Supplementary Material for "On Causal Discovery with Cyclic Additive Noise Models"

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1 Uniqueness of fixed points

Lemma 1 *Let $f : X \to X$ be a mapping, d a metric on X and suppose that f^N is a d-contraction for some $N \in \mathbb{N}$. Then f has a unique fixed point x_∞ and for any $x \in X$, the sequence $x, f(x), f^2(x), \dots$ obtained by iterating f converges to x_∞.*

Proof. Take any $x \in X$. Consider the N sequences obtained by iterating f^N, starting respectively in x, $f(x)$, …, $f^{N-1}(x)$:

$$\begin{aligned}
&x, f^N(x), f^{2N}(x), \dots \\
&f(x), f^{N+1}(x), f^{2N+1}(x), \dots \\
&\vdots \\
&f^{N-1}(x), f^{2N-1}(x), f^{3N-1}(x), \dots
\end{aligned}$$

Each sequence converges to x_∞ since f^N is a d-contraction with fixed point x_∞. But then the sequence $x, f(x), f^2(x), \dots$ must converge to x_∞. □

2 Identifiability

Theorem 1 *Let $p_{X,Y}$ be induced by two additive Gaussian noise models, $\mathcal{M}$ and $\tilde{\mathcal{M}}$:*

$$\begin{aligned}
&\mathcal{M}: && \tilde{\mathcal{M}}: \\
&X = f_X(Y) + E_X && X = \tilde{f}_X(Y) + \tilde{E}_X \\
&Y = f_Y(X) + E_Y && Y = \tilde{f}_Y(X) + \tilde{E}_Y \\
&E_X \perp\!\!\!\perp E_Y && \tilde{E}_X \perp\!\!\!\perp \tilde{E}_Y \\
&E_X \sim \mathcal{N}(0, \sigma_X^2) && \tilde{E}_X \sim \mathcal{N}(0, \tilde{\sigma}_X^2) \\
&E_Y \sim \mathcal{N}(0, \sigma_Y^2) && \tilde{E}_Y \sim \mathcal{N}(0, \tilde{\sigma}_Y^2).
\end{aligned}$$

Assuming that $\sup_{x,y} |f'_X(y) f'_Y(x)| < 1$ *and similarly* $\sup_{x,y} \left|\tilde{f}'_X(y)\tilde{f}'_Y(x)\right| < 1$*, then the two corresponding causal graphs coincide:* $\mathcal{G}_{\mathcal{M}} = \mathcal{G}_{\tilde{\mathcal{M}}}$*, i.e.:*

$$f_X \text{ is constant} \iff \tilde{f}_X \text{ is constant}, \qquad \text{and} \qquad f_Y \text{ is constant} \iff \tilde{f}_Y \text{ is constant},$$

or the models are of the following very special form:

- *either:* f_X*,* $\tilde{f}_X$*,* f_Y*,* $\tilde{f}_Y$ *are all affine,*
- *or: one model (say* $\tilde{\mathcal{M}}$*) is acyclic, the other is cyclic, and the following equations hold:*

$$f_Y(x) = Cx + D \text{ with } C \neq 0, f_X(y) = \frac{\tilde{\alpha}_X}{\alpha_X}\tilde{f}_X(y) - \frac{\alpha_Y}{\alpha_X}Cy + \frac{\alpha_Y}{\alpha_X}CD, \tilde{f}_Y(x) = \tilde{D} \quad (1)$$

and $\tilde{f}_X$ *satisfies the following differential equation:*[1]

$$\begin{aligned} &-\frac{1}{\alpha_X}(\tilde{\alpha}_X\tilde{f}_X - \alpha_Y Cy + \alpha_Y CD)(\tilde{\alpha}_X\tilde{f}'_X - \alpha_Y C) + \tilde{\alpha}_X\tilde{f}_X\tilde{f}'_X \\ &= \alpha_Y(y - D) - \tilde{\alpha}_Y(y - \tilde{D}) + C\frac{\tilde{\alpha}_X\tilde{f}''_X}{\alpha_X - (\tilde{\alpha}_X\tilde{f}'_X - \alpha_Y C)C}. \end{aligned} \quad (2)$$

where $\alpha_X = \sigma_X^{-2}$ *and* $\alpha_Y = \sigma_Y^{-2}$*.*

Proof. Writing $\pi_{\dots}(\cdots) := \log p_{\dots}(\cdots)$ for the logarithms of densities, we can reexpress the induced density $p_{X,Y}$ for the bivariate case as:

$$\pi_{X,Y}(x,y) = \pi_{E_X}\big(x - f_X(y)\big) + \pi_{E_Y}\big(y - f_Y(x)\big) + \log|1 - f'_X(y)f'_Y(x)| \quad (3)$$

Partial differentiation with respect to x and y yields the following equation, which will be the equation on which we base our identifiability proof:

$$\frac{\partial^2 \pi_{X,Y}}{\partial x \partial y} = -\pi''_{E_X}\big(x - f_X(y)\big)f'_X(y) - \pi''_{E_Y}\big(y - f_Y(x)\big)f'_Y(x) - \frac{f''_X(y)f''_Y(x)}{\big(1 - f'_X(y)f'_Y(x)\big)^2} \quad (4)$$

We will now specialize to Gaussian noise and prove identifiability of the causal graph. We assume:

$$E_X \sim \mathcal{N}(0, \alpha_X^{-1}), \quad E_Y \sim \mathcal{N}(0, \alpha_Y^{-1})$$

where α_X, α_Y are the precisions (inverse variances) of the Gaussian noise variables. Then, equation (4) simplifies to:

$$\frac{\partial^2 \pi_{X,Y}}{\partial x \partial y} = \alpha_X f'_X(y) + \alpha_Y f'_Y(x) - \frac{f''_X(y)f''_Y(x)}{\big(1 - f'_X(y)f'_Y(x)\big)^2} \quad (5)$$

A similar equation holds for the other model:

$$\frac{\partial^2 \pi_{X,Y}}{\partial x \partial y} = \tilde{\alpha}_X \tilde{f}'_X(y) + \tilde{\alpha}_Y \tilde{f}'_Y(x) - \frac{\tilde{f}''_X(y)\tilde{f}''_Y(x)}{\big(1 - \tilde{f}'_X(y)\tilde{f}'_Y(x)\big)^2} \quad (6)$$

The proof strategy will be to equate the r.h.s. of (5) with that of (6) and to rewrite the resulting equation in the form

$$\Phi_1(x)\Psi_1(y) + \Phi_2(x)\Psi_2(y) + \cdots + \Phi_k(x)\Psi_k(y) = 0 \quad (7)$$

where the functionals $\Phi_i(x)$ and $\Psi_i(y)$ depend only on x and y, respectively:

$$\Phi_i(x) = \Phi_i(x, f'_Y, f''_Y, \tilde{f}'_Y, \tilde{f}''_Y), \qquad \Psi_i(y) = \Psi_i(y, f'_X, f''_X, \tilde{f}'_X, \tilde{f}''_X),$$

We then adopt the solution method from [1, Supplement S.4.3] that gives a general method for solving functional-differential equations of the form (7). The idea behind the solution method is to repeatedly divide by one of the functionals and differentiate with respect to the corresponding

[1] Or similar equations with the roles of X and Y reversed.

variable. Each time, another functional-differential equation of the form (7) is obtained, but with one fewer term. For the case $k = 2$, one can write down all possible solutions:

$$0 = \Psi_1 + C\Psi_2, \quad 0 = C\Phi_1 - \Phi_2, \qquad C \in \mathbb{R}; \tag{8a}$$
$$0 = \Psi_2, \quad 0 = \Phi_1; \tag{8b}$$
$$0 = \Psi_1, \quad 0 = \Psi_2; \tag{8c}$$
$$0 = \Phi_1, \quad 0 = \Phi_2; \tag{8d}$$

Solving the corresponding ordinary differential equations and substituing them into the original functional-differential equation then gives all solutions to the original equation.

We consider two cases: (i) model $\tilde{\mathcal{M}}$ has zero "arrows", i.e., $\tilde{f}'_X = 0$ and $\tilde{f}'_Y = 0$; (ii) model $\tilde{\mathcal{M}}$ has one "arrow", say, $\tilde{f}'_X \neq 0$, $\tilde{f}'_Y = 0$. In both cases we show that generically, model $\mathcal{M}$ must coincide with model $\tilde{\mathcal{M}}$. This implies that if at least one of both models has zero or one arrows, their causal graphs must coincide generically. But then the same must hold if one of the models has two arrows.

(i) $\tilde{\mathcal{M}}$ has zero arrows

We assume $\tilde{f}'_X = \tilde{f}'_Y = 0$. Then the r.h.s. of (6) vanishes, and we obtain the equation

$$0 = \left(\alpha_X f'_X(y) + \alpha_Y f'_Y(x)\right)\left(1 - f'_X(y) f'_Y(x)\right)^2 - f''_X(y) f''_Y(x). \tag{9}$$

Renaming $\phi(x) := f'_Y(x)$, $\psi(y) := f'_X(y)$, we can write:

$$\begin{aligned} 0 &= (\alpha_X\psi + \alpha_Y\phi)\left(1 - \phi\psi\right)^2 - \phi'\psi' \\ &= \alpha_X\psi + \alpha_Y\phi + \psi^2(-2\alpha_X\phi + \alpha_Y\phi^3) + \phi^2(-2\alpha_Y\psi + \alpha_X\psi^3) - \phi'\psi' \end{aligned}$$

which is of the form (7) with

$$\begin{aligned} &\Psi_1 = \alpha_X\psi && \Phi_1 = 1 \\ &\Psi_2 = 1 && \Phi_2 = \alpha_Y\phi \\ &\Psi_3 = \psi^2 && \Phi_3 = -2\alpha_X\phi + \alpha_Y\phi^3 \\ &\Psi_4 = -2\alpha_Y\psi + \alpha_X\psi^3 && \Phi_4 = \phi^2 \\ &\Psi_5 = \psi' && \Phi_5 = -\phi' \end{aligned}$$

After differentiating with respect to x, we obtain again an equation of the form (7) with

$$\begin{aligned} &\Psi_1 = 1 && \Phi_1 = \alpha_Y\phi' \\ &\Psi_2 = \psi^2 && \Phi_2 = -2\alpha_X\phi' + 3\alpha_Y\phi^2\phi' \\ &\Psi_3 = -2\alpha_Y\psi + \alpha_X\psi^3 && \Phi_3 = 2\phi\phi' \\ &\Psi_4 = \psi' && \Phi_4 = -\phi'' \end{aligned}$$

After differentiating with respect to y, we obtain again an equation of the form (7) with

$$\begin{aligned} &\Psi_1 = 2\psi\psi' && \Phi_1 = -2\alpha_X\phi' + 3\alpha_Y\phi^2\phi' \\ &\Psi_2 = -2\alpha_Y\psi' + 3\alpha_X\psi^2\psi' && \Phi_2 = 2\phi\phi' \\ &\Psi_3 = \psi'' && \Phi_3 = -\phi'' \end{aligned}$$

We now assume $\psi'' \neq 0$ everywhere and divide by ψ'', and subsequently differentiate with respect to y:

$$\begin{aligned} &\Psi_1 = \frac{\partial}{\partial y}\left(2\frac{\psi\psi'}{\psi''}\right) && \Phi_1 = -2\alpha_X\phi' + 3\alpha_Y\phi^2\phi' \\ &\Psi_2 = \frac{\partial}{\partial y}\left(-2\frac{\alpha_Y\psi'}{\psi''} + 3\alpha_X\frac{\psi^2\psi'}{\psi''}\right) && \Phi_2 = 2\phi\phi' \end{aligned}$$

Now we study the four possible solutions:

(8a) The first solution implies $C(-2\alpha_X\phi' + 3\alpha_Y\phi^2\phi') - 2\phi\phi' = 0$, and hence $\phi'\big(C(-2\alpha_X + 3\alpha_Y\phi^2) - 2\phi\big) = 0$, i.e., $\phi' = 0$. Substituting this into the original equation yields $\psi' = 0$. However, this is in contradiction with our previous assumption that $\psi'' \neq 0$.

(8b) A similar reasoning as for solution (8a) applies.

(8c) Integrating with respect to y, we obtain

$$C\psi'' = (-2\psi - 2\alpha_Y + 3\alpha_X\psi^2)\psi'$$

Integrating one more time, we obtain

$$C\psi' = -\psi^2 - 2\alpha_Y\psi + \alpha_X\psi^3 + D$$

For the case $C = 0$, we find that ψ is constant, but this violates $\psi'' \neq 0$. For $C \neq 0$, we have the solution

$$\frac{C}{-\psi^2 - 2\alpha_Y\psi + \alpha_X\psi^3 + D}\psi' = 1$$

which can be solved analytically to find x as a function of $\psi(x)$. However, it turns out not to have any real-valued solutions.

(8d) A similar reasoning as for solution (8a) applies.

Therefore, this does not yield any solution.

The final possibility we have to consider is $\psi'' = 0$. This gives an equation of the form (7) with

$$\Psi_1 = 2\psi\psi' \qquad \Phi_1 = -2\alpha_X\phi' + 3\alpha_Y\phi^2\phi'$$
$$\Psi_2 = -2\alpha_Y\psi' + 3\alpha_X\psi^2\psi' \qquad \Phi_2 = 2\phi\phi'$$

Reasoning similarly as before, we now find the only possible solution: ψ and ϕ are constants satisfying $0 = \alpha_X\psi + \alpha_Y\phi$. Thus f_X, f_Y, $\tilde{f}_X$ and $\tilde{f}_Y$ are affine functions.

(ii) $\tilde{\mathcal{M}}$ has one arrow

Now let us deal with the case that the true model contains one arrow, i.e., we assume without loss of generality that $\tilde{f}'_Y = 0$ but $\tilde{f}'_X \neq 0$. Then we obtain from (5) and (6), writing $\tilde{\psi} := \frac{\tilde{\alpha}_X}{\alpha_X}\tilde{f}'_X \neq 0$,$\psi := f'_X$,$\phi := f'_Y$:

$$0 = \Big(\alpha_X(\psi - \tilde{\psi}) + \alpha_Y\phi\Big)\big(1 - \phi\psi\big)^2 - \phi'\psi' \tag{10}$$

This is a functional equation of the form (7) with:

$$\Psi_1 = \alpha_X(\psi - \tilde{\psi}) \qquad \Phi_1 = 1$$
$$\Psi_2 = \big(\alpha_Y - 2\alpha_X(\psi - \tilde{\psi})\psi\big) \qquad \Phi_2 = \phi$$
$$\Psi_3 = \big(-2\alpha_Y + \alpha_X(\psi - \tilde{\psi})\psi\big)\psi \qquad \Phi_3 = \phi^2$$
$$\Psi_4 = \alpha_Y\psi^2 \qquad \Phi_4 = \phi^3$$
$$\Psi_5 = \psi' \qquad \Phi_5 = -\phi'$$

We start by differentiating with respect to x:

$$\Psi_1 = \big(\alpha_Y - 2\alpha_X(\psi - \tilde{\psi})\psi\big) \qquad \Phi_1 = \phi'$$
$$\Psi_2 = \big(-2\alpha_Y + \alpha_X(\psi - \tilde{\psi})\psi\big)\psi \qquad \Phi_2 = 2\phi\phi'$$
$$\Psi_3 = \alpha_Y\psi^2 \qquad \Phi_3 = 3\phi^2\phi'$$
$$\Psi_4 = \psi' \qquad \Phi_4 = -\phi''$$

Now we divide by ϕ' (assuming that $\phi' \neq 0$ everywhere) and differentiate with respect to x:

$$\Psi_1 = \big(-2\alpha_Y + \alpha_X(\psi - \tilde{\psi})\psi\big)\psi \qquad \Phi_1 = 2\phi'$$
$$\Psi_2 = \alpha_Y\psi^2 \qquad \Phi_2 = 3\phi\phi'$$
$$\Psi_3 = \psi' \qquad \Phi_3 = -\frac{\phi'''}{\phi'} + \left(\frac{\phi''}{\phi'}\right)^2$$

Again we divide by ϕ' (assuming that $\phi' \neq 0$ everywhere) and differentiate with respect to x:

$$\Psi_1 = \alpha_Y \psi^2 \qquad \Phi_1 = 3\phi'$$

$$\Psi_2 = \psi' \qquad \Phi_2 = -\frac{\partial}{\partial x}\left(\frac{\phi'''}{(\phi')^2} + \frac{(\phi'')^2}{(\phi')^3}\right)$$

We investigate each possible solution:

(8a) This would imply $\psi = 0$ or $\psi' = A\psi^2$, which implies $(1/\psi)' = -A$ and therefore $\psi(y) = (B - Ay)^{-1}$. We consider both cases:

- $\psi = 0$. Substitution into the original equation yields $\alpha_X \tilde{\psi} = \alpha_Y \phi$ which implies $\phi = C$. This is in contradiction with our assumption that $\phi' \neq 0$.
- Substituting $\psi(y) = (B - Ay)^{-1}$ into the original equation yields (with $A \neq 0$):

  $$0 = \Big(\alpha_X\big((B - Ay)^{-1} - \tilde{\psi}(y)\big) + \alpha_Y \phi(x)\Big)\big(1 - \phi(x)(B - Ay)^{-1}\big)^2 - A(B - Ay)^{-2}\phi'(x)$$

  Multiplying with $(B - Ay)^2$:

  $$0 = \Big(\alpha_X\big((B - Ay)^{-1} - \tilde{\psi}(y)\big) + \alpha_Y \phi(x)\Big)\big((B - Ay) - \phi(x)\big)^2 - A\phi'(x)$$

  Writing out the products:

  $$\begin{aligned} 0 &= \Big(\alpha_X\big((B - Ay)^{-1} - \tilde{\psi}(y)\big)\Big)(B - Ay)^2 \\ &\quad + \phi(x)\Big(-2\alpha_X\big((B - Ay)^{-1} - \tilde{\psi}(y)\big)(B - Ay) + \alpha_Y(B - Ay)^2\Big) \\ &\quad + \phi^2(x)\Big(\alpha_X\big((B - Ay)^{-1} - \tilde{\psi}(y)\big) - 2\alpha_Y(B - Ay)\Big) \\ &\quad + \alpha_Y \phi^3(x) - A\phi'(x) \end{aligned}$$

  Differentiating with respect to y:

  $$\begin{aligned} 0 &= \alpha_X \frac{\partial}{\partial y}\Big((B - Ay) - \tilde{\psi}(y)(B - Ay)^2\Big) \\ &\quad + \phi(x)\frac{\partial}{\partial y}\Big(-2\alpha_X\big((B - Ay)^{-1} - \tilde{\psi}(y)\big)(B - Ay) + \alpha_Y(B - Ay)^2\Big) \\ &\quad + \phi^2(x)\frac{\partial}{\partial y}\Big(\alpha_X\big((B - Ay)^{-1} - \tilde{\psi}(y)\big) - 2\alpha_Y(B - Ay)\Big) \end{aligned}$$

  Differentiating with respect to x:

  $$\begin{aligned} 0 &= \phi'(x)\frac{\partial}{\partial y}\Big(-2\alpha_X\big((B - Ay)^{-1} - \tilde{\psi}(y)\big)(B - Ay) + \alpha_Y(B - Ay)^2\Big) \\ &\quad + 2\phi'(x)\phi(x)\frac{\partial}{\partial y}\Big(\alpha_X\big((B - Ay)^{-1} - \tilde{\psi}(y)\big) - 2\alpha_Y(B - Ay)\Big) \end{aligned}$$

  This is again a functional-differential equation of the form (7), with $k = 2$ terms. The solutions are therefore $\phi' = 0$ (however, this is in contradiction with our earlier assumption) and

  $$\begin{aligned} &\frac{\partial}{\partial y}\Big(-2\alpha_X\big((B - Ay)^{-1} - \tilde{\psi}(y)\big)(B - Ay) + \alpha_Y(B - Ay)^2\Big) = 0, \\ &\frac{\partial}{\partial y}\Big(\alpha_X\big((B - Ay)^{-1} - \tilde{\psi}(y)\big) - 2\alpha_Y(B - Ay)\Big) = 0 \end{aligned}$$

  or equivalently

  $$\begin{aligned} &2\alpha_X \tilde{\psi}'(y)(B - Ay) - 2A\alpha_X \tilde{\psi}(y) - 2A\alpha_Y(B - Ay) = 0, \\ &A\alpha_X(B - Ay)^{-2} + 2\alpha_Y A = \alpha_X \tilde{\psi}'(y) \end{aligned}$$

  and therefore

  $$\begin{aligned} &(B - Ay)^{-1} = \tilde{\psi}(y), \\ &A\big(\alpha_X(B - Ay)^{-2} + 2\alpha_Y\big) = \alpha_X \tilde{\psi}'(y) \end{aligned}$$

  which is a contradiction.

(8b) This implies $\psi' = 0$, i.e., $\psi = D$. Solution into the original equation yields $\alpha_X(\tilde{\psi} - D) = \alpha_Y \phi$ which implies $\phi = C$, which contradicts our assumption $\phi' \neq 0$.

(8c) Again this contradicts $\phi' \neq 0$.

(8d) This directly contradicts $\phi' \neq 0$.

Finally, we have to check the candidate solution $\phi' = 0$. This is indeed a possible solution:

$$\phi = C, \psi - \tilde{\psi} = -\frac{\alpha_Y}{\alpha_X} C$$

or

$$f'_Y(x) = C, \qquad \sigma_{E_X}^{-2} f'_X(y) - \tilde{\sigma}_{E_X}^{-2} \tilde{f}'_X = -\sigma_{E_Y}^{-2} C$$

However, in general, this need not be a solution to the fundamental equation (3).

If $\phi = C = 0$, then both models coincide. So assume $\phi = C \neq 0$. First, note that $\phi = C$ with (10) implies that $\psi = \tilde{\psi} - C\frac{\alpha_Y}{\alpha_X}$, and therefore $\alpha_X f_X = \tilde{\alpha}_X \tilde{f}_X - \alpha_Y C y + A$. Let $f_Y(x) = Cx + D$. Remember that we assumed $\tilde{f}'_Y(x) = 0$; let $\tilde{f}_Y(x) = \tilde{D}$. Differentiating (3) and the corresponding equation for $\tilde{\mathcal{M}}$ with respect to x yields:

$$\begin{aligned}\frac{\partial \pi_{X,Y}}{\partial x} &= -\alpha_X(x - f_X(y)) + \alpha_Y(y - f_Y(x))f'_Y(x) - \frac{f'_X(y)f''_Y(x)}{1 - f'_X(y)f'_Y(x)} \\ &= -\tilde{\alpha}_X(x - \tilde{f}_X(y)) + \tilde{\alpha}_Y(y - \tilde{f}_Y(x))\tilde{f}'_Y(x) - \frac{\tilde{f}'_X(y)\tilde{f}''_Y(x)}{1 - \tilde{f}'_X(y)\tilde{f}'_Y(x)}\end{aligned}$$

i.e.,

$$-\alpha_X(x - f_X(y)) + \alpha_Y(y - Cx - D)C = -\tilde{\alpha}_X(x - \tilde{f}_X(y))$$

or in other words:

$$x(\tilde{\alpha}_X - \alpha_X - \alpha_Y C^2) + y(\alpha_Y C) - \alpha_Y CD = -\alpha_X f_X(y) + \tilde{\alpha}_X \tilde{f}_X(y) = \alpha_Y C y - A,$$

which implies

$$x(\tilde{\alpha}_X - \alpha_X - \alpha_Y C^2) = -A + \alpha_Y CD$$

and hence

$$\tilde{\alpha}_X - \alpha_X - \alpha_Y C^2 = 0, A = \alpha_Y CD.$$

Differentiating (3) and the corresponding equation for $\tilde{\mathcal{M}}$ with respect to y yields:

$$\begin{aligned}\frac{\partial \pi_{X,Y}}{\partial y} &= \alpha_X(x - f_X(y))f'_X(y) - \alpha_Y(y - f_Y(x)) - \frac{f''_X(y)f'_Y(x)}{1 - f'_X(y)f'_Y(x)} \\ &= \tilde{\alpha}_X(x - \tilde{f}_X(y))\tilde{f}'_X(y) - \tilde{\alpha}_Y(y - \tilde{f}_Y(x)) - \frac{\tilde{f}''_X(y)\tilde{f}'_Y(x)}{1 - \tilde{f}'_X(y)\tilde{f}'_Y(x)}\end{aligned}$$

i.e.,

$$\begin{aligned}&x\alpha_X f'_X - \alpha_X f_X f'_X - \alpha_Y(y - Cx - D) - \frac{f''_X C}{1 - f'_X C} \\ &= x\tilde{\alpha}_X \tilde{f}'_X - \tilde{\alpha}_X \tilde{f}_X \tilde{f}'_X - \tilde{\alpha}_Y(y - \tilde{D}),\end{aligned}$$

in other words

$$-\alpha_X f_X f'_X + \tilde{\alpha}_X \tilde{f}_X \tilde{f}'_X - \frac{f''_X C}{1 - f'_X C} = \alpha_Y(y - D) - \tilde{\alpha}_Y(y - \tilde{D}).$$

Eliminating f'_X, f_X and f''_X:

$$\begin{aligned}&-\frac{1}{\alpha_X}(\tilde{\alpha}_X \tilde{f}_X - \alpha_Y C y + \alpha_Y CD)(\tilde{\alpha}_X \tilde{f}'_X - \alpha_Y C) + \tilde{\alpha}_X \tilde{f}_X \tilde{f}'_X - C\frac{\tilde{\alpha}_X \tilde{f}''_X}{\alpha_X - (\tilde{\alpha}_X \tilde{f}'_X - \alpha_Y C)C} \\ &= \alpha_Y(y - D) - \tilde{\alpha}_Y(y - \tilde{D})\end{aligned}$$

This second-order nonlinear differential equation in $\tilde{f}_X$ implies that this can only be a solution in very special cases: all model parameters have to be chosen in a very specific, non-generic way. □

References

[1] A.D. Polyanin and V.F. Zaitsev. *Handbook of Nonlinear Partial Differential Equations*. Chapman & Hall / CRC, 2004.