Exercises MLSS 2018: *Causality*

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1. Simpson's Paradox

You are investigating the effectiveness of a drug against a deadly disease. You are given access to data collected by health insurance companies about their customers. You divide the diseased customers into two groups: those that took the drug ("treatment group"), and those that didn't take the drug ("control group"). Some of the customers recovered, others unfortunately didn't recover. The reasons why some patients were treated and others were not, are unknown to you. You find the following numbers:

	Recovery	No recovery	Total	Recovery rate
Drug	20	20	40	...%
No drug	16	24	40	...%
Total	36	44	80	

1a. Calculate the recovery rates (in %) for both treatment and control group.
1b. If you were diseased, would you take the drug, or not?

Upon closer inspection of the data, you notice something peculiar when you group patients according to gender:

Males	Recovery	No recovery	Total	Recovery rate
Drug	18	12	30	...%
No drug	7	3	10	...%
Total	25	15	40	

Females	Recovery	No recovery	Total	Recovery rate
Drug	2	8	10	...%
No drug	9	21	30	...%
Total	11	29	40	

2a. Calculate the recovery rates (in %) for both the treatment and the control groups, for both subpopulations (males and females).
2b. In light of these numbers, would you take the drug if you were diseased, or not?

3. What would be your advice to a diseased patient with unknown gender?

This phenomenon is known as Simpson's paradox. A lot has been written about this paradox, but it dissolves once you recognize that you should not make the mistake of interpreting correlations as causations, as we'll see later today.

2. Paths, colliders, blocked paths and d-separation

Definition 1 (Paths, Ancestors) *Let $\mathcal{G}$ be a directed mixed graph.*

- *A **path** q in $\mathcal{G}$ is a sequence of adjacent edges in $\mathcal{G}$ in which no node occurs more than once.*

- *A path consisting of directed edges $X_{i_1} \to X_{i_2} \to X_{i_3} \to \cdots \to X_{i_k}$ that all point in the same direction is called a* ***directed*** *path.*
- *If there is a directed path from X to Y (or if $X = Y$), X is called a* ***ancestor*** *of Y.*
- *The ancestors of Y are denoted $\mathrm{an}_{\mathcal{G}}(Y)$, and include Y.*

Definition 2 (Colliders, Blocked Paths, d-separation) *Let $\mathcal{G}$ be a directed mixed graph, and q a path on $\mathcal{G}$.*

- *A* ***collider*** *on q is a (non-endpoint) node X on q with precisely two arrowheads pointing towards X on the adjacent edges:*

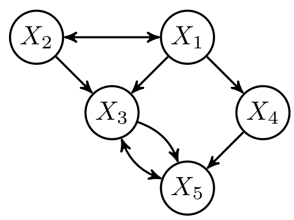
$$\to X \leftarrow, \quad \to X \leftrightarrow, \quad \leftrightarrow X \leftarrow, \quad \leftrightarrow X \leftrightarrow$$

- *A* ***non-collider*** *on q is any node on the path which is not a collider.*

A set of nodes $\boldsymbol{S}$ in $\mathcal{G}$ is said to ***block*** *q if q contains a non-collider which is in $\boldsymbol{S}$, or a collider which is* not *an ancestor of $\boldsymbol{S}$.*

For three sets $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ of nodes in $\mathcal{G}$, we say that ***$\boldsymbol{X}$ and $\boldsymbol{Y}$ are d-separated by $\boldsymbol{Z}$*** *iff all paths between a node in $\boldsymbol{X}$ and a node in $\boldsymbol{Y}$ are blocked by $\boldsymbol{Z}$, and write $\boldsymbol{X} \perp_{\mathcal{G}} \boldsymbol{Y} \mid \boldsymbol{Z}$.*

Consider the following directed mixed graph $\mathcal{G}$:



1a. Is $X_3 \to X_5 \leftrightarrow X_3$ a path? Is it a directed path?
1b. Is $X_3 \leftrightarrow X_5$ a path? Is it a directed path?
1c. Is $X_5 \leftarrow X_3 \leftarrow X_1$ a path? Is it a directed path?
1d. What are the ancestors of X_4?

Consider the path $X_2 \leftrightarrow X_1 \to X_3 \leftrightarrow X_5 \leftarrow X_4$ on $\mathcal{G}$.

2a. Which nodes on the path are colliders?
2b. Which nodes on the path are non-colliders?
2c. Does $\{X_3\}$ block this path? Does $\{X_5\}$ block this path? Does $\{X_3, X_5\}$ block this path?
2d. Does X_1 d-separate X_2 from X_4?
2e. Is $X_1 \perp X_5 \mid \{X_3, X_4\}$?

3. The Back-Door Criterion

Theorem 1 (Back-Door Criterion (Pearl, 2000)) *For an* ***acyclic*** *SCM $\mathcal{M}$, variables X, Y and set of variables $\boldsymbol{H}$: if*

1. *$X, Y \notin \boldsymbol{H}$;*
2. *X is not an ancestor of any variable in $\boldsymbol{H}$ in $\mathcal{G}(\mathcal{M})$;*
3. *$\boldsymbol{H}$ blocks all* ***back-door paths*** *$X \leftarrow \ldots Y$ and $X \leftrightarrow \ldots Y$ in $\mathcal{G}(\mathcal{M})$ (i.e., all paths between X and Y that start with an arrowhead at X).*

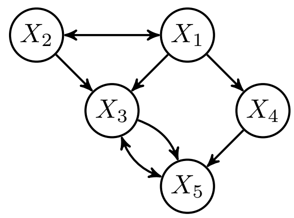
then $\boldsymbol{H}$ *is called* admissible for adjustment *to find the causal effect of* X *on* Y*, and this causal effect is given by:*

$$p_{\mathcal{M}}\big(y \,|\, \mathrm{do}(X=x)\big) = \int p_{\mathcal{M}}(y \,|\, x, \boldsymbol{h}) p_{\mathcal{M}}(\boldsymbol{h})\, d\boldsymbol{h} \left(= \sum_{\boldsymbol{h}} p_{\mathcal{M}}(y \,|\, x, \boldsymbol{h}) p_{\mathcal{M}}(\boldsymbol{h}) \right).$$

For the special case $\boldsymbol{H} = \emptyset$*, this should be read as:*

$$p_{\mathcal{M}}\big(y \,|\, \mathrm{do}(X=x)\big) = p_{\mathcal{M}}(y \,|\, x).$$

Consider an SCM $\mathcal{M}$ with the following functional graph $\mathcal{G}(\mathcal{M})$:



1a. Give a set that is admissible for adjustment to find the causal effect of X_4 *on* X_5*.*
1b. Provide an expression for this causal effect in terms of the observational distribution.

2a. Give a set that is admissible for adjustment to find the causal effect of X_1 *on* X_5*.*
2b. Provide an expression for this causal effect in terms of the observational distribution.

3. Is $\emptyset$ *admissible for adjustment to find the causal effect of* X_1 *on* X_4*? If so, provide an expression for this causal effect in terms of the observational distribution.*

4a. Which sets are admissible for adjustment to find the causal effect of X_3 *on* X_5*?*
4b. Which sets are admissible for adjustment to find the causal effect of X_5 *on* X_4*?*

4. Simpson's Paradox: resolution

This exercise continues where exercise 1 ended. We will make use of causal reasoning with SCMs to resolve Simpson's paradox.

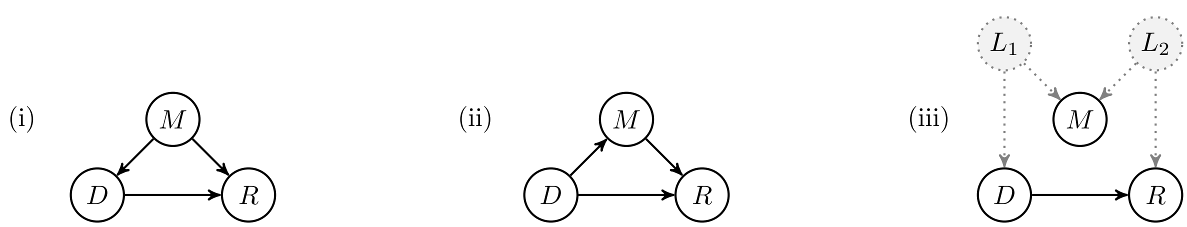


Figure 1: Different hypothetical functional graphs, where R stands for *Recovery*, D for taking the *Drug*, and M has different interpretations in cases (i), (ii) and (iii).

Suppose you believe that an SCM with functional graph as in Figure 1(i) applies, where M denotes gender of the patient (male/female).

1a. Apply the back-door criterion to obtain a formula that expresses $p(r \,|\, \mathrm{do}(D=d))$ *in terms of observable quantities (i.e., in terms of marginal or conditional distributions where the* do*-operator does not appear).*
1b. Is $p(r \,|\, \mathrm{do}(D=d)) = p(r \,|\, d)$ *in this case?*
1c. What would be your advice for a patient with unknown gender?

Now suppose that instead, you believe that an SCM with functional graph as in Figure 1(ii) applies. Intuitively, this would be quite unlikely, as we know that most drugs don't change gender, but we could have used a slightly different story where the variable M has a different interpretation (for example, "blood pressure"), and then this causal structure would also be a plausible one.

2a. Again, use the back-door criterion to express $p(r \mid \mathrm{do}(D = d))$ in terms of observable quantities.
2b. Is $p(r \mid \mathrm{do}(D = d)) = p(r \mid d)$ in this case?
2c. What would be your advice for a patient with unknown M (say, blood pressure) in this case?

Finally, suppose that you believe that the SCM has the functional graph of Figure 1(iii).

3a. Invent an interpretation of M and the two latent variables L_1, L_2 yourself that could match the causal model depicted in Figure 1(iii).
3b. Express $p(r \mid \mathrm{do}(D = d))$ in terms of observable *quantities.*
3c. Is $p(r \mid \mathrm{do}(D = d)) = p(r \mid d)$ in this case?
3d. Again, what would be your advice for a patient with unknown M in this case?

Conclusion: whether or not you should prescribe the drug depends on which causal model you believe to apply to this situation. The fact that different causal models will lead to different conclusions should not be paradoxical, it is another illustration that "correlation does not imply causation".

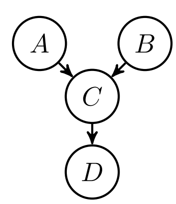
5. Y-structure

Theorem 2 (Global Markov Property) *For an* ***acyclic*** *SCM, the following* Global Markov Property *holds:*

$$\boldsymbol{X}, \boldsymbol{Y} \underset{\mathcal{G}(\mathcal{M})}{\perp} \boldsymbol{Z} \qquad \Longrightarrow \qquad \boldsymbol{X} \underset{p_{\mathcal{M}}}{\perp\!\!\!\perp} \boldsymbol{Y} \mid \boldsymbol{Z}$$

for all subsets $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ of nodes.

Given an SCM $\mathcal{M}$ with the following functional graph:



Which (conditional) independences in $p_{\mathcal{M}}$ are implied by the Global Markov Property?

6. LCD

Theorem 3 (Cooper, 1997) *Given an acyclic SCM $\mathcal{M}$ with three endogenous variables I, X, Y. If:*

1. $X \to I \notin \mathcal{G}(\mathcal{M})$,
2. $Y \to I \notin \mathcal{G}(\mathcal{M})$,
3. $I \not\perp\!\!\!\perp_{p_{\mathcal{M}}} X$,
4. $X \not\perp\!\!\!\perp_{p_{\mathcal{M}}} Y$,
5. $I \perp\!\!\!\perp_{p_{\mathcal{M}}} Y \mid X$,
6. *Faithfulness holds, i.e., the Global Markov Property gives* all *(conditional) independences in $p_{\mathcal{M}}$.*

Then $X \to Y \in \mathcal{G}(\mathcal{M})$ and $X \leftrightarrow Y \notin \mathcal{G}(\mathcal{M})$.

Prove this theorem by considering for each possible ADMGs $\mathcal{G}(\mathcal{M})$ whether it satisfies the assumptions. (Hint: could there be an edge between I and Y?)