v_i \neq v_j$ then v_j is also Z-σ-open on π' as v_j points left to a node in the same strongly connected component as v_j.

So in all cases π' stays Z-σ-open. □

Proposition 4. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{L})$ be a DMG. For $Z \subseteq \mathcal{V}$, and $v_i, v_j \in \mathcal{V}$, the following are equivalent:*

1. *there exists a Z-σ-open path between v_i and v_j in $\mathcal{G}$;*
2. *there exists a Z-σ-open walk between v_i and v_j in $\mathcal{G}$.*

Proof. $1 \implies 2$ is trivial.

$2 \implies 1$: Let π be a $Z - \sigma$-open walk between v_0 and v_n in $\mathcal{G}$. If a node w occurs more than once on π, let v_i be the first node on π and v_j be the last node on π that are in $\mathrm{Sc}_{\mathcal{G}}(w)$. We now use Lemma 25 to construct a new walk π' from π by replacing the subwalk between v_i and v_j of π by a particular directed path in $\mathrm{Sc}_{\mathcal{G}}(w)$ between v_i and v_j in such a way that π' is still Z-σ-open. On π', the number of nodes that occurs more than once is at least one less than on π, and all nodes within $\mathrm{Sc}_{\mathcal{G}}(w)$ occur within a single segment. This replacement procedure can be repeated until no nodes occur more than once. We have then obtained a Z-σ-open path between v_0 and v_n. □

A.3 σ-MARKOV EQUIVALENCE

With selection nodes $\mathcal{S}$, a σ-inducing path generalizes the notion of inducing paths in DAGs and plays a crucial role in Definition 5.

Definition 22 (σ-inducing Paths [Mooij and Claassen, 2020])**.** *Let $\mathcal{G} = (\mathcal{V}^+, \mathcal{E}, \mathcal{L})$ be a DMG with nodes $\mathcal{V}^+ = \mathcal{V} \cup \mathcal{S}$ and let $v_i, v_j \in \mathcal{V}$ be distinct nodes. A walk π in $\mathcal{G}$ between v_i and v_j is called σ-*inducing given *$\mathcal{S}$ if each collider on π is in $Anc_{\mathcal{G}}(\{v_i, v_j\} \cup \mathcal{S})$, and each non-endpoint non-collider on π is unblockable. If it is a path, it is called a σ-*inducing path given *$\mathcal{S}$ between v_i and v_j.*

Proposition 5 (Proposition 1 in [Mooij and Claassen, 2020])**.** *Let $\mathcal{G} = (\mathcal{V}^+, \mathcal{E}, \mathcal{L})$ be a DMG with nodes $\mathcal{V}^+ = \mathcal{V} \cup \mathcal{S}$ and let $v_i, v_j \in \mathcal{V}$ be distinct nodes. Then the following are equivalent:*

1. *There is a σ-inducing path given $\mathcal{S}$ in $\mathcal{G}$ between v_i and v_j;*
2. *There is a σ-inducing walk given $\mathcal{S}$ in $\mathcal{G}$ between v_i and v_j;*
3. *$v_i \underset{\mathcal{G}}{\overset{\sigma}{\perp}} v_j \mid (Z \cup \mathcal{S})$ for all $Z \subseteq \mathcal{V} \backslash \{v_i, v_j\}$;*
4. *$v_i \underset{\mathcal{G}}{\overset{\sigma}{\perp}} v_j \mid (Z \cup \mathcal{S})$ for all $Z = (Anc_{\mathcal{G}}(\{v_i, v_j\} \cup \mathcal{S})) \backslash \{v_i, v_j\}$.*

Proof. The proof is similar to that of Theorem 4.2 in [Richardson and Spirtes, 2002].

$1 \implies 2$ is trivial.

$2 \implies 3$: Assume the existence of a σ-inducing walk given $\mathcal{S}$ between v_i and v_j in $\mathcal{G}$. Let $Z \subseteq \mathcal{V} \backslash \{v_i, v_j\}$. Consider all walks in $\mathcal{G}$ between v_i and v_j with the property that all colliders on it are in $\text{Anc}_{\mathcal{G}}(\{v_i, v_j\} \cup \mathcal{S} \cup Z)$, and each non-endpoint non-collider on it is not in $\mathcal{S} \cup Z$ or is unblockable. Such walks exist, since the σ-inducing walk is one. Let μ be such a walk with a minimal number of colliders. We show that all colliders on μ must be in $\text{Anc}_{\mathcal{G}}(\mathcal{S} \cup Z)$. Suppose on the contrary the existence of a collider v_k on μ that is not ancestor of $\mathcal{S} \cup Z$. It is either ancestor of v_i or of v_j, by assumption. If $v_j \in J$, it cannot be ancestor of v_j, and hence must be ancestor of v_i. Otherwise, we can assume it to be ancestor of v_i without loss of generality. Then there is a directed path π from v_k to v_i in $\mathcal{G}$ that does not pass through any node of $\mathcal{S} \cup Z$. Then the subwalk of μ between v_k and v_j can be concatenated with the directed path π into a walk between v_i and v_j that has the property, but has fewer colliders than μ: a contradiction. Therefore, μ is σ-open given $\mathcal{S} \cup Z$. Hence, v_i and v_j are σ-connected given $\mathcal{S} \cup Z$.

$3 \implies 4$ is trivial.

$4 \implies 1$: Suppose that v_i and v_j are σ-connected given $Z = (\text{Anc}_{\mathcal{G}}(\{v_i, v_j\} \cup \mathcal{S})) \backslash \{v_i, v_j\}$. Let π be a path between v_i and $\{v_j\}$ that is σ-open given Z. We show that π must be a σ-inducing path given $\mathcal{S}$. First, all colliders on π are in $\text{Anc}_{\mathcal{G}}(Z)$, and hence in $\text{Anc}_{\mathcal{G}}(\{v_i, v_j\} \cup \mathcal{S})$. Second, let v_k be any non-endpoint non-collider on π. Then there must be a directed subpath of π starting at v_k that ends either at the first collider on π next to v_k or at an end node of π, and hence v_k must be in Z. Since π is σ-open given Z, v_k must be unblockable. Hence, all non-endpoint non-colliders on π must be unblockable. □

Lemma 26. *Let $\mathcal{G} = (\mathcal{V}^+, \mathcal{E}, \mathcal{L})$ be a DMG with nodes $\mathcal{V}^+ = \mathcal{V} \cup \mathcal{S}$ and let $v_i, v_j \in \mathcal{V}$ be distinct nodes. If there exists a σ-inducing path given $\mathcal{S}$ between v_i and v_j in $\mathcal{G}$, and all σ-inducing paths given $\mathcal{S}$ in $\mathcal{G}$ between v_i and v_j are out of v_j, then $v_j \in \text{Anc}_{\mathcal{G}}(\{v_i\} \cup \mathcal{S})$.*

Proof. Let μ be a σ-inducing path given $\mathcal{S}$ between v_i and v_j in $\mathcal{G}$. It must be of the form $v_i \cdots v_k \leftarrow v_j$ (with possibly $v_k = v_i$). First we show that v_k cannot be in $\text{Sc}_{\mathcal{G}}(v_j)$. If $v_k \in \text{Sc}_{\mathcal{G}}(v_j)$, then let π be a directed path in $\mathcal{G}$ from v_k to v_j that is entirely contained in $\text{Sc}_{\mathcal{G}}(v_j)$. Let m be the node on μ closest to v_i that is also on π (possibly $m = v_k$). The subpath of π between v_j and m can be concatenated with the subpath of μ between m and v_i into a walk between v_j and v_i. This must be a σ-inducing path given $\mathcal{S}$ between v_i and v_j that is into v_j by construction: contradiction. Hence v_k cannot be in $\text{Sc}_{\mathcal{G}}(v_j)$.

If μ is a directed path all the way to v_i, then clearly, $v_j \in \text{Anc}_{\mathcal{G}}(\{v_i\} \cup \mathcal{S})$. Otherwise, it must contain a collider. Let v_l be the collider on μ closest to v_j. v_l must be ancestor of v_i or v_j or $\mathcal{S}$. In the first and third cases, clearly $v_j \in \text{Anc}_{\mathcal{G}}(\{v_i\} \cup \mathcal{S})$. In the second case, all nodes on the subpath of μ between v_j and v_l must be in $\text{Sc}_{\mathcal{G}}(v_j)$, a contradiction. □

Lemma 27. *Let $\mathcal{G} = (\mathcal{V}^+, \mathcal{E}, \mathcal{L})$ be a DMG with nodes $\mathcal{V}^+ = \mathcal{V} \cup \mathcal{S}$ and let $v_i, v_j \in \mathcal{V}$ be distinct nodes. If there exists a σ-inducing path given $\mathcal{S}$ between v_i and v_j in $\mathcal{G}$ into v_j, and $v_i \notin \text{Anc}_{\mathcal{G}}(\{v_j\} \cup \mathcal{S})$, then there exists a σ-inducing path given $\mathcal{S}$ between v_i and v_j in $\mathcal{G}$ that is both into v_i and into v_j.*

Proof. Let μ be a σ-inducing path given $\mathcal{S}$ between v_i and v_j in $\mathcal{G}$ into v_j. If μ is into v_i, we are done. Therefore, suppose it is of the form $v_i \to \cdots \ast\!\!\to v_j$. It cannot be a directed path, since $v_i \notin \text{Anc}_{\mathcal{G}}(\{v_j\} \cup \mathcal{S})$. Therefore, there must be a collider v_k on μ such that μ is of the form $v_i \to \cdots \to v_k \leftarrow\!\!\ast \cdots \ast\!\!\to v_j$ (with the subpath between v_i and v_k directed). Then $v_k \in \text{Anc}_{\mathcal{G}}(\{v_i\})$, and hence all nodes on μ between v_i and v_k must be in $\text{Sc}_{\mathcal{G}}(v_i)$. Let π be a directed path in $\mathcal{G}$ from v_k to v_i that is entirely contained in $\text{Sc}_{\mathcal{G}}(v_i)$. Let v_l be the node on μ closest to v_j that is also on π (possibly $v_l = v_k$). Then $v_l \neq v_j$, because otherwise $v_j \in \text{Sc}_{\mathcal{G}}(v_i)$, contradicting $v_i \notin \text{Anc}_{\mathcal{G}}(\{v_j\} \cup \mathcal{S})$. The non-trivial subpath of π between v_i and v_l can be concatenated with the non-trivial subpath of μ between v_l and v_j into a walk between v_i and v_j. This must be a σ-inducing path given $\mathcal{S}$ between v_i and v_j that is both into v_i and into v_j. □

Definition 23 (σ-Markov Equivalence Given $\mathcal{S}$)**.** *Two DMGs $\mathcal{G}_1, \mathcal{G}_2$ with the same node set $\mathcal{V}^+ = \mathcal{V} \cup \mathcal{S}$ are said to be* σ-Markov equivalent given $\mathcal{S}$ *if, for any three subsets of nodes $X, Y, Z \subseteq \mathcal{V}$, it holds that*

$$X \text{ is } \sigma\text{-separated from } Y \text{ given } Z \cup \mathcal{S} \text{ in } \mathcal{G}_1$$

if and only if

$$X \text{ is } \sigma\text{-separated from } Y \text{ given } Z \cup \mathcal{S} \text{ in } \mathcal{G}_2.$$

B PROOFS

B.1 PROOFS IN SECTION 3

Lemma 1 (Fundamental Property of σ-MAGs)**.** *Let $\mathcal{H}$ be a σ-MAG. If $\mathcal{H}$ contains a triple of the form $a \ast\!\!\rightarrow b — c$, then the edge between a and c is of the same type as the edge between a and b, and the neighborhoods of b and c are both complete.*

Proof. By the σ-completeness of σ-MAGs, a and c are adjacent in $\mathcal{H}$. Suppose the edge between a and b is $a \rightarrow b$. Since $\mathcal{H}$ is ancestral and contains a path $a \rightarrow b — c$, it follows that $\mathcal{H}$ does not contain an edge of the form $a \leftarrow\!\!\ast\, c$. Therefore, $\mathcal{H}$ must contain either $a \rightarrow c$ or $a — c$. If $\mathcal{H}$ contains the edge $a — c$, then from the path $b — c — a$, we conclude that $\mathcal{H}$ should not contain the edge $b \leftarrow\!\!\ast\, a$, which is a contradiction. Thus, the edge between a and c must be $a \rightarrow c$ as well.

Now we suppose the edge between a and b is $a \leftrightarrow b$.

1. If $\mathcal{H}$ contains the edge $a — c$, then from the path $b — c — a$, it follows that $\mathcal{H}$ would not contain an edge of the form $a \ast\!\!\rightarrow b$, which is a contradiction.
2. If $\mathcal{H}$ contains the edge $a \rightarrow c$, then from the path $a \rightarrow c — b$, it follows that $\mathcal{H}$ would not contain an edge of the form $b \ast\!\!\rightarrow a$, which is a contradiction.
3. If $\mathcal{H}$ contains the edge $a \leftarrow c$, then from the path $b — c \rightarrow a$, it follows that $\mathcal{H}$ would not contain an edge of the form $a \ast\!\!\rightarrow b$, which is a contradiction.

Thus, we conclude that $\mathcal{H}$ must contain the edge $a \leftrightarrow c$.

If $\mathcal{H}$ also contains $b — d$, then by the σ-maximality of σ-MAGs, c and d must be adjacent in $\mathcal{H}$. Since $\mathcal{H}$ contains both paths $d — b — c$ and $c — b — d$, it follows that $\mathcal{H}$ cannot contain an edge of the form $c \ast\!\!\rightarrow d$ or $d \ast\!\!\rightarrow c$. Thus, the edge between c and d must be $c — d$. Hence, we conclude that $\mathrm{Nbh}_{\mathcal{H}}(b)$ is complete. Similarly, since $\mathcal{H}$ also contains the triple $a \ast\!\!\rightarrow c — b$, as we have shown, it follows by symmetry that the neighborhood of c must also be complete. □

Lemma 2. *Let $\mathcal{H}$ be a σ-MAG. If $\mathcal{H}$ contains an anterior path that starts with a directed edge:*

$$v_0 \rightarrow v_1 \rightarrow\!\!\ast \cdots \rightarrow\!\!\ast\, v_n$$

for $n \geq 2$, then v_0 belongs to the ancestors of v_n in $\mathcal{H}$.

Proof. Start with v_2. If the edge between v_1 and v_2 is directed, then clearly $v_0 \in \mathrm{Anc}_{\mathcal{H}}(v_2)$. If instead $v_1 - v_2$ is present, then by Lemma 1, $\mathcal{H}$ also contains the directed edge $v_0 \rightarrow v_2$, given the triple $v_0 \rightarrow v_1 — v_2$. Now, suppose that for some $2 \leq k \leq n$, we have $v_0 \in \mathrm{Anc}_{\mathcal{H}}(v_k)$. If the edge between v_k and v_{k+1} is directed, then it directly follows that $v_0 \in \mathrm{Anc}_{\mathcal{H}}(v_{k+1})$. Since $v_0 \in \mathrm{Anc}_{\mathcal{H}}(v_k)$, there exists a directed path from v_0 to v_k in $\mathcal{H}$. Let the last edge on this path be $a \rightarrow v_k$. If the edge between v_k and v_{k+1} is undirected, then from the triple $a \rightarrow v_k — v_{k+1}$, it follows that $a \rightarrow v_{k+1}$. Consequently, $\mathcal{H}$ contains a directed path from v_0 to v_{k+1}, so we conclude that $v_0 \in \mathrm{Anc}_{\mathcal{H}}(v_{k+1})$. Thus, by induction, $v_0 \in \mathrm{Anc}_{\mathcal{H}}(v_n)$, completing the proof. □

Lemma 3. *Let $\mathcal{H}$ be a σ-MAG with nodes $\mathcal{V}$. Then, there exists a DMG $\mathcal{G}$ with nodes $\mathcal{V}^+ = \mathcal{V} \cup \mathcal{S}$, such that $\mathcal{H}$ represents $\mathcal{G}$ given $\mathcal{S}$.*

Proof. We construct the DMG $\mathcal{G}$ as follows. It has nodes $\mathcal{V}^+ = \mathcal{V} \cup \mathcal{S}$, where

$$\mathcal{S} = \{s_{\{a,b\}} : a — b \in \mathcal{H}, \mathrm{Nbh}_{\mathcal{H}}(a) \text{ or } \mathrm{Nbh}_{\mathcal{H}}(b) \text{ is incomplete}\}.$$

First, we include all (bi)directed edges $a \ast\!\!\rightarrow b$ for $a, b \in \mathcal{V}$ that are present in $\mathcal{H}$ as edges in $\mathcal{G}$. For undirected edges $a — b$ in $\mathcal{H}$, we treat them as follows: If the neighborhood of a or b is incomplete, we introduce a new node $s_{\{a,b\}}$ and add the edges $a \rightarrow s_{\{a,b\}} \leftarrow b$. On the other hand, if both of the neighborhoods of a and b are complete, we replace the undirected edge $a - b$ with the directed edges $a \rightarrow b$ and $a \leftarrow b$.

We need to show that $\mathcal{H}$ represents $\mathcal{G}$ given $\mathcal{S}$. For that we need to show:

1. Two distinct nodes $a, b \in \mathcal{V}$ are adjacent in $\mathcal{H}$ if and only if there exists a σ-inducing path between a and b given $\mathcal{S}$ in $\mathcal{G}$.

2. If $a \leftarrow\!\!* \, b$ in $\mathcal{H}$, then $a \notin \mathrm{Anc}_{\mathcal{G}}(\{b\} \cup \mathcal{S})$.

3. If $a \multimap b$ in $\mathcal{H}$, then $a \in \mathrm{Anc}_{\mathcal{G}}(\{b\} \cup \mathcal{S})$.

We now demonstrate that for $a \leftarrow\!\!* \, b \in \mathcal{H}$, it holds that $a \notin \mathrm{Anc}_{\mathcal{G}}(\mathcal{S})$. Assume, for the sake of contradiction, that $a \in \mathrm{Anc}_{\mathcal{G}}(\mathcal{S})$, which implies the existence of a shortest directed path from a to some $s \in \mathcal{S}$ in $\mathcal{G}$. Suppose this path has more than one edge, meaning it takes the form $a \to \cdots \to c \to d \to s$, where a and c might be the same, and all nodes except s on the path belong to $\mathcal{V}$. By the construction of $\mathcal{G}$, there exists a node $e \in \mathcal{V}$ such that $d \to s \leftarrow e$ with $s = s_{\{d,e\}}$ in $\mathcal{G}$, the edge $d - e$ is present in $\mathcal{H}$, and the neighborhood of d or e is incomplete. If $c \to d$ is also present in $\mathcal{H}$, then by Lemma 1, $\mathrm{Nbh}_{\mathcal{H}}(d)$ and $\mathrm{Nbh}_{\mathcal{H}}(e)$ must be complete, which is a contradiction. Thus, the edge between c and d must be undirected in $\mathcal{H}$. Moreover, by the construction of $\mathcal{G}$, we only have $c \to d$ in $\mathcal{G}$ given $c - d$ in $\mathcal{H}$ when the neighborhoods of c and d are complete. So, c and e are connected by an undirected edge in $\mathcal{H}$, and there exists a node $s_{\{c,e\}} \in \mathcal{S}$ such that $\mathcal{G}$ contains $c \to s_{\{c,e\}} \leftarrow e$ because $\mathrm{Nbh}_{\mathcal{H}}(e)$ is incomplete. This leads to a shorter directed path $a \to \cdots \to c \to s_{\{c,e\}}$ from a to $\mathcal{S}$, which is a contradiction. The only remaining possibility is that we have $a \to s$ in $\mathcal{G}$, where $s \in \mathcal{S}$. By the construction of $\mathcal{G}$, there exists a node $f \in \mathcal{V}$ such that $a \to s \leftarrow f$ in $\mathcal{G}$ with $s = s_{\{a,f\}}$, the edge $a - f$ is present in $\mathcal{H}$, and the neighborhood of a or f is incomplete. Combining this with the edge $b *\!\!\to a$ in $\mathcal{H}$, we obtain a triple $b *\!\!\to a - f$ in $\mathcal{H}$, which, by Lemma 1, implies that $\mathrm{Nbh}_{\mathcal{H}}(a)$ and $\mathrm{Nbh}_{\mathcal{H}}(f)$ must be complete, leading to a contradiction. Thus, we conclude that $a \notin \mathrm{Anc}_{\mathcal{G}}(\mathcal{S})$.

Next, we will show that for $a \leftarrow\!\!* \, b \in \mathcal{H}$, it holds that $a \notin \mathrm{Anc}_{\mathcal{G}}(b)$. Assume the contrary, that $a \in \mathrm{Anc}_{\mathcal{G}}(b)$. This implies the existence of a shortest directed path from a to b in $\mathcal{G}$, which corresponds to an anterior path from a to b in $\mathcal{H}$. However, $\mathcal{H}$ also contains the edge $b *\!\!\to a$, which contradicts the definition of σ-MAGs. Therefore, $a \notin \mathrm{Anc}_{\mathcal{G}}(b)$. Thus, we conclude that if $a \leftarrow\!\!* \, b \in \mathcal{H}$, then $a \notin \mathrm{Anc}_{\mathcal{G}}(\{b\} \cup \mathcal{S})$.

Now suppose $a \multimap b$ in $\mathcal{H}$. Then either $a - b$ in $\mathcal{H}$ or $a \to b$ in $\mathcal{H}$. In the first case, by construction of $\mathcal{G}$, a must be an ancestor of either b or $\mathcal{S}$ in $\mathcal{G}$. In the second case, by construction of $\mathcal{G}$, a must be an ancestor of b. Hence, $a \in \mathrm{Anc}_{\mathcal{G}}(\{b\} \cup \mathcal{S})$.

Finally, we consider adjacency. If two distinct nodes $v_0, v_n \in \mathcal{V}$ are adjacent in $\mathcal{H}$, then v_0, v_n must also be adjacent in $\mathcal{G}$, or there must be a triple $v_0 \to s_{v_0,v_n} \leftarrow v_n$ in $\mathcal{G}$. In both cases, there exists a σ-inducing path given $\mathcal{S}$ between v_0 and v_n in $\mathcal{G}$. Next, suppose $v_0, v_n \in \mathcal{V}$ are not adjacent in $\mathcal{H}$. This implies that there should not be a σ-inducing path between v_0 and v_n given $\mathcal{S}$ in $\mathcal{G}$. We assume the opposite and let π be a shortest σ-inducing path given $\mathcal{S}$ with the smallest collider distance sum to $\{v_0, v_n\} \cup \mathcal{S}$ between v_0 and v_n in $\mathcal{G}$. Notice that there is no other σ-inducing path μ between v_0 and v_n in $\mathcal{G}$ given $\mathcal{S}$ such that μ has fewer edges than π, or μ has the same number of edges but a smaller collider distance sum to $\{v_0, v_n\} \cup \mathcal{S}$ than π.

Suppose π contains a node from $\mathcal{S}$. Then π must include a subpath of the form $v_k \to s_{\{v_k,v_{k+1}\}} \leftarrow v_{k+1}$. In this case, v_k and v_{k+1} are non-colliders, and both are blockable. This implies that π reduces to $v_0 \to s_{\{v_0,v_n\}} \leftarrow v_n$, and $v_0 - v_n$ is an edge in $\mathcal{H}$, which contradicts the assumption that v_0 and v_n are not adjacent in $\mathcal{H}$. Hence, π must only contain nodes from $\mathcal{V}$.

Suppose v_k is a non-endpoint non-collider on π. W.O.L.G., assume $v_{k-1} *\!\!-\!\!* \, v_k \to v_{k+1}$ is present on π. By the definition of a σ-inducing path, we know that v_k and v_{k+1} belong to the same strongly connected component in $\mathcal{G}$, which implies that there exists a directed path from v_{k+1} to v_k in $\mathcal{G}$. If the edge $v_k \to v_{k+1}$ is present in $\mathcal{H}$, then by the second property we have established, $v_{k+1} \notin \mathrm{Anc}_{\mathcal{G}}(v_k)$, which leads to a contradiction. Therefore, the undirected edge $v_k - v_{k+1}$ must be present in $\mathcal{H}$.

If we have $v_{k-1} \leftarrow v_k \to v_{k+1}$ on π, then similarly, we obtain $v_{k-1} - v_k$ in $\mathcal{H}$. By the construction of $\mathcal{G}$, $\mathrm{Nbh}_{\mathcal{H}}(\{v_k\})$ is complete, implying that v_{k-1} and v_{k+1} are connected by an undirected edge in $\mathcal{H}$ and their neighborhoods are both complete. Consequently, we can replace the triple $v_{k-1} \leftarrow v_k \to v_{k+1}$ on π with $v_{k-1} \to v_{k+1}$. The new path remains σ-inducing since v_{k+1} retains the same edge mark, and v_{k-1} now serves as an unblockable non-collider on the new path, regardless of whether it was a collider or a non-collider on π. This is contradictory, as we obtain a shorter σ-inducing path than π.

If $v_{k-1} *\!\!\to v_k$ is present in $\mathcal{H}$, then by the construction of $\mathcal{G}$, we also have $v_{k-1} *\!\!\to v_k$ in $\mathcal{G}$, and by Lemma 1, $\mathcal{G}$ contains the edge $v_{k-1} *\!\!\to v_{k+1}$. Thus, we can replace the triple $v_{k-1} *\!\!\to v_k \to v_{k+1}$ on π with $v_{k-1} *\!\!\to v_{k+1}$, obtaining a shorter σ-inducing path than π, which is contradictory. Now, consider the last case where $v_{k-1} \to v_k$ on π and $v_{k-1} - v_k$ in $\mathcal{H}$. Similarly as for the case $v_{k-1} \leftarrow v_k \to v_{k+1}$ on π, v_{k-1} and v_{k+1} are connected by an undirected edge in $\mathcal{H}$ and their neighborhoods are both complete, so $\mathcal{G}$ contains the directed edge $v_{k-1} \to v_{k+1}$. Replacing $v_{k-1} \to v_k \to v_{k+1}$ on π with $v_{k-1} \to v_{k+1}$ yields a shorter σ-inducing path, which is contradictory. Thus, all non-end nodes on π must be colliders.

Given that π is a shortest σ-inducing path with the smallest collider distance to $\{v_0, v_n\} \cup \mathcal{S}$, all edges except possibly for the two end-edges on it are also present in $\mathcal{H}$, because they are bidirected. There are two cases:

1. Suppose π has more than one collider. Then the two end-edges must also be present in $\mathcal{H}$; otherwise, W.L.O.G., assume $v_0 - v_1 \leftrightarrow v_2$ appears in $\mathcal{H}$. By definition, this triple on π can be replaced with $v_0 \leftrightarrow v_2$, leading to a shorter σ-inducing path, contradicting minimality. Furthermore, each collider on π is not in $\mathrm{Anc}_{\mathcal{G}}(\mathcal{S})$ (as they have arrowheads pointing toward themselves in $\mathcal{H}$) but is in $\mathrm{Anc}_{\mathcal{G}}(\{v_0, v_n\})$. Thus, for any collider v_k in the triple $v_{k-1} \ast\!\!\rightarrow v_k \leftarrow\!\!\ast v_{k+1}$ on π, there exists a directed path from v_k to v_0 or v_n in $\mathcal{G}$. W.L.O.G., suppose there exists a shortest directed path $v_k \rightarrow q \rightarrow \cdots \rightarrow v_0$ from v_k to v_0 in $\mathcal{G}$. The corresponding path in $\mathcal{H}$ is $v_k -\!\!\ast\, q -\!\!\ast \cdots -\!\!\ast\, v_0$. If $v_k - q$ is present in $\mathcal{H}$, then by definition, $\mathcal{H}$ also contains the path $v_{k-1} \ast\!\!\rightarrow q \leftarrow\!\!\ast v_{k+1}$. Replacing the subpath of π between v_{k-1} and v_{k+1} with this path results in a σ-inducing path with a smaller collider distance sum to $\{v_i, v_j\} \cup \mathcal{S}$, which is contradictory. Thus, we have $v_k \rightarrow q$ in $\mathcal{H}$. Then, by Lemma 2, there exists a directed path from v_k to v_0 in $\mathcal{H}$. Notice that we also consider the case where there is a path consisting of a single edge $v_k \rightarrow v_0$ in $\mathcal{G}$. If $v_k - v_0$ is present in $\mathcal{H}$, then by Lemma 1 $\mathcal{H}$ also contains the edge $v_0 \leftarrow\!\!\ast v_{k+1}$, implying that the subpath of π from v_0 to v_{k+1} can be replaced by $v_0 \leftarrow\!\!\ast v_{k+1}$, contradicting the minimality of π. Hence, $v_k \rightarrow v_0$ must be present in $\mathcal{H}$.
2. Suppose π is $v_0 \ast\!\!\rightarrow v_1 \leftarrow\!\!\ast v_2$ with $v_n = v_2$. If either of the edges $v_0 \ast\!\!\rightarrow v_1$ or $v_1 \leftarrow\!\!\ast v_2$ is present in $\mathcal{H}$, then the other must also be present in $\mathcal{H}$, since if $v_0 - v_1$ or $v_1 - v_2$ is in $\mathcal{H}$, Lemma 1 ensures that v_0 and v_2 must be adjacent, which is contradictory. Moreover, if $v_0 - v_1 - v_2$ exists in $\mathcal{H}$, then by the construction of $\mathcal{G}$, v_0 and v_2 must be connected by an undirected edge in $\mathcal{H}$, again leading to a contradiction. Therefore, we only need to consider the case where the triple $v_0 \ast\!\!\rightarrow v_1 \leftarrow\!\!\ast v_2$ is explicitly present in $\mathcal{H}$. Additionally, by the second property, we have $v_1 \notin \mathrm{Anc}_{\mathcal{G}}(\{v_0, v_2\} \cup \mathcal{S})$, which contradicts the assumption that π is σ-inducing given $\mathcal{S}$.

Hence, π must contain more than one collider, and all colliders on π are ancestors of $\{v_0, v_n\}$ in $\mathcal{H}$. Consequently, π corresponds to an inducing path between v_0 and v_n in $\mathcal{H}$, contradicting the maximality of $\mathcal{H}$.

Consequently, we conclude that $\mathcal{H}$ represents $\mathcal{G}$ given $\mathcal{S}$. □

Lemma 4. *Let $\mathcal{H}$ be a σ-MAG that represents a DMG $\mathcal{G}$ given $\mathcal{S}$. Let a and b be distinct nodes in $\mathcal{H}$.*

1. *$a \in Ant_{\mathcal{H}}(b)$ implies $a \in Anc_{\mathcal{G}}(\{b\} \cup \mathcal{S})$.*
2. *If $\mathcal{H}$ contains the edge $a \ast\!\!\rightarrow b$, then there exists a σ-inducing path given $\mathcal{S}$ in $\mathcal{G}$ between a and b that is into b.*
3. *If $\mathcal{H}$ contains the edge $a \leftrightarrow b$, then there exists a σ-inducing path given $\mathcal{S}$ in $\mathcal{G}$ between a and b that is into both a and b.*

Proof. 1. Suppose $\mathcal{H}$ contains a directed path $a = v_0 -\!\!\ast \cdots -\!\!\ast\, v_n = b$. Note that for all $k = 0, 1, \ldots, n-1$, we have $v_k \in \mathrm{Anc}_{\mathcal{G}}(\{v_{k+1}\} \cup \mathcal{S})$. By induction, it follows that $a \in \mathrm{Anc}_{\mathcal{G}}(\{b\} \cup \mathcal{S})$.

2. Since a and b are adjacent in $\mathcal{H}$, and $\mathcal{H}$ represents $\mathcal{G}$ given $\mathcal{S}$, there exists a σ-inducing path given $\mathcal{S}$ between a and b in $\mathcal{G}$. Suppose, for contradiction, that all such σ-inducing paths given $\mathcal{S}$ between a and b in $\mathcal{G}$ are out of b. Then, by Lemma 26, it would follow that $b \in \mathrm{Anc}_{\mathcal{G}}(\{a\} \cup \mathcal{S})$, contradicting the orientation $a \ast\!\!\rightarrow b$ in $\mathcal{H}$. Therefore, there must exist a σ-inducing path given $\mathcal{S}$ between a and b in $\mathcal{G}$ that is into b.
3. Similarly, there exists a σ-inducing path given $\mathcal{S}$ between a and b in $\mathcal{G}$. Applying Lemma 27, if $a \leftrightarrow b$ is present in $\mathcal{H}$ so that $a \notin \mathrm{Anc}_{\mathcal{G}}(\{b\} \cup \mathcal{S})$, then there exists a σ-inducing path given $\mathcal{S}$ in $\mathcal{G}$ between a and b that is into both a and b.

□

Lemma 5. *Let $\mathcal{H}$ be a mixed graph with nodes $\mathcal{V}$ and edges $\mathcal{E}$ of the types $\{\rightarrow, \leftarrow, \leftrightarrow, -\}$. Let $\mathcal{G}$ be a DMG with nodes $\mathcal{V}^+ = \mathcal{V} \cup \mathcal{S}$. If $\mathcal{H}$ represents $\mathcal{G}$ given $\mathcal{S}$, then $\mathcal{H}$ is a σ-MAG.*

Proof. By the assumption that $\mathcal{H}$ represents $\mathcal{G}$ given $\mathcal{S}$, we know that between any two distinct nodes in $\mathcal{H}$, there is at most one edge, and no node is adjacent to itself.

Now we show that $\mathcal{H}$ is ancestral. Suppose $\mathcal{H}$ contains an anterior path from v_0 to v_n, namely $v_0 -\!\!\ast \cdots -\!\!\ast\, v_n$. By Lemma 4, $v_0 \in \mathrm{Anc}_{\mathcal{G}}(\{v_n\} \cup \mathcal{S})$. If $\mathcal{H}$ also contains an edge $v_n \ast\!\!\rightarrow v_0$, it would imply that $v_0 \notin \mathrm{Anc}_{\mathcal{G}}(\{v_n\} \cup \mathcal{S})$, which is a contradiction. Therefore, such edges cannot exist, meaning that $\mathcal{H}$ is ancestral.

We now continue to show that $\mathcal{H}$ is maximal, meaning that there is no inducing path between any two distinct non-adjacent nodes. Suppose, for contradiction, that there exists an inducing path π between two distinct non-adjacent nodes $v_0, v_n \in \mathcal{V}$. For every edge $v_k \ast\!\!-\!\!\ast v_{k+1}$ on π, where $k = 0, \ldots, n-1$, there exists a σ-inducing path μ_k between v_k and v_{k+1} given

$\mathcal{S}$ in $\mathcal{G}$. By Lemma 4, these σ-inducing paths can be chosen to be into v_k if the edge is $v_k \leftarrow\!\!* \; v_{k+1}$, into v_{k+1} if the edge is $v_k \; *\!\!\rightarrow v_{k+1}$, and both into v_k and v_{k+1} if the edge is $v_k \leftrightarrow v_{k+1}$. Now, concatenate all μ_k in the order of the edges in π to form a walk μ in $\mathcal{G}$ between v_0 and v_n. By definition, every non-endpoint node on π is a collider, and by the construction of μ, these nodes remain colliders on μ. Since all colliders on π belong to $\mathrm{Anc}_{\mathcal{H}}(\{v_0, v_n\})$, they must also be in $\mathrm{Anc}_{\mathcal{G}}(\{v_0, v_n\} \cup \mathcal{S})$ in $\mathcal{G}$. Similarly, all non-endpoint colliders on any μ_k are in $\mathrm{Anc}_{\mathcal{G}}(\{v_k, v_{k+1}\} \cup \mathcal{S})$ and, therefore, in $\mathrm{Anc}_{\mathcal{G}}(\{v_0, v_n\} \cup \mathcal{S})$. Thus, all colliders on μ are in $\mathrm{Anc}_{\mathcal{G}}(\{v_0, v_n\} \cup \mathcal{S})$. Additionally, all non-endpoint non-colliders on each μ_k are unblockable, meaning all non-endpoint non-colliders on μ are also unblockable. Hence, μ is a σ-inducing walk given $\mathcal{S}$ in $\mathcal{G}$. Consequently, by Proposition 5, there must exist a σ-inducing path given $\mathcal{S}$ in $\mathcal{G}$ between v_0 and v_n. Since $\mathcal{H}$ represents $\mathcal{G}$ given $\mathcal{S}$, it follows that v_0 and v_n must be adjacent in $\mathcal{H}$, contradicting our assumption.

The last step is to show that $\mathcal{H}$ is σ-complete. Suppose $\mathcal{H}$ contains a triple of the form $a \; *\!\!\rightarrow b - c$. Since $b \in \mathrm{Anc}_{\mathcal{G}}(\{c\} \cup \mathcal{S})$ and $b \notin \mathrm{Anc}_{\mathcal{G}}(\{a\} \cup \mathcal{S})$, we must have $b \in \mathrm{Anc}_{\mathcal{G}}(c)$. Moreover, we also know that $c \in \mathrm{Anc}_{\mathcal{G}}(\{b\} \cup \mathcal{S})$. Since $b \in \mathrm{Anc}_{\mathcal{G}}(c)$ and $b \notin \mathrm{Anc}_{\mathcal{G}}(\mathcal{S})$, it follows that $c \in \mathrm{Anc}_{\mathcal{G}}(b)$. Thus, b and c are in the same strongly connected component of $\mathcal{G}$. Next, we extend a σ-inducing path given $\mathcal{S}$ in $\mathcal{G}$ between a and b that is into b (which exists by Lemma 4) by appending the directed path from b to c within $\mathrm{Sc}_{\mathcal{G}}(b)$, thereby forming a σ-inducing walk given $\mathcal{S}$ between a and c. By Proposition 5, there exists a σ-inducing path given $\mathcal{S}$ between a and c, which implies that a and c are adjacent. Moreover, if there exists a node $d \in \mathcal{V}$ such that $b - d$ is present in $\mathcal{H}$, then by similar reasoning, from the triple $a \; *\!\!\rightarrow b - d$, we can infer that $d \in \mathrm{Sc}_{\mathcal{G}}(b)$, so c and d are in the same strongly connected component in $\mathcal{G}$. Thus, there exists a directed path from c to d in $\mathcal{G}$, which is also a σ-inducing path given $\mathcal{S}$, implying that c and d are adjacent. Therefore, we conclude that $\mathcal{H}$ is σ-complete. □

Theorem 1. *Let $\mathcal{H}$ be a mixed graph with nodes $\mathcal{V}$ and edges $\mathcal{E}$ of the types $\{\rightarrow, \leftarrow, \leftrightarrow, -\}$. The following equivalence holds:*

$$\begin{aligned}&\textit{There exists a DMG } \mathcal{G} \textit{ with nodes } \mathcal{V}^+ = \mathcal{V} \cup \mathcal{S},\\ &\textit{such that } \mathcal{H} \textit{ represents } \mathcal{G} \textit{ given } \mathcal{S}.\\ &\iff \mathcal{H} \textit{ is a } \sigma\textit{-MAG}.\end{aligned}$$

Proof. Obviously obtained by Lemma 3 and Lemma 5. □

B.2 PROOFS IN SECTION 4

Proposition 1. *Let $\mathcal{H}$ be a σ-MAG. For $Z \subseteq \mathcal{V}$, and $a, b \in \mathcal{V}$, the following are equivalent:*

1. *there exists a Z-m-open walk between a and b in $\mathcal{H}$;*
2. *there exists a Z-m-open path between a and b in $\mathcal{H}$.*

Proof. $1 \implies 2$: Suppose π is a Z-m-open walk between $v_0 = a$ and $v_n = b$ in $\mathcal{H}$. If a node v appears more than once on π, let $v_i = v$ be its first occurrence and $v_j = v$ its last occurrence on π. Consider removing the subwalk between v_i and v_j by replacing the subwalk

$$v_{i-1} *\!\!-\!\!* v_i *\!\!-\!\!* v_{i+1} *\!\!-\!\!* \cdots *\!\!-\!\!* v_{j-1} *\!\!-\!\!* v_j *\!\!-\!\!* v_{j+1}$$

(where v_{i-1} and v_{j+1} may not exist if v_i or v_j is an endpoint) with the shorter path $v_{i-1} *\!\!-\!\!* v *\!\!-\!\!* v_{j+1}$.

1. Suppose one of v_i and v_j is an endpoint of π; without loss of generality, let $v_i = v_0 = v$ be the starting node of π. Then the subwalk

$$v_0 *\!\!-\!\!* v_1 *\!\!-\!\!* \cdots *\!\!-\!\!* v_{j-1} *\!\!-\!\!* v_j *\!\!-\!\!* v_{j+1}$$

contains a repeated node $v = v_0 = v_j$. We can simplify this by replacing the entire subwalk from v_0 to v_j with the final edge of the subwalk, namely:

$$v_0 = v_j *\!\!-\!\!* v_{j+1}.$$

This replacement preserves the edge marks at v_{j+1}. Moreover, since v_0 is a non-collider and $v \notin Z$, the resulting walk remains Z-m-open.

2. Assume v_i and v_j are non-endpoint nodes on π.

(a) If both v_i and v_j are colliders on π, then we can simplify the walk by replacing the subwalk between v_{i-1} and v_{j+1} with the shorter walk

$$v_{i-1} \ast\!\!\rightarrow v_i = v_j \leftarrow\!\!\ast v_{j+1}.$$

The resulting walk remains Z-m-open since the repeated node v is still a collider in the new path and, by assumption, is in $\mathrm{Anc}_{\mathcal{H}}(Z)$.

(b) If v_i and v_j are both non-colliders on π, then we can attempt a similar simplification using the shorter walk $v_{i-1} \ast\!\!-\!\!\ast v_i = v_j \ast\!\!-\!\!\ast v_{j+1}$. If the repeated node v remains a non-collider on the new path and $v \notin Z$, then the new walk appears to be Z-m-open. However, we must account for the possibility that the shorter walk takes the form $v_{i-1} - v_i = v_j \leftarrow\!\!\ast v_{j+1}$ or $v_{i-1} \ast\!\!\rightarrow v_i = v_j - v_{j+1}$, which could block the path due to the edge marks. In such cases, a more careful analysis is needed to ensure the resulting walk remains Z-m-open. Without loss of generality, suppose the first case holds, i.e., $v_{i-1} - v_i = v_j \leftarrow\!\!\ast v_{j+1}$ exists. Then the original walk π must contain the subwalk

$$v_{k-1} \leftarrow v_k - \cdots - v_{i-1} - v_i -\!\!\ast \cdots \leftarrow v_j \leftarrow\!\!\ast v_{j+1},$$

where v_{k-1} may not exist if $v_k = v_0$ (i.e., the walk starts at v_k), and possibly $v_k = v_{i-1}$. By Lemma 1, since $\mathcal{H}$ contains the undirected walk $v_k - \cdots - v_i = v_j$, it must also contain the edge $v_k \leftarrow\!\!\ast v_{j+1}$. Therefore, we can simplify π by replacing the subwalk between v_k and v_{j+1} with the edge $v_k \leftarrow\!\!\ast v_{j+1}$. This results in a shorter walk that remains Z-m-open, because v_k, being a non-collider on both the original and new walks, is not in Z, and the edge marks at other nodes are preserved.

Now consider another special case in which the repeated node v becomes a collider on the new path, if we replace the subwalk of π between v_{i-1} and v_{j+1} with $v_{i-1} \ast\!\!-\!\!\ast v_i = v_j \ast\!\!-\!\!\ast v_{j+1}$. In this case, we must have:

$$v_{i-1} \ast\!\!\rightarrow v_i \rightarrow v_{i+1} \quad \text{and} \quad v_{j-1} \leftarrow v_j \leftarrow\!\!\ast v_{j+1}.$$

Then, the original walk π must contain a directed subwalk from v_i to some collider v_l for some $i < l < j$ such that $v_l \in \mathrm{Anc}_{\mathcal{H}}(Z)$. Consequently, v_i (and thus v_j) is also an ancestor of Z. Therefore, the repeated node v satisfies the collider condition for m-openness, and the new path remains Z-m-open.

(c) If v_i and v_j have different collider statuses on π, without loss of generality, assume v_i is a non-collider and v_j is a collider on π. Then we have either

$$v_{i-1} \ast\!\!-\!\!\ast v_i -\!\!\ast v_{i+1} \quad \text{or} \quad v_{i-1} \ast\!\!- v_i \ast\!\!-\!\!\ast v_{i+1},$$

and

$$v_{j-1} \ast\!\!\rightarrow v_j \leftarrow\!\!\ast v_{j+1}.$$

We attempt to simplify the walk by replacing the subwalk of π between v_{i-1} and v_{j+1} with the shorter walk $v_{i-1} \ast\!\!-\!\!\ast v_i = v_j \leftarrow\!\!\ast v_{j+1}$. This replacement preserves Z-m-openness because the repeated node v (i.e., $v_i = v_j$) was a non-collider at one occurrence and a collider at the other, satisfying both conditions: $v \notin Z$ and $v \in \mathrm{Anc}_{\mathcal{H}}(Z)$. However, if $\mathcal{H}$ contains the path

$$v_{i-1} - v_i = v_j \leftarrow\!\!\ast v_{j+1},$$

then the new walk would be blocked. In this case, similarly to the earlier scenario where both v_i and v_j are non-colliders, there exists a node v_k such that the subwalk of π from v_k to v_i is undirected:

$$v_k - \cdots - v_i = v_j,$$

and by Lemma 1, $\mathcal{H}$ must contain the edge $v_k \leftarrow\!\!\ast v_{j+1}$. Therefore, we can replace the entire subwalk of π from v_k to v_{j+1} with the single edge $v_k \leftarrow\!\!\ast v_{j+1}$, preserving the Z-m-openness of the walk.

This replacement procedure can be iteratively applied until no node occurs more than once on the walk. And each replacement preserves Z-m-openness, so the final path remains Z-m-open. The resulting walk is then a Z-m-open *path* between v_0 and v_n.

$2 \implies 1$ is trivial since paths are walks. ☐

Lemma 6. *Let $\mathcal{G}$ be a DMG with nodes $\mathcal{V}^+ = \mathcal{V} \cup \mathcal{S}$, and let $\mathcal{H}$ be a σ-MAG that represents $\mathcal{G}$ given $\mathcal{S}$. Let $a, b \in \mathcal{V}$ and $Z \subseteq \mathcal{V}$. If π is a m-open path given Z between a and b in $\mathcal{H}$, then every node on π is in $\mathrm{Anc}_{\mathcal{G}}(\{a, b\} \cup Z \cup \mathcal{S})$.*

Proof. W.L.O.G., assume $Z \cap \{a, b\} = \emptyset$. Suppose π is in the form $a = v_0 \ast\!\!-\!\!\ast \cdots \ast\!\!-\!\!\ast v_n = b$, and let v_k be an arbitrary node on the path π. We will prove $v_k \in \mathrm{Anc}_{\mathcal{G}}(\{v_0, v_n\} \cup Z \cup \mathcal{S})$ based on its position and properties.

If v_k is an endpoint, then $v_k \in \{v_0, v_n\}$, which is straightforward.

If v_k is a collider, it follows that $v_k \in \mathrm{Anc}_{\mathcal{H}}(Z)$. Consequently, $v_k \in \mathrm{Anc}_{\mathcal{G}}(Z \cup \mathcal{S})$.

Now consider the case where v_k is a non-collider. Without loss of generality, assume the subpath around v_k takes the form

$$v_0 \ast\!\!-\!\!\ast \cdots \ast\!\!-\!\!\ast v_{k-1} \ast\!\!-\!\!\ast v_k \multimap v_{k+1} \ast\!\!-\!\!\ast \cdots \ast\!\!-\!\!\ast v_n.$$

If there is no edge of the form $\leftarrow\!\!\ast$ on the subpath of π between v_k and v_n, then $v_k \in \mathrm{Anc}_{\mathcal{G}}(\{v_n\} \cup \mathcal{S})$.

Suppose there exists such an edge, with the corresponding node v_l in the configuration $v_{l-1} \multimap v_l \leftarrow\!\!\ast v_{l+1}$ on the subpath of π between v_k and v_n. In this case, $v_k \in \mathrm{Anc}_{\mathcal{G}}(\{v_l\} \cup \mathcal{S})$. Since π is Z-m-open, we have $v_{l-1} \to v_l \leftarrow\!\!\ast v_{l+1}$. Then v_l is a collider, so $v_l \in \mathrm{Anc}_{\mathcal{H}}(Z)$, implying $v_k \in \mathrm{Anc}_{\mathcal{G}}(Z \cup \mathcal{S})$.

From these cases, we conclude that every node on the path π belongs to $\mathrm{Anc}_{\mathcal{G}}(\{v_0, v_n\} \cup Z \cup \mathcal{S})$. □

Lemma 7. *Let $\mathcal{G}$ be a DMG with nodes $\mathcal{V}^+ = \mathcal{V} \cup \mathcal{S}$, and let $\mathcal{H}$ be a σ-MAG that represents $\mathcal{G}$ given $\mathcal{S}$. Let $a, b \in \mathcal{V}$ and $Z \subseteq \mathcal{V}$. If there exists a Z-m-open path between a and b in $\mathcal{H}$, then there exists a $(Z \cup \mathcal{S})$-σ-open path in $\mathcal{G}$ between a and b.*

Proof. W.L.O.G., assume $Z \cap \{a, b\} = \emptyset$. Let π be a Z-m-open path between a and b in $\mathcal{H}$ as follows:

$$a = v_0 \ast\!\!-\!\!\ast v_1 \ast\!\!-\!\!\ast \cdots \ast\!\!-\!\!\ast v_{n-1} \ast\!\!-\!\!\ast v_n = b.$$

For each $i = 1, \ldots, n$, let μ_i be a σ-inducing path given $\mathcal{S}$ in $\mathcal{G}$ between v_{i-1} and v_i, where μ_i is into v_{i-1} if $v_{i-1} \leftarrow\!\!\ast v_i$ on π, and into v_i if $v_{i-1} \ast\!\!\rightarrow v_i$ on π (Lemma 4 guarantees this). Concatenating all paths $(\mu_i)_{i=1,\ldots,n}$ gives a walk μ.

We aim to show that there exists a walk in $\mathcal{G}$ between v_0 and v_n that satisfies the following properties:

1. All colliders on the walk are in $\mathrm{Anc}_{\mathcal{G}}(\{v_0, v_n\} \cup Z \cup \mathcal{S})$.
2. All non-colliders on the walk are either not in $Z \cup \mathcal{S}$ or are unblockable.

Such a walk exists because μ satisfies these properties, as we now verify.

By Lemma 6, $v_i \in \mathrm{Anc}_{\mathcal{G}}(\{v_0, v_n\} \cup Z \cup \mathcal{S})$ for all $i = 1, \ldots, n$. This holds in particular for all v_i that are colliders on μ. Furthermore, every non-endpoint collider on μ_i is in $\mathrm{Anc}_{\mathcal{G}}(\{v_{i-1}, v_i\} \cup \mathcal{S})$, and hence also in $\mathrm{Anc}_{\mathcal{G}}(\{v_0, v_n\} \cup Z \cup \mathcal{S})$. For non-colliders, observe that all non-endpoint non-colliders on μ_i are unblockable. Now consider v_i that are non-colliders on μ. By construction, such v_i are also non-colliders on π. Since π is Z-m-open, we have $v_i \notin Z$, and obviously $v_i \notin \mathcal{S}$. Therefore, all non-colliders are either not in $Z \cup \mathcal{S}$ or are unblockable.

Let ν be a walk satisfying the above properties, with the minimal number of colliders, and such that v_0 and v_n appear only once on ν. We claim that all colliders on ν must be in $\mathrm{Anc}_{\mathcal{G}}(Z \cup \mathcal{S})$.

Suppose, for contradiction, that there exists a collider v_k on ν such that $v_k \notin \mathrm{Anc}_{\mathcal{G}}(Z \cup \mathcal{S})$ but $v_k \in \mathrm{Anc}_{\mathcal{G}}(\{v_0, v_n\})$. Then there must exist a directed path from v_k to v_0 that does not pass through v_n, or a directed path from v_k to v_n that does not pass through v_0. Without loss of generality, assume the former. Notice that every node on the directed path from v_k to v_0 is not in $Z \cup \mathcal{S}$. Replacing the subpath of ν from v_k to v_0 with this directed path would result in a walk with fewer colliders between v_0 and v_n that still satisfies the conditions, contradicting the minimality of ν.

Thus, all colliders on ν must be in $\mathrm{Anc}_{\mathcal{G}}(Z \cup \mathcal{S})$. Consequently, ν is a σ-open walk given $Z \cup \mathcal{S}$ in $\mathcal{G}$ between v_0 and v_n. Therefore, by Proposition 4, there must exist a $Z \cup \mathcal{S}$-σ-open path between a and b in $\mathcal{G}$. □

Lemma 8. *Let $\mathcal{G}$ be a DMG with nodes $\mathcal{V}^+ = \mathcal{V} \cup \mathcal{S}$, and let $\mathcal{H}$ be a σ-MAG that represents $\mathcal{G}$ given $\mathcal{S}$. Let $a, b \in \mathcal{V}$ and $Z \subseteq \mathcal{V}$. If there exists a $(Z \cup \mathcal{S})$-σ-open path between a and b in $\mathcal{G}$, then there exists a Z-m-open path in $\mathcal{H}$ between a and b.*

Proof. W.L.O.G., assume $Z \cap \{a, b\} = \emptyset$. This proof follows the approach used in proving Lemma 18 from [Spirtes and Richardson, 1996] and incorporates the segment-based σ-separation introduced in [Forré and Mooij, 2017] (Definition 21).

To begin, given the strongly connected components of $\mathcal{G}$, we pick a σ-open path π conditioned on $Z \cup \mathcal{S}$, uniquely expressed in segment form as:

$$\sigma_0 \mathrel{\ast\!\!-\!\!\ast} \sigma_1 \mathrel{\ast\!\!-\!\!\ast} \cdots \mathrel{\ast\!\!-\!\!\ast} \sigma_{n-1} \mathrel{\ast\!\!-\!\!\ast} \sigma_n,$$

where $\sigma_{0,l} = a$ and $\sigma_{n,r} = b$. By the definition of segment-based σ-separation:

1. $a, b \notin Z \cup \mathcal{S}$.
2. For all non-collider segments, their endpoints corresponding to outgoing directed edges are not in $Z \cup \mathcal{S}$.
3. For all collider segments, their nodes intersect non-trivially with $\text{Anc}_{\mathcal{G}}(Z \cup \mathcal{S})$.

We construct a sequence of nodes Q_0 based on the segments of π. Initialize $Q_0(1) = a$. For each segment σ_i: If σ_i is a non-collider segment, include its endpoints corresponding to outgoing directed edges in Q_0. If σ_i is a collider segment and its intersection with $\text{Anc}_{\mathcal{G}}(\mathcal{S})$ is empty, add an arbitrary endpoint from its ends to Q_0. Finally, add b to Q_0 if it is not already included, and set $Q_0(m) = b$.

We now show that for $i = 1, \ldots, m-1$, the nodes $Q_0(i)$ and $Q_0(i+1)$ are adjacent in $\mathcal{H}$. By construction, the subpath of π between $Q_0(i)$ and $Q_0(i+1)$ includes only unblockable non-colliders or colliders that are ancestors of $\mathcal{S}$. Such a path is σ-inducing given $\mathcal{S}$, which guarantees that $Q_0(i)$ and $Q_0(i+1)$ are adjacent in $\mathcal{H}$. Consequently, Q_0 forms a path π_0 from a to b in $\mathcal{H}$:

$$a = Q_0(1) \mathrel{\ast\!\!-\!\!\ast} \cdots \mathrel{\ast\!\!-\!\!\ast} Q_0(m) = b.$$

Next, we show that if $Q_0(i)$ originates from a non-collider segment on π, it remains a non-collider on π_0. Without loss of generality, assume $Q_0(i)$ is the right endpoint of a non-collider segment, with an outgoing directed edge to the right. In this case, $Q_0(i)$ is either an ancestor of the right endpoint of the next non-collider segment or a collider segment's endpoint, which belongs to $\text{Anc}_{\mathcal{G}}(Z \cup \mathcal{S})$. Thus, $Q_0(i)$ and $Q_0(i+1)$ are connected by an edge $Q_0(i) \mathrel{-\!\!\ast} Q_0(i+1)$ in $\mathcal{H}$. Moreover, if $Q_0(i) \notin \text{Anc}_{\mathcal{G}}(\mathcal{S})$, meaning that $Q_0(i)$ has a directed edge into the segment of π containing $Q_0(i+1)$, then $Q_0(i+1)$ cannot be an ancestor of $Q_0(i)$, as they lie in different strongly connected components. Similarly, suppose $Q_0(i)$ is the left endpoint of a non-collider segment on π, with an outgoing directed edge $\leftarrow$ to the left. Then $Q_0(i-1) \mathrel{\ast\!\!-} Q_0(i)$ holds in $\mathcal{H}$. Moreover, if $Q_0(i) \notin \mathcal{S}$, it follows that $Q_0(i-1) \notin \text{Anc}_{\mathcal{G}}(Q_0(i))$.

For nodes $Q_0(i)$ originating from collider segments, we cannot guarantee that $Q_0(i)$ will remain a collider on π_0. To resolve this, we employ the following algorithm to remove problematic nodes from the sequence Q_0:

$j \leftarrow 0$
repeat
    $I \leftarrow \{0 < i < m \colon Q_j(i) \in \text{Anc}_{\mathcal{G}}(\{Q_j(i-1), Q_j(i+1)\})$ and $Q_j(i)$ is from a collider segment on $\pi\}$
    if $I \neq \emptyset$ **then**
        form sequence Q_{j+1} from Q_j by removing some $Q_j(i)$ with $i \in I$
        $j \leftarrow j+1, m \leftarrow m-1$
    end if
until $I = \emptyset$
$k \leftarrow j$

We now show that, in each intermediate sequence Q_j, every pair of consecutive nodes are adjacent in $\mathcal{H}$, thus forming a valid path π_j in $\mathcal{H}$. This property has already been shown for Q_0. Now, assume the property holds for Q_j, and consider the $(j+1)$-th step in which the node $Q_j(i)$ is removed. In this case, we know that $Q_j(i) \in \text{Anc}_{\mathcal{G}}(\{Q_j(i-1), Q_j(i+1)\})$. The subpath of π between $Q_j(i-1)$ and $Q_j(i+1)$ is σ-inducing given $\mathcal{S}$, since $Q_j(i)$ is either a collider in $\text{Anc}_{\mathcal{G}}(\{Q_j(i-1), Q_j(i+1)\})$, or an unblockable non-collider on π, and all other nodes on this subpath are either unblockable non-colliders or colliders in $\text{Anc}_{\mathcal{G}}(\{Q_j(i-1), Q_j(i+1)\} \cup \mathcal{S})$. Consequently, we conclude that $Q_{j+1}(i-1) \equiv Q_j(i-1)$ and $Q_{j+1}(i) \equiv Q_j(i+1)$ are adjacent in $\mathcal{H}$.

We now show that if $Q_k(i)$ comes from a non-collider segment on π, it remains a non-collider on π_k. Moreover, if $Q_k(i) \notin \text{Anc}_{\mathcal{G}}(\mathcal{S})$, then $Q_k(i+1) \notin \text{Anc}_{\mathcal{G}}(Q_k(i))$ if it is the right endpoint with edge $\rightarrow$, and $Q_k(i-1) \notin \text{Anc}_{\mathcal{G}}(Q_k(i))$ if it is the left endpoint with edge $\leftarrow$. This has been proved for Q_0. Suppose the result holds for Q_j. Then, if $Q_j(i)$ is from a non-collider segment on π, it remains a non-collider on π_j. Furthermore, $Q_j(i)$ retains the same edge marks on π_{j+1}, unless $Q_j(i) \mathrel{-\!\!\ast} Q_j(i+1)$ is present on π_j and $Q_j(i+1)$ is removed at the $(j+1)$-th step, or $Q_j(i-1) \mathrel{\ast\!\!-} Q_j(i)$ and $Q_j(i-1)$ is removed at the $(j+1)$-th step. Without loss of generality, we assume the former case. Since $Q_j(i) \mathrel{-\!\!\ast} Q_j(i+1)$ is in $\mathcal{H}$, $Q_j(i) \in \text{Anc}_{\mathcal{G}}(\{Q_j(i+1)\} \cup \mathcal{S})$. If $Q_j(i)$ is an ancestor of $\mathcal{S}$, it is obviously a non-collider on π_{j+1}. Therefore, we need only consider the case where $Q_j(i) \notin \mathrm{Anc}_{\mathcal{G}}(\mathcal{S})$ but $Q_j(i) \in \mathrm{Anc}_{\mathcal{G}}(Q_j(i+1))$. Since $Q_j(i+1)$ is removed at the $(j+1)$-th step, we know that $Q_j(i+1)$ is an ancestor of either $Q_j(i)$ or $Q_j(i+2)$. Given that $Q_j(i+1) \notin \mathrm{Anc}_{\mathcal{G}}(Q_j(i))$, we conclude that $Q_j(i+1) \in \mathrm{Anc}_{\mathcal{G}}(Q_j(i+2))$. Thus, $Q_{j+1}(i) \equiv Q_j(i)$ is also an ancestor of $Q_{j+1}(i+1) \equiv Q_j(i+2)$, meaning $Q_{j+1}(i)$ is a non-collider on π_{j+1}. Additionally, if $Q_{j+1}(i) \notin \mathrm{Anc}_{\mathcal{G}}(\mathcal{S})$, we have $Q_{j+1}(i+1) \notin \mathrm{Anc}_{\mathcal{G}}(Q_{j+1}(i))$. Otherwise, this would imply that $Q_j(i+1) \in \mathrm{Anc}_{\mathcal{G}}(Q_j(i))$, which leads to a contradiction.

Given that all nodes from non-collider segments on π remain non-colliders on π_k and lie outside $Z \cup \mathcal{S}$, we now consider nodes on π_k originating from collider segments. For any such node $Q_k(i)$, we have $Q_k(i) \notin \mathrm{Anc}_{\mathcal{G}}(\{Q_k(i-1), Q_k(i+1)\})$; otherwise, it would have been removed by the algorithm. Moreover, since all such $Q_k(i)$ lie in collider segments on π, we have $Q_k(i) \in \mathrm{Anc}_{\mathcal{G}}(Z) \setminus \mathrm{Anc}_{\mathcal{G}}(\mathcal{S})$. Therefore, $Q_k(i)$ must be a collider on π_k, and in particular, satisfies $Q_k(i) \in \mathrm{Ant}_{\mathcal{H}}(Z)$.

So far, we have shown that all non-colliders on π_k are not in Z. To prove that π_k is Z-m-open, we only need to check all potentially present subpaths of π_k in the form of $Q_k(i-1) \ast\!\!\rightarrow Q_k(i) - Q_k(i+1)$ or $Q_k(i-1) - Q_k(i) \leftarrow\!\!\ast\, Q_k(i+1)$. Suppose it contained a subpath of the form $Q_k(i-1) \ast\!\!\rightarrow Q_k(i) - Q_k(i+1)$. Then $Q_k(i), Q_k(i+1)$ must be in the same strongly connected component in $\mathcal{G}$, and both are not in $\mathrm{Anc}_{\mathcal{G}}(\mathcal{S})$. Since $Q_k(i)$ lies in a non-collider segment on π, but we have shown that if $Q_k(i) \notin \mathrm{Anc}_{\mathcal{G}}(\mathcal{S})$, then $Q_k(i+1) \notin \mathrm{Anc}_{\mathcal{G}}(Q_k(i))$, this leads to a contradiction. One can show in a similar way that it cannot contain a subpath $Q_k(i-1) - Q_k(i) \leftarrow\!\!\ast\, Q_k(i+1)$ either.

We now have shown the existence of a path π_k in $\mathcal{H}$ between a and b that almost qualifies for being m-open given Z, except that some of its colliders may not be in $\mathrm{Anc}_{\mathcal{H}}(Z)$ (yet all colliders are in $\mathrm{Ant}_{\mathcal{H}}(Z)$. By Lemma 1, there exist nodes in $\mathcal{H}$ that lie in $\mathrm{Anc}_{\mathcal{H}}(Z)$, allowing us to replace the problematic colliders on π_k with these nodes while preserving their collider status on the path. As a result, we obtain an m-open path π'_k between a and b in $\mathcal{H}$. ☐

Theorem 2. *Let $\mathcal{G}$ be a DMG with nodes $\mathcal{V}^+ = \mathcal{V} \cup \mathcal{S}$, and let $\mathcal{H}$ be a σ-MAG that represents $\mathcal{G}$ given $\mathcal{S}$. Let $X, Y, Z \subseteq \mathcal{V}$ be subsets of the nodes. We have the following equivalence:*

$$X \underset{\mathcal{H}}{\overset{m}{\perp}} Y \mid Z \qquad \Longleftrightarrow \qquad X \underset{\mathcal{G}}{\overset{\sigma}{\perp}} Y \mid Z \cup \mathcal{S}.$$

Proof. Obviously obtained by Lemma 7 and Lemma 8. ☐

Proposition 2. *Let $\mathcal{H}$ be a σ-MAG with nodes $\mathcal{V}$ and let $a, b \in \mathcal{V}$ be distinct nodes. Then the following are equivalent:*

1. *There is an inducing path in $\mathcal{H}$ between a and b;*
2. *There is an inducing walk in $\mathcal{H}$ between a and b;*
3. *$a \underset{\mathcal{H}}{\overset{m}{\not\perp}} b \mid Z$ for all $Z \subseteq \mathcal{V} \backslash \{a, b\}$;*
4. *$a \underset{\mathcal{H}}{\overset{m}{\not\perp}} b \mid Z$ for $Z = Anc_{\mathcal{H}}(\{a, b\}) \backslash \{a, b\}$.*

Proof. $1 \implies 2$ is trivial.

$2 \implies 3$: Assume there exists an inducing walk between a and b in $\mathcal{H}$. Let $Z \subseteq \mathcal{V} \setminus \{a, b\}$. Consider all walks in $\mathcal{H}$ between a and b such that all colliders on the walk lie in $\mathrm{Anc}_{\mathcal{H}}(\{a, b\} \cup Z)$, all non-endpoint non-colliders are not in Z, and there is no arrowhead into an undirected edge along the walk. Such walks exist, since the inducing walk is one of them. Let μ be such a walk with a minimal number of colliders. We claim that all colliders on μ must lie in $\mathrm{Anc}_{\mathcal{H}}(Z)$. Suppose, for contradiction, that there exists a collider c on μ that is not an ancestor of Z. By assumption, c must be an ancestor of either a or b. Without loss of generality, assume $c \in \mathrm{Anc}_{\mathcal{H}}(a)$. Therefore, there exists a directed walk π from c to a in $\mathcal{H}$ that does not pass through any node in Z. Concatenating this walk with the subwalk of μ between c and b yields another walk between a and b with the same properties but fewer colliders, contradicting the minimality of μ. Hence, all colliders on μ must lie in $\mathrm{Anc}_{\mathcal{H}}(Z)$, implying that μ is m-open given Z. By Proposition 1, this ensures the existence of a Z-m-open path between a and b, and thus a and b are m-connected given Z.

$3 \implies 4$ is trivial.

$4 \implies 1$: Suppose that a and b are m-connected given $Z = \mathrm{Anc}_{\mathcal{H}}(\{a, b\}) \setminus \{a, b\}$. Let π be a path between a and b that is m-open given Z. We claim that π must be an inducing path. First, all colliders on π are in $\mathrm{Anc}_{\mathcal{H}}(Z)$, and hence must be in $\mathrm{Anc}_{\mathcal{H}}(\{a, b\})$. Second, all non-endpoint nodes on π must be colliders. Suppose, for contradiction, that there exists a non-collider c on π. Then there must be a directed subpath of π starting at c and ending either at the first collider after c or at one of the endpoints a or b. This implies that $c \in \text{Anc}_{\mathcal{H}}(\{a, b\})$, and hence $c \in Z$. But this contradicts the assumption that π is m-open given Z, as it would contain a non-collider in Z. Therefore, all non-endpoint nodes on π are colliders in $\text{Anc}_{\mathcal{H}}(\{a, b\})$, and thus π is an inducing path. □

Lemma 9. *Let $\mathcal{H}$ be a σ-MAG with nodes $\mathcal{V}$ and let $a, b \in \mathcal{H}$ be distinct. If there exists an inducing path between a and b in $\mathcal{H}$, and all inducing paths in $\mathcal{H}$ between a and b are out of b, then $b \in \text{Ant}_{\mathcal{H}}(a)$.*

Proof. Let μ be an inducing path between a and b in $\mathcal{H}$. If μ contains only the two endpoints a and b, then μ must be of the form $a \ast\!\!- b$, which implies that $b \in \text{Ant}_{\mathcal{H}}(a)$. If μ contains more than two nodes, then it must be of the form $a \ast\!\!\rightarrow\!\!\ast \cdots \ast\!\!\rightarrow\!\!\ast\, c \leftarrow b$, where c is a collider lying in $\text{Anc}_{\mathcal{H}}(\{a, b\})$. Since $\mathcal{H}$ is ancestral, c cannot be an ancestor of b; hence, it must be that $c \in \text{Anc}_{\mathcal{H}}(a)$, which implies $b \in \text{Anc}_{\mathcal{H}}(a)$. □

Lemma 10. *Let $\mathcal{H}$ be a σ-MAG with nodes $\mathcal{V}$ and let $a, b \in \mathcal{V}$ be distinct. If there exists an inducing path between a and b in $\mathcal{H}$ that is into b, and $a \notin \text{Anc}_{\mathcal{H}}(b)$, then there exists an inducing path between a and b in $\mathcal{H}$ that is both into a and into b.*

Proof. Let μ be an inducing path between a and b in $\mathcal{H}$ that is into b. If μ contains only the two endpoints a and b, then it must be of the form $a \ast\!\!\rightarrow b$. Since $a \notin \text{Anc}_{\mathcal{H}}(b)$, μ must be of the form $a \leftrightarrow b$. If μ contains more than two nodes, then it must be of the form $a \ast\!\!\rightarrow c \leftrightarrow \cdots \leftrightarrow b$, where c is a collider that lies in $\text{Anc}_{\mathcal{H}}(\{a, b\})$. Since $\mathcal{H}$ is ancestral, c cannot be an ancestor of a. Moreover, if μ is not into a, then c cannot be an ancestor of b either, because $a \notin \text{Anc}_{\mathcal{H}}(b)$, which contradicts the assumption that μ is an inducing path. Therefore, we conclude that μ is into both a and b. □

B.3 PROOFS IN SECTION 5

Lemma 11. *In a σ-MAG $\mathcal{H}$ with nodes $\mathcal{V}$, let $\pi = (a, v_0, \ldots, v_n, b, c)$ be a discriminating path for b. Then, the following hold:*

1. *If b is a collider on π, then for any subset of nodes $Z \subseteq \mathcal{V} \backslash \{a, c\}$ such that a and c are m-separated given Z, we have $b \notin Z$.*
2. *If b is a non-collider on π, then for any subset of nodes $Z \subseteq \mathcal{V} \backslash \{a, c\}$ such that a and c are m-separated given Z, we have $b \in Z$.*

Proof. Suppose $Z \subseteq \mathcal{V} \setminus \{a, c\}$ is a subset of nodes such that a and c are m-separated given Z. By the definition of discriminating paths, v_0 is a collider on π and has a directed edge into c. Then, $\mathcal{H}$ contains a path:

$$a \ast\!\!\rightarrow v_0 \rightarrow c.$$

Since v_0 is a non-collider on this path, Z must contain v_0 to m-block the path. Furthermore, $\mathcal{H}$ contains another path:

$$a \ast\!\!\rightarrow v_0 \leftrightarrow v_1 \rightarrow c.$$

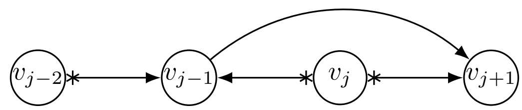
Since $v_0 \in Z$, we have $v_0 \in \text{Anc}_{\mathcal{H}}(Z)$. Moreover, v_1 is a non-collider on this path, implying that Z must also contain v_1. By induction, all nodes v_k for $k = 0, \ldots, n$ belong to Z. Consequently, all nodes on π, except for b, do not m-block π, so b must m-block π. Thus, we conclude:

1. If b is a *collider* on π, then $b \notin Z$.
2. If b is a *non-collider* on π, then $b \in Z$.

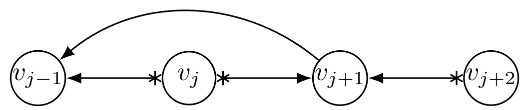
□

Lemma 12. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two σ-MAGs with the same node set $\mathcal{V}$. If $\mathcal{H}_1$ and $\mathcal{H}_2$ are m-Markov equivalent, then they satisfy Condition 1.*

Proof. Suppose that $\mathcal{H}_1$ and $\mathcal{H}_2$ do not satisfy Condition 1. Assume that a and b are adjacent in $\mathcal{H}_1$ but not in $\mathcal{H}_2$. Since the edge between a and b in $\mathcal{H}_1$ is an inducing path, by Proposition 2, it follows that $a \overset{m}{\underset{\mathcal{H}_1}{\not\perp}} b \mid Z$ for $Z = \text{Anc}_{\mathcal{H}_1}(\{a, b\}) \setminus \{a, b\}$. As $\mathcal{H}_1$ and $\mathcal{H}_2$ are m-Markov equivalent, we also have $a \overset{m}{\underset{\mathcal{H}_2}{\not\perp}} b \mid Z$ for the same set Z. Applying Proposition 2 again, this



(a) One possible case when $v_{j-1} \rightarrow v_{j+1}$



(b) The other possible case when $v_{j-1} \leftarrow v_{j+1}$

Figure 6: Illustration for Proposition 3.

implies the existence of an inducing path between a and b in $\mathcal{H}_2$, contradicting the maximality of $\mathcal{H}_2$. Therefore, $\mathcal{H}_1$ and $\mathcal{H}_2$ must have the same adjacencies.

Now assume that v_k is an unshielded collider of the form $v_{k-1} \ast\!\!\rightarrow v_k \leftarrow\!\!\ast v_{k+1}$ in $\mathcal{H}_1$, but not in $\mathcal{H}_2$. Since v_{k-1} and v_{k+1} are non-adjacent in $\mathcal{H}_2$, there exists no inducing path between them. By Proposition 2, we have $v_{k-1} \overset{m}{\underset{\mathcal{H}_2}{\perp}} v_{k+1} \mid Z$, where $Z = \mathrm{Anc}_{\mathcal{H}_2}(\{v_{k-1}, v_{k+1}\}) \setminus \{v_{k-1}, v_{k+1}\}$. Since $\mathcal{H}_1$ and $\mathcal{H}_2$ are m-Markov equivalent, it follows that $v_{k-1} \overset{m}{\underset{\mathcal{H}_1}{\perp}} v_{k+1} \mid Z$ holds for the same set Z. In $\mathcal{H}_2$, the path between v_{k-1} and v_{k+1} passes through v_k as a non-collider (either $v_{k-1} \ast\!\!-\!\!\ast v_k -\!\!\ast v_{k+1}$ or $v_{k-1} \ast\!\!- v_k \ast\!\!-\!\!\ast v_{k+1}$), which implies that $v_k \in Z$. However, in $\mathcal{H}_1$, v_k is a collider on the path $v_{k-1} \ast\!\!\rightarrow v_k \leftarrow\!\!\ast v_{k+1}$, which implies $v_k \notin Z$. This contradiction shows that $\mathcal{H}_1$ and $\mathcal{H}_2$ must have the same unshielded colliders.

Finally, assume π is a discriminating path between a and c for a node b in $\mathcal{H}_1$, and let π' be the corresponding path in $\mathcal{H}_2$, which is also a discriminating path for b. By similar reasoning, we have $a \overset{m}{\underset{\mathcal{H}_1, \mathcal{H}_2}{\not\perp}} c \mid Z$, for $Z = \mathrm{Anc}_{\mathcal{H}_1}(\{a, c\}) \setminus \{a, c\}$, since a and c are not adjacent. If b is a collider on π but not on π', then by Lemma 11, it follows that $b \notin Z$ in $\mathcal{H}_1$, whereas $b \in Z$ in $\mathcal{H}_2$. This contradiction implies that b is a collider on π if and only if it is a collider on π'. □

Proposition 3. *Let π be a shortest m-open path between v_0 and v_n given Z in a σ-MAG $\mathcal{H}$. If v_j is a covered node on π, then:*

1. *If $v_{j-1} \rightarrow v_{j+1}$, there exists a unique index $i < j$ such that the subpath of π between v_i and v_{j+1} is a discriminating path for v_j.*
2. *If $v_{j-1} \leftarrow v_{j+1}$, there exists a unique index $i > j$ such that the subpath of π between v_{j-1} and v_i is a discriminating path for v_j.*

Proof. This proof is inspired by the Lemma 9 in [Spirtes and Richardson, 1996], with modifications to adapt it to σ-MAGs.

If v_j is a covered node on π, then $\mathcal{H}$ must contain one of the six subgraphs described in Lemma 14. In particular, if $v_{j-1} \rightarrow v_{j+1}$, then π contains the subgraph Figure 6a. Similarly, if $v_{j-1} \leftarrow v_{j+1}$, then π contains the subpath Figure 6b. Since these two cases are symmetric, we assume W.L.O.G. the former.

We aim to find a node v_k with $k \leq j-1$ such that the subpath of π between v_k and v_{j+1} consists of at least $n \geq 2$ edges, is into v_k, all non-endpoint nodes on the subpath (except the endpoints and v_j) are colliders, and every node on the subpath (excluding v_j and v_{j+1}) has a directed edge into v_{j+1} in $\mathcal{H}$. Such nodes do exist, as the subpath of π between v_{j-1} and v_{j+1} satisfies these conditions for $n = 2$. We will show that the subpath of π between v_{k-1} and v_{j+1} is either a discriminating path for v_j, or a subpath with $n+1$ edges that satisfies the required conditions, in which q_{k-1} is a collider on π and $\mathcal{H}$ contains the edge $q_{k-1} \rightarrow q_{j+1}$.

By Lemma 13, v_k and v_{j+1} are adjacent in $\mathcal{H}$ but not on π, and we have $v_k \rightarrow v_{j+1}$, so there must be an edge $v_{k-1} \ast\!\!\rightarrow v_k$ on π. Notice that all non-endpoint nodes between v_{k-1} and v_j are colliders on π and have directed edges into v_{j+1}. If the subpath of π between v_{k-1} and v_{j+1} is not a discriminating path for v_j, then v_{k-1} and v_{j+1} must be adjacent. Applying Lemma 13 again, the edge between v_{k-1} and v_{j+1} must be directed. If we have $v_{k-1} \leftarrow v_{j+1}$ in $\mathcal{H}$, then a directed cycle or an almost directed cycle would exist, such as $v_k \to v_{j+1} \to v_{k-1} \ast\!\!\to v_k$, which leads to a contradiction. Therefore, we conclude that $v_{k-1} \to v_{j+1}$, and there is an edge $v_{k-2} \ast\!\!\to v_{k-1} \leftrightarrow v_k$ on π.

If the subpath of π between v_{k-1} and v_{j+1} is not a discriminating path for v_j, then the subpath between v_{k-2} and v_{j+1} might be a discriminating path for v_j. If it is not, we move to the next node on the left. Notice that if the subpath of π between v_1 and v_{j+1} satisfies the conditions above, then v_0 is not adjacent to v_{j+1}. Otherwise, by Lemma 13, the edge between v_0 and v_{j+1} must be $v_0 \leftarrow v_{j+1}$, which would create a directed or almost directed cycle $v_1 \to v_{j+1} \to v_0 \ast\!\!\to v_1$ in $\mathcal{H}$. Since π is of finite length, it follows that there must exist a node v_i between v_0 and v_j such that the subpath of π between v_i and v_{j+1} is a discriminating path for v_j.

We will show that this discriminating path is unique. Let π_i denote this path. No subpath of π_i can be a discriminating path for v_j since all nodes on π_i, except its endpoints, are adjacent to v_{j+1}. Moreover, no subpath of π that contains π_i can be a discriminating path for v_j since v_i is not adjacent to v_{j+1}. Therefore, we conclude that π_i is unique. □

Lemma 13. *Let π be a shortest m-open path between v_0 and v_n given Z in a σ-MAG $\mathcal{H}$. Suppose there exists an edge $v_i \ast\!\!-\!\!\ast v_j$ ($i < j$) in $\mathcal{H}$ that is not part of π. Define π' as the path obtained by replacing the subpath between v_i and v_j on π with the edge $v_i \ast\!\!-\!\!\ast v_j$. Then, one of the following conditions must hold:*

1. *$v_{i-1} \ast\!\!\to v_i \leftarrow\!\!\ast v_{i+1}$ appears on π and $v_i \to v_j$ exists in $\mathcal{H}$.*
2. *$v_{j-1} \ast\!\!\to v_j \leftarrow\!\!\ast v_{j+1}$ appears on π and $v_i \leftarrow v_j$ exists in $\mathcal{H}$.*

Proof. This proof is inspired by the Lemma 7 in [Spirtes and Richardson, 1996], with modifications to adapt it to σ-MAGs.

Since π is a shortest m-open path between v_0 and v_n given Z, it follows that π' is m-blocked by Z. Therefore, one of the following must hold:

1. There exists a subpath of the form:
   - $v_{i-1} \ast\!\!\to v_i - v_j$, or
   - $v_{i-1} - v_i \leftarrow\!\!\ast v_j$, or
   - $v_i - v_j \leftarrow\!\!\ast v_{j+1}$, or
   - $v_i \ast\!\!\to v_j - v_{j+1}$.
2. One of v_i or v_j is either a non-collider on π' that lies in Z, or a collider on π' that does not belong to $\mathrm{Anc}_H(Z)$.

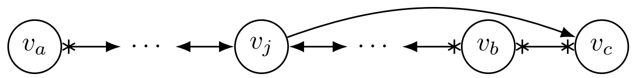
Assume one of the above subpaths is present on π'. Suppose we have $v_{i-1} \ast\!\!\to v_i - v_j$ on π'. If v_j is an endpoint of π', then we can substitute $v_{i-1} \ast\!\!\to v_i - v_j$ with $v_{i-1} \ast\!\!\to v_j$ and obtain a shorter Z-m-open path, leading to a contradiction. If we have $v_j \leftarrow\!\!\ast v_{j+1}$ on π', then we can substitute $v_{i-1} \ast\!\!\to v_i - v_j \leftarrow\!\!\ast v_{j+1}$ with $v_{i-1} \ast\!\!\to v_i \leftarrow\!\!\ast v_{j+1}$, which results in a Z-m-open path because v_i is either a collider or has a directed subpath into a collider on π. If we have $v_j \multimap v_{j+1}$ on π', then we can substitute $v_{i-1} \ast\!\!\to v_i - v_j \multimap v_{j+1}$ with $v_{i-1} \ast\!\!\to v_j \multimap v_{j+1}$, which results in a shorter Z-m-open path because v_j is a non-collider on π and π'. The case $v_i - v_j \leftarrow\!\!\ast v_{j+1}$ is analogous. Suppose we have $v_{i-1} - v_i \leftarrow\!\!\ast v_j$ on π'. Since there exists a node v_k on π' such that $v_{k-1} \leftarrow v_k - \cdots - v_i \leftarrow\!\!\ast v_j$ (where v_{k-1} may not appear), we can substitute $v_k - \cdots - v_i \leftarrow\!\!\ast v_j$ with $v_k \leftarrow\!\!\ast v_j$, which produces a shorter Z-m-open path since v_k is a non-collider on π. The case $v_i \ast\!\!\to v_j - v_{j+1}$ is similar. Hence, one of v_i and v_j must block π'.

So the above subpaths cannot exist on π'. If v_i is a non-collider on π but not on π', then we have $v_{i-1} \ast\!\!\to v_i \to v_{i+1}$ on π and $v_i \leftarrow\!\!\ast v_j$ on π'. If there is no collider on the subpath between v_i and v_j on π, then v_i is an ancestor of v_j, which leads to a directed cycle or an almost directed cycle in $\mathcal{H}$ since we have $v_i \leftarrow\!\!\ast v_j$. It follows that there must be a collider on the subpath, and v_i is an ancestor of the first collider on the subpath, implying $v_i \in \mathrm{Anc}_{\mathcal{H}}(Z)$. Hence, v_i does not block π'.

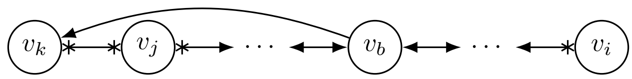
Similarly, if v_j is a non-collider on π but not on π', then v_j does not block π'. Therefore, either v_i is a collider on π but not on π', or v_j is a collider on π but not on π'. Hence, we have $v_{i-1} \ast\!\!\to v_i \leftarrow\!\!\ast v_{i+1}$ on π and $v_i \to v_j$ in $\mathcal{H}$, or we have $v_{j-1} \ast\!\!\to v_j \leftarrow\!\!\ast v_{j+1}$ on π and $v_i \leftarrow v_j$ in $\mathcal{H}$. □

Lemma 14. *Let π be a shortest m-open path between v_0 and v_n given Z in a σ-MAG $\mathcal{H}$. Suppose π contains the subpath $v_{k-1} \ast\!\!-\!\!\ast v_k \ast\!\!-\!\!\ast v_{k+1}$, where v_{k-1} and v_{k+1} are adjacent in $\mathcal{H}$. Then, $\mathcal{H}$ must contain one of the subgraphs shown in Figure 5.*

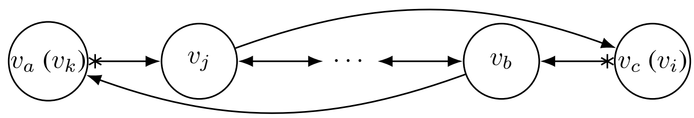
Proof. Directly applying Lemma 13, and noting that $\mathcal{H}$ does not contain a directed cycle or an almost directed cycle, we obtain the remaining six cases. □



(a) v_j on the path π_b



(b) v_b on the path π_j



(c) Subpath of π between v_a (v_k) and v_c (v_i)

Figure 7: Illustrations for Lemma 15.

Lemma 15. *In a σ-MAG $\mathcal{H}$, if π is a shortest m-open path between v_0 and v_n given Z, then no pair of distinct covered nodes v_b and v_j on π can satisfy both of the following conditions: v_b is a covered node on the discriminating subpath of π for v_j, and v_j is a covered node on the discriminating subpath of π for v_b.*

Proof. This proof is inspired by the Lemma 10 in [Spirtes and Richardson, 1996], with modifications to adapt it to σ-MAGs.

Suppose, for the sake of contradiction, that v_b is a covered node on the discriminating subpath of π between v_i and v_k for v_j (denoted by π_j, where v_j is adjacent to v_k), and that v_j is a covered node on the discriminating subpath of π between v_a and v_c for v_b (denoted by π_b, where v_b is adjacent to v_c).

W.L.O.G., assume that $a < b = c - 1$. Since v_b and v_j are two distinct non-endpoint nodes on π_j and π_b, respectively, we must have the ordering $k + 1 = j < i$. Otherwise, a contradiction arises: namely, $i < b < j$ and $a < j < b$ cannot simultaneously hold. Therefore, the paths π_b and π_j are illustrated in Figures 7a and 7b.

Since v_b is a non-endpoint node on π_j and distinct from v_j, and since v_b and v_c are adjacent on π, it follows that v_c also lies on π_j. Consequently, we must have $v_c \neq v_k$. Moreover, if $v_c = v_j$, then v_j would be the endpoint of π_b, v_k, without being a covered node, contradicting the assumption. Thus, we conclude that $v_c \neq v_j$.

Now, suppose that $v_c = v_i$. Since v_j must be a non-endpoint node between v_a and v_b on π_b (where $v_j \neq v_a$ and $v_j \neq v_b$), and since v_k is adjacent to v_j, it follows that v_k lies between v_a and v_b on π_b ($v_k \neq v_b$). Moreover, because all nodes between v_a and v_b, except for v_a, are adjacent to $v_c = v_i$, we must have $v_k = v_a$; otherwise, v_k would be adjacent to v_i, contradicting the assumption that π_j is a discriminating path. Thus, we have $v_c = v_i$ and $v_k = v_a$ simultaneously. Then the subpath of π between v_a (v_k) and v_c (v_i) is in the form of Figure 7c.

Consequently, all nodes between v_a and v_c on π are colliders that belong to the ancestors of $\{v_a, v_c\}$. This implies the existence of an inducing path between v_a and v_c in $\mathcal{H}$, contradicting the maximality of σ-MAGs.

Since v_c lies on π_j but is distinct from v_i, v_j, and v_k, it follows that $v_c \to v_k$ in $\mathcal{H}$. Similarly, we obtain $v_k \to v_c$ in $\mathcal{H}$, leading to a directed cycle in $\mathcal{H}$, which is a contradiction. □

Lemma 16. *In a σ-MAG $\mathcal{H}$, if π is a shortest m-open path between v_0 and v_n given Z, then no triple of distinct covered nodes v_i, v_j, v_k on π can satisfy all of the following conditions: v_i is a covered node on the discriminating subpath of π for v_j, v_j is a covered node on the discriminating subpath of π for v_k, and v_k lies between v_i and v_j on π.*

Proof. This proof follows from Lemma 11 in [Spirtes and Richardson, 1996], which we reproduce here for the reader's convenience.

Suppose, for the sake of contradiction, that there exists such a triple satisfying all conditions. Let π_j and π_k be the discriminating subpaths of π for v_j and v_k, respectively. Since v_i is a covered node on π_j, every node between v_i and v_j must also lie on π_j. Consequently, v_k is also on π_j. Since both v_i and v_j are non-endpoint nodes on π_j, it follows that v_k is also a non-endpoint node on π_j. Thus, v_{k-1} and v_{k+1}, which are adjacent to v_k on π, must both be on π_j, meaning that v_k remains a covered node on π_j. By hypothesis, v_j is a covered node on π_k, which contradicts Lemma 15. □

Lemma 17. *In a σ-MAG $\mathcal{H}$, if π is a shortest m-open path between v_0 and v_n given Z, then no quadruple of distinct covered nodes v_i, v_j, v_k, v_l on π can satisfy all of the following conditions: v_i is a covered node on the discriminating subpath of π for v_j, v_k is a covered node on the discriminating subpath of π for v_l, v_l lies between v_i and v_j on π, and v_j lies between v_l and v_k on π.*

Proof. This proof follows from Lemma 12 in [Spirtes and Richardson, 1996], which we reproduce here for the reader's convenience.

Suppose, for the sake of contradiction, that there exists such a quadruple satisfying all conditions. Let π_j and π_l be the discriminating subpaths of π for v_j and v_l, respectively. Since v_l lies between v_i and v_j, and v_i is on π_j, it follows that v_l is also a non-endpoint node on π_j. Moreover, since v_l is a covered node on π, both of the nodes adjacent to v_l must be on π_j, implying that v_l is a covered node on π_j. Similarly, v_j is a covered node on π_l, contradicting Lemma 15. □

Lemma 18. *In a σ-MAG $\mathcal{H}$, if π is a shortest m-open path between a and b given Z, then no sequence of distinct covered nodes $v_0, v_1, \dots, v_n$ on π with $n \geq 2$ can satisfy the following conditions: for each $i = 0, \dots, n-1$, the node v_i is a covered node on the discriminating subpath of π for v_{i+1}, and v_n is a covered node on the discriminating subpath of π for v_0.*

Proof. This proof follows the same structure as Lemma 13 in [Spirtes and Richardson, 1996], with modifications to adapt it to the framework of σ-MAGs.

Suppose, for the sake of contradiction, that there exists such a sequence of nodes satisfying all conditions. Without loss of generality, assume that v_0 is to the right of v_n on π. Let k be the largest index such that v_k is to the right of v_0 on π, if such a node exists; otherwise, let $k = 0$. Now, we will show that v_{k+1} ($k < n$) is to the left of v_n. If $k = 0$, then every other node in the sequence lies to the left of v_0 on π, so v_1 is to the left of v_0. By Lemma 15, we know that $v_1 \neq v_n$. Furthermore, by Lemma 16, v_1 is not between v_0 and v_n, which implies that $v_1 = v_{k+1}$ is on the left side of v_n on π. If $k \neq 0$, then we must have $v_{k+1} \neq v_n$, as otherwise, this would contradict Lemma 16. Moreover, applying Lemma 17, we conclude that v_{k+1} is not between v_0 and v_n. Hence, v_{k+1} lies to the left of v_n on π.

Now, we will show that there exists a node v_l with $l \geq k+1$ on π such that v_l is to the right of v_k, leading to a contradiction since k is the largest index for which v_k is to the right of v_0. By Lemma 16, v_{k+2} is not between v_{k+1} and v_k on π, implying that $v_{k+2} \neq v_n$ ($k+2 \leq n$, as we have already established that $v_{k+1} \neq v_n$). If v_{k+2} is to the right of v_k, then we are done. Otherwise, consider the case where v_{k+2} is to the left of v_{k+1} on π. Since v_n is to the right of v_{k+1} and $k+2 < m$, there must exist a node v_l with $l \geq k+3$ such that v_{l-1} is to the left of v_{k+1} and v_l is to the right of v_{k+1} on π. By Lemma 17, v_l is not between v_{k+1} and v_k, implying that $v_l \neq v_n$. Thus, v_l is to the right of v_k on π, leading to a contradiction. □

Lemma 19. *If two σ-MAGs $\mathcal{H}_1$ and $\mathcal{H}_2$ with the same nodes $\mathcal{V}$ satisfy Condition 1, and if $\pi = (a, v_0, \dots, v_n, b, c)$ is a discriminating path for b in $\mathcal{H}_1$, then let π' be the corresponding path to π in $\mathcal{H}_2$. If every node on π', except for the endpoints and b, is a collider, then π' is a discriminating path for b in $\mathcal{H}_2$.*

Proof. This proof is inspired by the Lemma 15 in [Spirtes and Richardson, 1996], with modifications to adapt it to σ-MAGs.

By definition, in $\mathcal{H}_1$, a is not adjacent to c, v_0 is a collider on π, and v_0 is an unshielded non-collider on $a \ast\!\!\to v_0 \to c$. Since $\mathcal{H}_1$ and $\mathcal{H}_2$ have the same adjacencies, it follows that in $\mathcal{H}_2$, a is also not adjacent to c. Additionally, by hypothesis, v_0 remains a collider on π', and v_0 is an unshielded non-collider on $a \ast\!\!\to v_0 \to\!\!\ast\, c$, as $\mathcal{H}_1$ and $\mathcal{H}_2$ share the same unshielded colliders. The edge between v_0 and c cannot be undirected; otherwise, a and c would be adjacent by definition. Thus, we conclude that $v_0 \to c$ in $\mathcal{H}_2$.

Now, suppose that for $0 \leq i \leq j-1$ with $1 \leq j \leq n$, we have $v_i \to c$ in $\mathcal{H}_2$, and by hypothesis, v_i is a collider on π'. Let π'_j be the concatenation of the subpath of π' between a and v_j and the edge between v_j and c in $\mathcal{H}_2$ (since $\mathcal{H}_1$ and $\mathcal{H}_2$ have the same adjacencies, v_j and c must be adjacent). Every node on π'_j between v_0 and v_{j-1} is a collider and a parent of c in $\mathcal{H}_2$, implying that π'_j is a discriminating path for v_j in $\mathcal{H}_2$. Let π_j be the corresponding path in $\mathcal{H}_1$; then π_j is also a discriminating path for v_j, and v_j is a non-collider on π_j (since we have $v_j \to c$ on π_j). Hence, v_j must also be a non-collider on π'_j because v_j is a collider on π'_j in $\mathcal{H}_2$ if and only if it is a collider on π_j in $\mathcal{H}_1$. It follows that $v_j \to\!\!\ast\, c$ holds in $\mathcal{H}_2$, given that v_j is a collider on π' by hypothesis. Moreover, $v_j - c$ is not possible; otherwise, we would have $v_{j-1} \leftrightarrow c$, which is contradictory. Therefore, we conclude that $v_j \to c$ in $\mathcal{H}_2$.

By induction, all nodes $v_0, \ldots, v_n$ have a directed edge into c in $\mathcal{H}_2$. Given that every node on π', except for the endpoints and b, is a collider, it follows that π' is a discriminating path for b in $\mathcal{H}_2$. □

Lemma 20. *If two σ-MAGs $\mathcal{H}_1$ and $\mathcal{H}_2$ with the same nodes $\mathcal{V}$ satisfy Condition 1, and if π is a shortest m-open path between v_0 and v_n given Z in $\mathcal{H}_1$, with π' being the corresponding path to π in $\mathcal{H}_2$, then v_k is a collider on π if and only if v_k is a collider on π'.*

Proof. This proof is inspired by the Lemma 16 in [Spirtes and Richardson, 1996], with modifications to adapt it to σ-MAGs.

If v_k is not a covered node on π, then since $\mathcal{H}_1$ and $\mathcal{H}_2$ have the same unshielded colliders, v_k is a collider on π if and only if v_k is a collider on π'.

Suppose v_k is a covered node on π. By Proposition 3, π contains a unique discriminating subpath for v_k. Let π_k denote the discriminating subpath of π for v_k.

Now, suppose v_k is a zero-order covered node on π. Then all nodes on π_k, except for the endpoints and v_k, are unshielded colliders in $\mathcal{H}_1$. Since $\mathcal{H}_1$ and $\mathcal{H}_2$ have the same unshielded colliders, all unshielded colliders on π_k remain unshielded colliders on π'_k, the corresponding path to π_k in $\mathcal{H}_2$. Hence, by Lemma 19, π'_k is a discriminating path in $\mathcal{H}_2$. It follows that v_k is a collider on π if and only if v_k is a collider on π', by the third hypothesis of Theorem 3.

Now, suppose that for $0 \leq i < j$, the i^{th}-order covered nodes on π are oriented in the same way as on π'. Suppose v_k is a j^{th}-order covered node on π. By the induction hypothesis, all colliders on π_k, possibly except for v_k, are either not covered or are covered nodes of order less than j. Consequently, these nodes remain colliders on π'_k. Hence, by Lemma 19, π'_k is also a discriminating path for v_k in $\mathcal{H}_2$. It follows that v_k is a collider on π if and only if v_k is a collider on π'. □

Lemma 21. *If a σ-MAG $\mathcal{H}$ contains a path $a \ast\!\!\rightarrow b \rightarrow c$ and an edge $a \ast\!\!-\!\!\ast c$, then:*

1. *The edge between a and c is oriented as $a \ast\!\!\rightarrow c$.*
2. *If $a \ast\!\!-\!\!\ast c$ has a different edge mark at a than $a \ast\!\!\rightarrow b$, then the edges are oriented as $a \leftrightarrow b \rightarrow c$ and $a \rightarrow c$.*

Proof. This proof is inspired by the Lemma 19 in [Spirtes and Richardson, 1996], with modifications to adapt it to σ-MAGs.

The edge between a and c cannot be undirected; otherwise, $\mathcal{H}$ would also contain an edge $b \rightarrow a$, which is contradictory. If the edge between a and c were $a \leftarrow c$, the structure $b \rightarrow c \rightarrow a \ast\!\!\rightarrow b$ would form a directed cycle or an almost directed cycle, which is also contradictory. Thus, we conclude that $a \ast\!\!\rightarrow c$ must exist in $\mathcal{H}$.

If $a \ast\!\!-\!\!\ast c$ has a different edge mark at a than $a \ast\!\!\rightarrow b$, then there are two possible cases:

1. $a \leftrightarrow b$ and $a \rightarrow c$,
2. $a \rightarrow b$ and $a \leftrightarrow c$.

The second case leads to an almost cycle $a \rightarrow b \rightarrow c \leftrightarrow a$, which is contradictory. Therefore, we consider only the first case. □

Lemma 22. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two σ-MAGs with the same node set $\mathcal{V}$ that satisfy Condition 1. Suppose π is a shortest m-open path between v_0 and v_n given Z in $\mathcal{H}_1$ and has the smallest collider distance sum to Z. If v_i is a collider on π, q is an ancestor of Z in $\mathcal{H}_1$, and there is an edge $v_i \rightarrow q$ on a shortest directed path from v_i to Z in $\mathcal{H}_1$, then the edge $v_i \rightarrow q$ must also be present in $\mathcal{H}_2$.*

Proof. This proof follows the same structure as Lemma 20 in [Spirtes and Richardson, 1996], with modifications to adapt it to the framework of σ-MAGs.

If v_i is a collider on π, then by Lemma 20, both $\mathcal{H}_1$ and $\mathcal{H}_2$ contain the structure $v_{i-1} \ast\!\!\rightarrow v_i \leftarrow\!\!\ast v_{i+1}$. If v_{i-1} and q are not adjacent in $\mathcal{H}_1$, then v_i is an unshielded non-collider on $v_{i-1} \ast\!\!\rightarrow v_i \rightarrow q$ in $\mathcal{H}_1$, implying that v_i is also an unshielded non-collider on $v_{i-1} \ast\!\!\rightarrow v_i -\!\!\ast q$ in $\mathcal{H}_2$. Since the edge between v_i and q cannot be undirected in $\mathcal{H}_2$ (otherwise, v_{i-1} and q would be adjacent in $\mathcal{H}_2$), it follows that we must have $v_i \rightarrow q$ in $\mathcal{H}_2$. Similarly, if v_{i+1} and q are not adjacent in $\mathcal{H}_1$, then we conclude that $v_i \rightarrow q$ in $\mathcal{H}_2$ as well. Now, suppose that both v_{i-1} and v_{i+1} are adjacent to q.

There exists a node u on π between v_0 and v_{i-1} that satisfies at least one of the following conditions:

(i) u is not adjacent to q.

(ii) The edge between u and q is not into q.

(iii) u has the same collider/non-collider status on π and on the concatenation of the subpath of π between v_0 and u and the edge between u and q.

Such a node must exist since v_0 satisfies either condition (i) if v_0 is not adjacent to q in $\mathcal{H}_1$, or condition (iii) if v_0 is adjacent to q in $\mathcal{H}_1$. Let v_{i-m_1} (where $m_1 \geq 1$) be the closest such node to the left of v_i. Similarly, there exists a node on π between v_{i+1} and v_n that satisfies at least one of conditions (i) or (ii), or the following condition:

(iii') u has the same collider/non-collider status on π and on the concatenation of the subpath of π between v_n and u and the edge between u and q.

Let v_{i+m_2} (where $m_2 \geq 1$) be the closest such node to the right of v_i.

We aim to show that every non-endpoint node between v_{i-m_1} and v_i on π is a collider and has a directed edge into q. If $m_1 = 1$, this holds trivially since there are no non-endpoint nodes between v_{i-m_1} and v_i. Thus, we assume $m_1 \geq 2$. We will show that for $1 \leq k \leq m_1 - 1$, v_{i-k} is a collider on π and there exists an edge $v_{i-k} \to q$ in $\mathcal{H}_1$. Since v_{i-1} lies between v_{i-m_1} and v_i but does not satisfy the above conditions, $\mathcal{H}_1$ contains the edge $v_{i-1} *\!\!\to q$, and v_{i-1} must have a different collider/non-collider status on π compared to the concatenation of the subpath of π from v_0 to v_{i-1} together with the edge between v_{i-1} and q. This implies that $\mathcal{H}_1$ also contains the edge $v_{i-2} *\!\!\to v_{i-1}$, and that the edge between v_{i-1} and v_i has a different edge mark at v_{i-1} compared to the edge between v_{i-1} and q in $\mathcal{H}_1$. By Lemma 21, it follows that $v_{i-1} \to q$ and $v_{i-1} \leftrightarrow v_i$. Thus, v_{i-1} is a collider on π and has a directed edge into q. This settles the case when $m_1 = 2$. Now assume $m_1 \geq 3$. Suppose that for $1 \leq l \leq k-1$, v_{i-l} is a collider on π and there exists an edge $v_{i-l} \to q$ in $\mathcal{H}_1$. Since v_{i-l} lies between v_{i-m_1} and v_i, we have $v_{i-k} *\!\!\to q$, and v_{i-k} exhibits a different collider/non-collider status on π compared to the concatenation of the subpath of π from v_0 to v_{i-k} and the edge between v_{i-k} and q. This implies that $\mathcal{H}_1$ contains the edge $v_{i-k-1} *\!\!\to v_{i-k}$, and that the edge between v_{i-k} and v_{i-k+1} has a different edge mark than the edge between v_{i-k} and q in $\mathcal{H}_1$. By Lemma 21, it follows that $v_{i-k} \to q$ and $v_{i-k-1} *\!\!\to v_{i-k} \leftrightarrow v_{i-k+1}$. Hence, every non-endpoint node between v_{i-m_1} and v_i (if any) is a collider on π and has a directed edge into q. Similarly, every non-endpoint node between v_i and v_{i+m_2} is a collider on π and has a directed edge into q.

Suppose v_{i-m_1} is adjacent to q, and we have $v_{i-m_1+1} \to q$ and $v_{i-m_1} *\!\!\to v_{i-m_1+1}$ by induction. Then, by Lemma 21, it follows that $v_{i-m_1} *\!\!\to q$. By hypothesis, v_{i-m_1} retains the same collider/non-collider status on π and on the concatenation of the subpath of π between v_0 and v_{i-m_1}, along with the edge between v_{i-m_1} and q in $\mathcal{H}_1$. Similarly, if v_{i+m_2} is adjacent to q, then v_{i+m_2} has the same collider/non-collider status on π and on the concatenation of the subpath of π between v_{i+m_2} and v_n, along with the edge between v_{i+m_2} and q. Then, denote the concatenation of the subpath of π between v_0 and v_{i-m_1}, the edge between v_{i-m_1} and q, the edge between q and v_{i+m_2}, and the subpath of π between v_{i+m_2} and v_n by μ. Notice that there does exist subpaths of the form $v_{i-m_1-1} - v_{i-m_1} \leftrightarrow q$ or $q \leftrightarrow v_{i+m_2} - v_{i+m_2+1}$ on μ, since otherwise $\mathcal{H}_1$ contains an almost directed cycle $v_{i-m_1} \to v_{i-m_1+1} \to q \leftrightarrow v_{i-m_1}$ or $v_{i+m_2} \to v_{i+m_2+1} \to q \leftrightarrow v_{i+m_2}$. Thus, μ is m-open given Z and is shorter than π, which is a contradiction. The only exception occurs when $m_1 = m_2 = 1$, in which case the concatenated path has the same length as π, but a smaller sum of distances from colliders to Z, which is also a contradiction. It follows that at least one of v_{i-m_1} or v_{i+m_2} is not adjacent to q.

W.L.O.G., suppose v_{i-m_1} is not adjacent to q. Since we assume that v_{i-1} is adjacent to v_i in $\mathcal{H}_1$, it follows that $m_1 \geq 2$. Denote by π_i the concatenation of the subpath of π between v_{i-m_1} and v_i along with the edge $v_i \to q$ in $\mathcal{H}_1$, and let π_i' be the corresponding path in $\mathcal{H}_2$. By definition, π_i is a discriminating path for v_i in $\mathcal{H}_1$, and v_i is a non-collider on this path. By Lemma 20, all colliders on π remain colliders on π', the corresponding path to π in $\mathcal{H}_2$. Furthermore, by Lemma 19, π_i' is a discriminating path for v_i. Hence, v_i is a non-collider on π_i' by assumption. Since $\mathcal{H}_2$ contains the edges $v_{i-1} \leftrightarrow v_i$ and $v_{i-1} \to q$, we cannot have $v_i — q$ in $\mathcal{H}_2$. Therefore, we conclude that $v_i \to q$ in $\mathcal{H}_2$.

□

Lemma 23. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two σ-MAGs with the same node set $\mathcal{V}$ that satisfy Condition 1. Suppose π is a shortest m-open path between v_0 and v_n given Z in $\mathcal{H}_1$ and has the smallest collider distance sum to Z. If v_k is a collider on π, and $q \in Z$ is the endpoint of a shortest directed path μ from v_k to Z in $\mathcal{H}_1$, then v_k is an ancestor of q in $\mathcal{H}_2$.*

Proof. This proof is inspired by the Lemma 21 in [Spirtes and Richardson, 1996], with modifications to adapt it to σ-MAGs.

Suppose μ is in the form $v_k = u_0 \to u_1 \to \cdots \to u_m = q$. Let π' and μ' be the corresponding paths to π and μ, respectively, in $\mathcal{H}_2$. By Lemma 22, the first edge on μ' is directed and points out of u_0. Suppose there exists a subpath of μ' in the form

$u_i \to u_{i+1} - \cdots - u_j \leftrightarrow\!\!* \; u_{j+1}$, with $j \geq i+1$. By Lemma 1, this implies that $\mathcal{H}_2$ contains a triple $u_i \to u_j \leftarrow\!\!* \; u_{j+1}$. Since μ contains the edge $u_j \to u_{j+1}$, and $\mathcal{H}_1$ and $\mathcal{H}_2$ share the same unshielded colliders, it follows that u_i and u_{j+1} must be adjacent in $\mathcal{H}_2$, and consequently, also adjacent in $\mathcal{H}_1$. Notice that μ contains a subpath $u_i \to \cdots \to u_j \to u_{j+1}$. If the edge between u_i and u_{j+1} were $u_i \leftarrow\!\!* \; u_{j+1}$, this would lead to a directed or almost directed cycle, which is a contradiction. If the edge were undirected, i.e., $u_i - u_{j+1}$, then by Lemma 1, it would also contain the edge $u_j \to u_i$, creating a directed cycle, which is again a contradiction. If the edge were $u_i \to u_{j+1}$, replacing the subpath of π between u_i and u_{j+1} with $u_i \to u_{j+1}$ would yield a shorter directed path from v_k to q, which contradicts the assumption that π is the shortest path. Therefore, no such subpath exists in μ', and we conclude that μ' must be in the form $u_0 \to u_1 \to\!\!* \cdots \to\!\!* \; u_m$. By Lemma 2, this implies that v_k is an ancestor of q in $\mathcal{H}_2$. □

Lemma 24. *If two σ-MAGs $\mathcal{H}_1, \mathcal{H}_2$ with the same node set $\mathcal{V}$ satisfy Condition 1, then they are m-Markov equivalent.*

Proof. Suppose $\mathcal{H}_1, \mathcal{H}_2$ satisfy Condition 1. For any subsets of the nodes $X, Y, Z \subseteq \mathcal{V}$, if we have $X \overset{m}{\underset{\mathcal{H}_1}{\not\perp}} Y \mid Z$ in $\mathcal{H}_1$, then there exists a shortest m-open path π given Z with the smallest collider distance sum to Z from a node $v_0 \in X$ to a node $v_n \in Y$ in $\mathcal{H}_1$. Let π' be the corresponding path in $\mathcal{H}_2$. By Lemma 20, we know that all colliders on π remain colliders on π'. Furthermore, by Lemma 23, these colliders are still ancestors of Z in $\mathcal{H}_2$. Additionally, by Lemma 20, all non-colliders on π remain non-colliders on π', and they do not belong to Z since π is Z-m-open.

We now verify whether π' contains a subpath of the form $v_i \;*\!\!\leftrightarrow v_{i+1} - v_{i+2}$ or $v_i - v_{i+1} \leftrightarrow\!\!* \; v_{i+2}$. Without loss of generality, assume the former case holds. By Lemma 1, $\mathcal{H}_2$ also contains an edge of the form $v_i \;*\!\!\leftrightarrow v_{i+2}$. Since $\mathcal{H}_1$ and $\mathcal{H}_2$ share the same adjacencies, v_{i+1} is a covered node on π. Applying Lemma 13 and Lemma 20, we deduce that v_{i+2} is a non-collider on π, and v_i is a collider on π with a directed edge $v_i \to v_{i+2}$. By Lemma 20, v_i must also be a collider on π', implying the existence of the edges:

$$v_i \leftrightarrow v_{i+1} \quad \text{on } \pi', \quad \text{and} \quad v_i \leftrightarrow v_{i+2} \quad \text{in } \mathcal{H}_2.$$

Furthermore, by Proposition 3, there exists a node v_j with $0 \leq j < i$ such that the subpath μ of π between v_j and v_{i+2} is a discriminating path for v_{i+1}. Applying Lemma 19, the corresponding path μ' in $\mathcal{H}_2$ is also a discriminating path for v_{i+1}. However, this contradicts the presence of the edge $v_i \leftrightarrow v_{i+2}$ in $\mathcal{H}_2$. Thus, π' cannot contain such subpaths, and it follows that π' is m-open given Z in $\mathcal{H}_2$. The case where $X \overset{m}{\underset{\mathcal{H}_2}{\not\perp}} Y \mid Z$ in $\mathcal{H}_2$ is analogous. We therefore conclude that $\mathcal{H}_1$ and $\mathcal{H}_2$ are m-Markov equivalent. □

Theorem 3. *Two σ-MAGs $\mathcal{H}_1, \mathcal{H}_2$ with the same nodes $\mathcal{V}$ are m-Markov equivalent if and only if $\mathcal{H}_1, \mathcal{H}_2$ satisfy Condition 1.*

Proof. Obviously obtained by Lemma 12 and Lemma 24. □

Theorem 4. *Let $\mathcal{G}_1, \mathcal{G}_2$ be two DMGs with the same nodes $\mathcal{V}^+ = \mathcal{V} \cup \mathcal{S}$, and let $\mathcal{H}_1, \mathcal{H}_2$ be two σ-MAGs that represents $\mathcal{G}_1, \mathcal{G}_2$ given $\mathcal{S}$ respectively. $\mathcal{G}_1, \mathcal{G}_2$ are σ-Markov equivalent given $\mathcal{S}$ if and only if $\mathcal{H}_1, \mathcal{H}_2$ satisfy Condition 1.*

Proof. Obviously obtained by Theorem 2 and Theorem 3. □