Non-wellfounded proofs have been very successful in the proof theory of modal fixed point logics. They play a key role in the known completeness proofs for Kozen's axiomatisation of the modal $\mu$-calculus [1, 5], and have been applied to various other logics to establish metalogical properties such as completeness and interpolation.

There is a strong connection between non-wellfounded proof theory and automata theory. In particular, the set of proofs of a given endsequent can often be seen as a tree language recognizable by a parity tree automaton. It follows, by results originally due to Rabin [6], that the emptiness problem for this language (and hence the question of whether the endsequent has a proof) is decidable. Moreover, any nonempty such language contains a tree that is *regular*, *i.e.* one that has only finitely many subtrees. Because regular trees can be represented as finite trees with backedges, regular proofs are often referred to as *cyclic proofs*.

An active line of research aims to study the non-wellfounded proof theory of modal fixed point logics directly, without having to appeal to results from automata theory. This is usually achieved by equipping sequents with extra structure, called *annotations*. For instance, by adding annotations to a non-wellfounded derivation system for the modal $\mu$-calculus, Jungteerapanich and Stirling obtain direct proofs of decidability, the fact that any provable sequent has a regular proof, and the small model property [8, 3].

Crucially, Jungteerapanich and Stirling use the fact that any provable annotated sequent has an infinitary proof such that on each branch the first repeated sequent (including the annotations) is a so-called *good repeat*. We shall call these proofs *concise*.

Recent work on non-wellfounded annotated derivation systems has seen completeness proofs that do not immediately yield concise proofs. For example in [4] and [2], which use game-theoretical arguments for their completeness proofs, and in [7], which uses canonical models. This raises the question: can every infinitary annotated proof be made concise? In ongoing work in progress we answer this question positively in a weak abstract setting that captures most of the known infinitary annotated derivation systems for modal fixed point logics.

More precisely, we say that an *infinitary annotated proof system* $\mathsf{P}$ for some ranked alphabet $\Sigma$ consists of:

(i) An equivalence relation $\equiv$ on $\Sigma$.

(ii) A relation $R \subseteq \Sigma \times \Sigma^*$ such that:

  (a) If $aRw$, then $\mathsf{length}(w) = \mathsf{ar}(a)$.

  (b) If $aRw$ and $w$ is componentwise $\equiv$-equivalent to $w'$, then $aRw'$.

(iii) A subset $G$ of $\Sigma$ consisting of *good words* such that:

$$\text{if } w_1 \cdot w_2 \cdot w_3 \in G \text{ and } w_2 \notin G, \text{ then } w_1 \cdot w_3 \in G.$$

(iv) A subset $I$ of $\Sigma^\infty$ consisting of *good infinite words* such that:

$$\text{if } w_0 \cdot w_1 \cdot w_2 \cdots \in I, \text{ then } w_n \in G \text{ for some } n \geq 0.$$

A $\Sigma$-labelled tree $T$ with labelling function $l$ is said to be a $\mathsf{P}$-*preproof* if for every node $u \in T$ it holds that $l(u)Rl(u \cdot 0) \cdots l(u \cdot (\mathsf{ar}(u) - 1))$. A $\mathsf{P}$-*proof* is a $\mathsf{P}$-preproof of which the word induced by each infinite branch belongs to $I$. A node $u$ of a $\mathsf{P}$-proof $T$ is called a *repeat* if $l(u) \equiv l(v)$ for some $v < u$. If $u$ is a repeat, we write

$\widehat{u}$ for the least deep such $v$. If $u, v$ are nodes in a tree such that $u \leq v$, we write $p(u, v)$ for the finite upward path from $u$ to $v$ (inclusive). A repeat $u$ in $T$ is called *good* whenever the word induced by $p(\widehat{u}, u)$ belongs to $G$. Finally, a $\mathsf{P}$-proof is said to be *concise* if the least deep repeat on any branch is good (when it exists).

We prove the following theorem.

**Theorem.** *Let* $\mathsf{P}$ *be an infinitary annotated proof system and let* $T$ *be a* $\mathsf{P}$*-proof such that the set* $\{l_T(u) : u \in T\}/\equiv$ *is finite. Then there is a concise* $\mathsf{P}$*-proof* $T'$ *with an* $\equiv$*-equivalent root label.*

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