On Causal Explanations of Quantum Correlations

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Quantum Foundations ↔ Causal Inference
Operational characterization of Quantum Theory

\[ p(a|P, M) \equiv \text{The probability of outcome } a \text{ given measurement } M \text{ and preparation } P \]
Operational characterization of Quantum Theory

Vector in Hilbert space

\[ |\psi\rangle \in \mathcal{H} \]

Hermitian operator

\[ \hat{A} \]

Eigenvectors

\[ \{ |a\rangle \} \]

Outcome

\[ \alpha \]

Preparation

\[ P \]

Measurement

\[ M \]

Probability

\[ p(a|P, M) = |\langle \psi |a \rangle|^2 \]
A set of Hermitian operators can only be jointly measured if they commute relative to the matrix commutator.

\[ [\hat{Q}, \hat{P}] \neq 0 \]

measure \( \hat{Q} \) measure \( \hat{P} \)

\[ [\hat{Q}_A - \hat{Q}_B, \hat{P}_A + \hat{P}_B] = 0 \]

measure \( \hat{Q}_A - \hat{Q}_B \) and \( \hat{P}_A + \hat{P}_B \)
Collapse rule

Measure $\hat{Q}$ find $q$
Collapse rule

Measure $\hat{Q}$ find $q$

Prepare $\hat{P}$ with value $p$
Collapse rule

Measure $\hat{P}$ find $p'$

Measure $\hat{Q}$ find $q$

Prepare $\hat{P}$ with value $p$

“Objective Randomness!”
Einstein-Podolsky-Rosen experiment

Measure $\hat{Q}_A$ find $q$

$|\psi_{EPR}\rangle$

$(\hat{Q}_B - \hat{Q}_A)|\psi_{EPR}\rangle_{AB} = 0$

$(\hat{P}_B + \hat{P}_A)|\psi_{EPR}\rangle_{AB} = 0$

collapses to $|q\rangle$
Einstein-Podolsky-Rosen experiment

\[ |\psi_{EPR}\rangle \]

\[ (\hat{Q}_B - \hat{Q}_A)|\psi_{EPR}\rangle_{AB} = 0 \]

\[ (\hat{P}_B + \hat{P}_A)|\psi_{EPR}\rangle_{AB} = 0 \]
Einstein-Podolsky-Rosen experiment

Measure $\hat{P}_A$ find $p$

\[ (\hat{Q}_B - \hat{Q}_A)|\psi_{EPR}\rangle_{AB} = 0 \]
\[ (\hat{P}_B + \hat{P}_A)|\psi_{EPR}\rangle_{AB} = 0 \]

"Spooky action at a distance"

collapses to $| - p \rangle$
A set of variables can only be **jointly known** if they commute relative to the Poisson bracket.

\[
\text{know } Q \quad \text{know } P \quad \text{know } Q_A - Q_B \text{ and } P_A + P_B
\]

\[
P(q, p) \propto \delta(q - a) \quad P(q, p) \propto \delta(p - b) \quad P_{EPR}(q_A, p_A, q_B, p_B) \propto \delta(q_A - q_B)\delta(p_A + p_B)
\]

Statistical theory of classical mechanics with an epistemic restriction
Einstein-Podolsky-Rosen experiment

\[ P_{EPR}(q_A, p_A, q_B, p_B) \propto \delta(q_A - q_B)\delta(p_A + p_B) \]

\[ Q_B - Q_A = 0 \]

\[ P_B + P_A = 0 \]
Einstein-Podolsky-Rosen experiment

Measure $Q_A$ find $q$

\[
P_{\text{EPR}}(q_A, p_A, q_B, p_B) \propto \delta(q_A - q_B)\delta(p_A + p_B)
\]

$Q_B - Q_A = 0$

$P_B + P_A = 0$
Einstein-Podolsky-Rosen experiment

Measure $P_A$ find $p$

\[
P_{\text{EPR}}(q_A, p_A, q_B, p_B) \propto \delta(q_A - q_B)\delta(p_A + p_B)
\]

\[
Q_B - Q_A = 0
\]

\[
P_B + P_A = 0
\]
Einstein-Podolsky-Rosen experiment

Measure $Q_A$ find $q$

\[
P_{\text{EPR}}(q_A, p_A, q_B, p_B) \propto \delta(q_A - q_B)\delta(p_A + p_B)
\]

$Q_B - Q_A = 0$

$P_B + P_A = 0$
Einstein-Podolsky-Rosen experiment

Measure $P_A$ find $p$

\[ P_{\text{EPR}}(q_A, p_A, q_B, p_B) \propto \delta(q_A - q_B)\delta(p_A + p_B) \]

\[ Q_B - Q_A = 0 \]

\[ P_B + P_A = 0 \]
Measure $Q$ find $q$

But this would violate the epistemic restriction!
Collapse Rule

Measure $Q$ find $q$

But this would violate the epistemic restriction!
Collapse Rule

Measure $Q'_A$ find $q_f$

$H_{AB} = P_A Q_B$

Prepare $Q_A$ with value $q_i$
Collapse Rule

\[ H_{AB} = P_A Q_B \]

Prepare \( Q_A \) with value \( q_i \)
Measure $Q_A'$ find $q_f$

$H_{AB} = P_A Q_B$

Prepare $Q_A$ with value $q_i$
“But our present quantum mechanical formalism is not purely epistemological; it is a peculiar mixture describing in part realities of Nature, in part incomplete human information about Nature --- all scrambled up by Heisenberg and Bohr into an omelette that nobody has seen how to unscramble.”

E.T. Jaynes
Simpson’s Paradox

\[ P(\text{recovery} \mid \text{do (treatment)}) \neq P(\text{recovery} \mid \text{observe (treatment)}) \]

Influence inference
Brief review of causal inference algorithms

J. Pearl, Causality: Models, Reasoning and Inference
P. Spirtes, C. Glymour, R. Scheines, Causation, Prediction and Search
Functional causal model

Causal Structure

Parameters

\[ P(W) \]
\[ P(S) \]
\[ P(T) \]
\[ P(\lambda) \]
\[ P(\mu) \]
\[ X = f(S, T, W, Y, \lambda) \]
\[ Y = g(T, W, \mu) \]
Reichenbach’s principle
If $X$ and $Y$ are dependent, then

$X \rightarrow Y$ or $X \leftarrow Y$ or $\lambda$

or

$X \rightarrow Y$ or $X \leftarrow Y$ or $\lambda$

Functional causal model

Causal Structure

\[
\begin{align*}
\lambda &\quad W \\
S &\quad X \\
Y &\quad T \\
\mu &\quad W
\end{align*}
\]

Parameters

\[
\begin{align*}
P(W) \\
P(S) \\
P(T) \\
P(\lambda) \\
P(\mu) \\
X &= f(S, T, W, Y, \lambda) \\
Y &= g(T, W, \mu)
\end{align*}
\]

- Parentless variables are independently distributed
Causal model

Causal Structure

Parameters

\[
P(W) \\
P(S) \\
P(T) \\
P(X|S,T,W,Y) \\
P(Y|T,W)
\]
Causal model

Causal Structure

Causal model

Causal Structure

Parameters

\[ P(W) \]
\[ P(S) \]
\[ P(T) \]
\[ P(X|S,T,W,Y) \]
\[ P(Y|T,W) \]

\[
\]

Causal inference algorithms seek to solve the inverse problem
Inferring facts about the causal structure from the conditional independences
Faithfulness (No fine-tuning)
A causal model of an observed distribution is fine-tuned if the conditional independences in the distribution only hold for a set of measure zero of the values of the parameters in the model.
Inferring facts about the causal structure from the strength of correlations
Strength of Correlations

\[ P(X, Y, Z) = \frac{1}{2} [000] + \frac{1}{2} [111] \]
Strength of Correlations

\[ P(X, Y, Z) = (1 - \epsilon)(\frac{1}{2}[000] + \frac{1}{2}[111]) + \epsilon(\text{other}) \]

- Janzing and Beth, IJQI 4, 347 (2006)
- Steudel and Ay, arXiv:1010:5720
- Branciard, Rosset, Gisin, Pironio, PRA 85, 3 (2012)
Strength of Correlations

\[ P(X, Y, Z) = \frac{1}{3} [001] + \frac{1}{3} [010] + \frac{1}{3} [100] \]

Joint work with Matt Pusey, Tobias Fritz, and Wah Loon Keng
A deficiency of many causal inference algorithms

Certain versions of Occam’s razor lead to incorrect causal conclusions

\[ P(X, Y, Z) = \frac{1}{3}[001] + \frac{1}{3}[010] + \frac{1}{3}[100] \]

Set of CI relations among \(X, Y, Z\) is the empty set

Set of faithful causal models for the given set of CI relations on observed variables

Set of faithful causal models for the given probability distribution over the observed variables
What are the causal structures for which CI relations do not capture all the constraints on the observed distribution?

A sufficient condition was found in:
Henson, Lal, Pusey, arXiv:1405.2572

4 nodes

5 nodes
6 nodes
Can we find a causal explanation of quantum correlations?

Chris Wood and RWS, arXiv:1208.4119
What $P(A,B,X,Y)$ is observed?
\[ P(X, Y) = \left( \frac{1}{2} [0] + \frac{1}{2} [1] \right) \left( \frac{1}{2} [0] + \frac{1}{2} [1] \right) \]

\[ P(A, B | X, Y) = \begin{cases} \frac{1}{2} [00] + \frac{1}{2} [11] & \text{if } XY = 0 \\ \frac{1}{2} [01] + \frac{1}{2} [10] & \text{if } XY = 1 \end{cases} \]
\[ P(X, Y) = \left( \frac{1}{2}[0] + \frac{1}{2}[1] \right) \left( \frac{1}{2}[0] + \frac{1}{2}[1] \right) \]

\[ P(A, B|X, Y) = \begin{cases} 
\frac{1}{2}[00] + \frac{1}{2}[11] & \text{if } XY = 0 \\
\frac{1}{2}[01] + \frac{1}{2}[10] & \text{if } XY = 1 
\end{cases} \]
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\[ P(X, Y) \]
\[ = \left( \frac{1}{2}[0] + \frac{1}{2}[1] \right) \left( \frac{1}{2}[0] + \frac{1}{2}[1] \right) \]

\[ P(A, B|X, Y) \]
\[ = \frac{1}{2}[00] + \frac{1}{2}[11] \quad \text{if} \ X Y = 0 \]
\[ = \frac{1}{2}[01] + \frac{1}{2}[10] \quad \text{if} \ X Y = 1 \]
\[ P(X, Y) \]
\[
\begin{align*}
P(X, Y) &= (0 \cdot 0 + \frac{1}{2}[1]) + \frac{1}{2}[1] \\
&= \frac{1}{2}[00] + \frac{1}{2}[01] + \frac{1}{2}[10] + \frac{1}{2}[11] \\
&= \frac{1}{2}[00] + \frac{1}{2}[01] + \frac{1}{2}[10] \quad \text{if } XY = 0 \\
&= \frac{1}{2}[01] + \frac{1}{2}[10] \quad \text{if } XY = 1
\end{align*}
\]

\[ P(A, B | X, Y) \]
\[
\begin{align*}
P(A, B | X, Y) &= \begin{cases} 
0 & \text{if } X = 0, Y = 1 \\
1 & \text{if } X = 0, Y = 0 \\
\frac{1}{2}[00] & \text{if } X = 1, Y = 1 \\
\frac{1}{2}[10] & \text{if } X = 1, Y = 0 \\
\frac{1}{2}[01] & \text{if } X = 0, Y = 1 \\
\frac{1}{2}[11] & \text{if } X = 1, Y = 1
\end{cases}
\end{align*}
\]

\[ x \perp y \]

\[ b \perp x \mid y \]

\[ a \perp y \mid x \]
\[ P(X, Y) \]
\[ = (\frac{1}{2}[00] + \frac{1}{2}[01]) \]
\[ X \perp Y \]
\[ B \perp X | Y \]
\[ A \perp Y | X \]
\[ = \frac{1}{2}[00] + \frac{1}{2}[10] \quad \text{if } XY = 1 \]
\[ P(X, Y) = (\frac{1}{2}[00] + \frac{1}{2}[1]) \]

\[ P(A, B) = \begin{cases} 
\frac{1}{2}[00] & \text{if } Y = 0 \\
\frac{1}{2}[01] + \frac{1}{2}[10] & \text{if } XY = 1 
\end{cases} \]

\[ X \perp Y \\
B \perp X \mid Y \\
A \perp Y \mid X \]
\[
P(X, Y) = (0, 0) \frac{1}{2} + (0, 1) \frac{1}{2} \text{ if } XY = 0
\]
\[
P(A, B) = \frac{1}{2} \text{ if } X = 0, Y = 0
\]
\[
P(A, B) = \frac{1}{2} \frac{1}{2} \frac{1}{2} \text{ if } X = 1, Y = 1
\]
\[
P(X, Y) = (\frac{1}{2}[00] + \frac{1}{2}[01] + \frac{1}{2}[10])
\]
\[
P(A, B, X, Y) = \frac{1}{2}[00] + \frac{1}{2}[10] \quad \text{if } XY = 1
\]
\[
\begin{align*}
X & \perp Y \\
B & \perp X | Y \\
A & \perp Y | X
\end{align*}
\]
\[
\begin{align*}
P(X, Y) &= \left(\frac{1}{2}[01] + \frac{1}{2}[1]\right) + \frac{1}{2}[1] \\
P(A, B | Y) &= \frac{1}{2}[00] + \frac{1}{2}[10] \quad \text{if } Y = 0 \\
&= \frac{1}{2}[01] + \frac{1}{2}[10] \quad \text{if } XY = 1
\end{align*}
\]
\[ P(X, Y) \]
\[ = \left( \frac{1}{2}[00] + \frac{1}{2}[01] \right) + \frac{1}{2}[1] \]
\[ X \perp Y \]
\[ B \perp X \mid Y \]
\[ A \perp Y \mid X \]
\[ P(A, B) \]
\[ = \frac{1}{4}[00] + \frac{1}{2}[01] + \frac{1}{4}[10] + \frac{1}{4}[11] \]
\[ = \frac{1}{2}[00] + \frac{1}{2}[11] \quad \text{if } XY = 0 \]
\[ = \frac{1}{2}[01] + \frac{1}{2}[10] \quad \text{if } XY = 1 \]
Nothing works!
• Reichenbach’s principle

• No fine-tuning

\[
P(X, Y) = (\frac{1}{2}[0] + \frac{1}{2}[1])(\frac{1}{2}[0] + \frac{1}{2}[1])
\]

\[
P(A, B|X, Y) = \frac{1}{2}[00] + \frac{1}{2}[11] \text{ if } XY = 0
\]

\[
= \frac{1}{2}[01] + \frac{1}{2}[10] \text{ if } XY = 1
\]

Contradiction with

\[
X \perp Y
\]

\[
B \perp X|Y
\]

\[
A \perp Y|X
\]
• Reichenbach’s principle

• No fine-tuning

Contradiction with quantum theory and experiment
• Reichenbach’s principle

• No fine-tuning

• In the causal model, unobserved nodes are described by classical variables and our knowledge of these is described by classical probability theory

Contradiction with quantum theory and experiment
Quantum Causal Models


cannot reproduce the quantum correlations

\[ P(A, B|X, Y) = \sum_{\lambda} P(A|\lambda, X)P(B|\lambda, Y)P(\lambda) \]

\[ \rho_{AB|XY} = Tr_S(\rho_{A|XS}\rho_{B|YS}\rho_S) \]

Can reproduce the quantum correlations

Quantum conditional independence

\[ \rho_{A|BC} = \rho_{A|C} \]
\[ \rho_{B|AC} = \rho_{B|C} \]
\[ \rho_{AB|C} = \rho_{A|C} \rho_{B|C} \]

Denote this \((A \perp B|C)\)

Actually, it is only this simple in special cases!
Modified Reichenbach’s principle
If $A$ and $B$ are quantum dependent, then

Modified Faithfulness (No fine-tuning)
A quantum causal model underlying an observed quantum state is unfaithful if the quantum conditional independences in the observed quantum state only hold for a set of measure zero of the values of the parameters in the model.
Quantum Causal Models

\[ P(X) \]
\[ P(Y) \]
\[ P(\lambda) \]
\[ P(A|\lambda, X) \]
\[ P(B|\lambda, Y) \]

\[ P(A, B|X, Y) = \sum_{\lambda} P(A|\lambda, X)P(B|\lambda, Y)P(\lambda) \]
\[ = \text{Tr}_S(\rho_{A|XS}\rho_{B|YS}\rho_S) \]
A Quantum Advantage for Causal Inference

Theory collaborators: Katja Ried, Dominik Janzing
Expt’l collaborators: Megan Agnew, Lydia Vermeyden, Kevin Resch
arXiv: 1406.5036
Classical causal inference

Direct cause

\[ \begin{align*}
A \rightarrow B
\end{align*} \]

Common cause

\[ \begin{align*}
A \rightarrow B \\
B \rightarrow A
\end{align*} \]

Passive observation of A

→ No information about causal structure
Classical causal inference

Direct cause

- Passive observation of A
  - No information about causal structure

Common cause

- Intervention on A
  - Complete solution of causal inference problem
Quantum causal inference

Direct cause

Passive observation of A → Still information about causal structure

Common cause

Intervention on A → Complete solution of causal inference problem
Quantum causal inference

Direct cause

Common cause
Quantum causal inference

Direct cause

Common cause
Experimental set-up

\[ p \in [0, 1] \]

a controlled parameter
What we would see classically

$p$ used in fit

expt'l value of $p$

prob. (1-p)

prob. $p$
Experimental Results

<table>
<thead>
<tr>
<th>prob. (1-p)</th>
<th>$\Phi^+$</th>
<th>exp'tl value of $p$</th>
<th>$\ln(\chi^2)$</th>
<th>rms deviation 0.032</th>
</tr>
</thead>
<tbody>
<tr>
<td>prob. $p$</td>
<td></td>
<td>$\Phi^+$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
A sketch of the origin of the quantum advantage

\[ P_{\text{id}}(q_B, p_B | q_A, p_A) \propto \delta(q_A - q_B)\delta(p_A - p_B) \]

\[ P_{\text{EPR}}(q_A, p_A, q_B, p_B) \propto \delta(q_A - q_B)\delta(p_A + p_B) \]

\[ Q_B - Q_A = 0 \]
\[ P_B + P_A = 0 \]
A sketch of the origin of the quantum advantage

\[ P_{\text{EPR}}(q_B, p_B | q_A, p_A) \propto \delta(q_A - q_B)\delta(p_A + p_B) \]

\[ \begin{align*}
Q_B - Q_A &= 0 \\
P_B - P_A &= 0
\end{align*} \]

Not allowed!
Conclusions

• The framework of causal inference provides a very elegant formulation of Bell’s theorem
• Quantum causal models are a promising avenue for achieving a causal explanation of quantum correlations
• Tools developed in the community working on Bell’s theorem are likely to be useful for improving causal inference algorithms
• Quantum theory provides an advantage for causal inference in certain contexts

References

C. Wood and RWS, The lesson of causal discovery algorithms for quantum correlations: Causal explanations of Bell-inequality violations require fine-tuning, arXiv:1208.4119
