Focus-style proofs for the two-way alternation-free μ-calculus

(joint work in progress with Yde Venema)

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Overview

- The (alternation-free) modal μ-calculus
  - Game semantics
  - Cyclic proofs for the alternation-free modal mu-calculus
  - Focus annotations
  - Completeness
- The two-way alternation-free modal μ-calculus
  - Problems for completeness
  - The solution: trace atoms
- Our results
- Conclusion and future work

The modal μ-calculus

- A set P of propositional variables.
- A set D of actions.

$$\varphi ::= p \mid \overline{p} \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \langle a \rangle \varphi \mid [a]\varphi \mid \mu x \varphi \mid \nu x \varphi$$

where $\overline{x}$ does not occur in φ.

Given a Kripke model $\mathbb{S} = (S, (R_a)_{a \in \mathrm{D}}, V)$ and a propositional variable x, a formula φ induces a function

$$\begin{aligned} [\![\varphi]\!]^{\mathbb{S}}_x &: \mathcal{P}(S) \to \mathcal{P}(S) \\ &: X \mapsto [\![\varphi]\!]^{\mathbb{S}[x \mapsto X]} \end{aligned}$$

$[\![\eta x \varphi]\!]^{\mathbb{S}}$ is the least/greatest fixed point of $[\![\varphi]\!]^{\mathbb{S}}_x$.

The alternation-free fragment

Definition

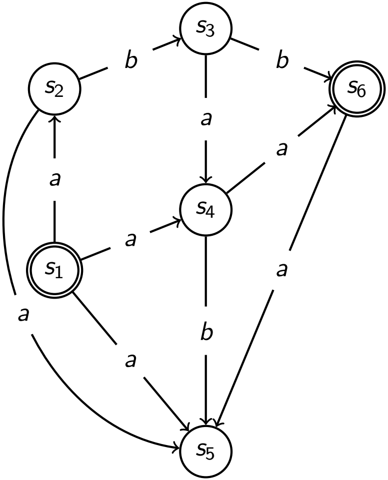
A formula ξ is *alternation free* if for every subformula $\eta x \varphi$ of ξ it holds that no free occurrence of x in φ is in the scope of an $\overline{\eta}$-operator in φ.

$\mu x \mu y(\langle a \rangle(x \vee p) \wedge \langle b \rangle y)$	$\mu x \nu y(\langle a \rangle(x \vee p) \wedge \langle b \rangle y)$
$\mu x(\langle a \rangle(x \vee p) \wedge \mu y \langle b \rangle y)$	$\mu x(\langle a \rangle(x \vee p) \wedge \nu y \langle b \rangle y)$

- The alternation-free modal μ-calculus subsumes PDL, CKL and many other extensions of modal logic by fixed point operators.

The evaluation game (example)

At position (φ, s), player $\exists$ wants to show that φ is true s, while player $\forall$ wishes to show that φ is false at s.



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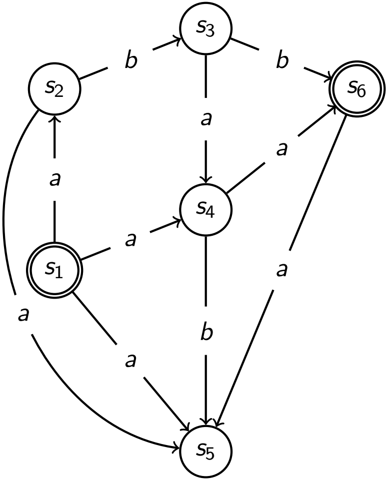

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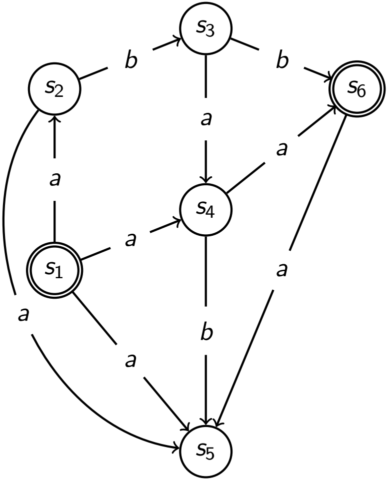

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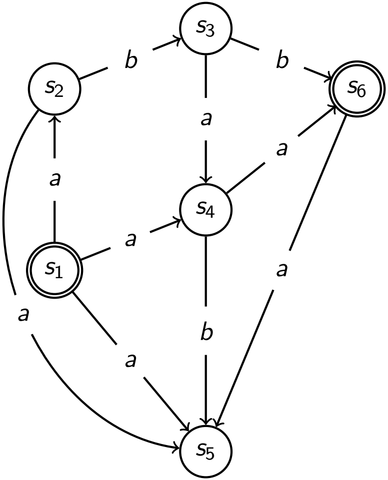

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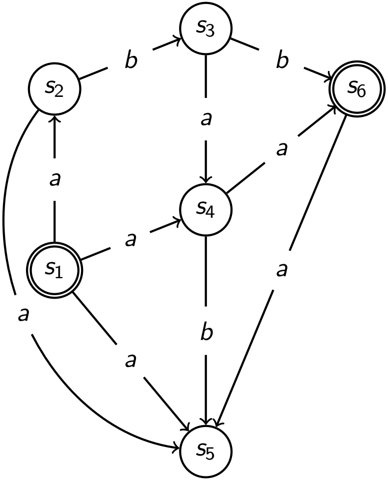

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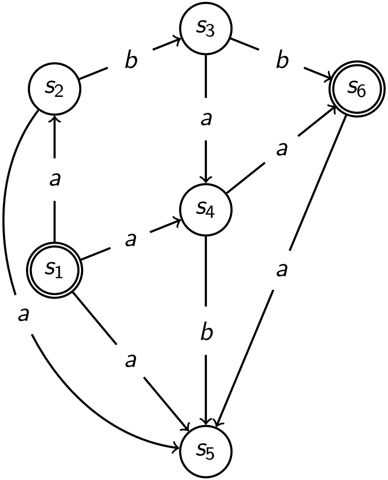

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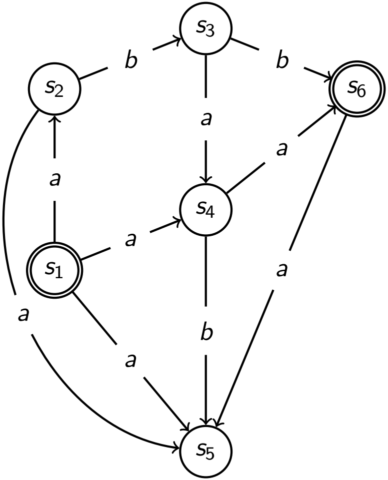

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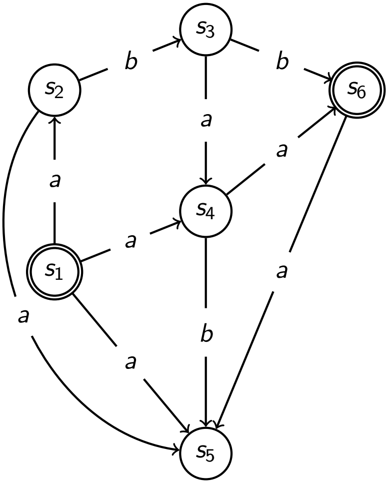

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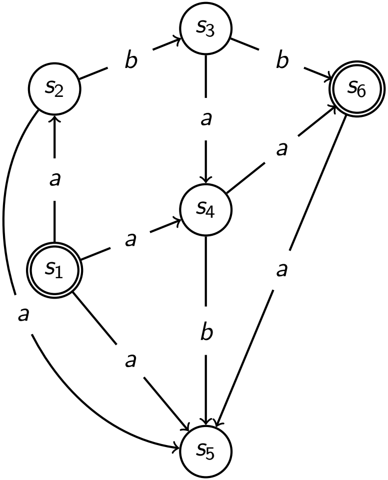

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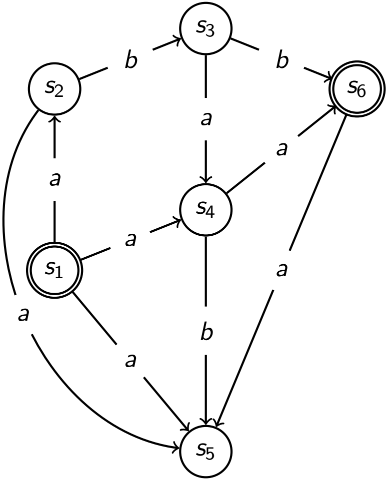

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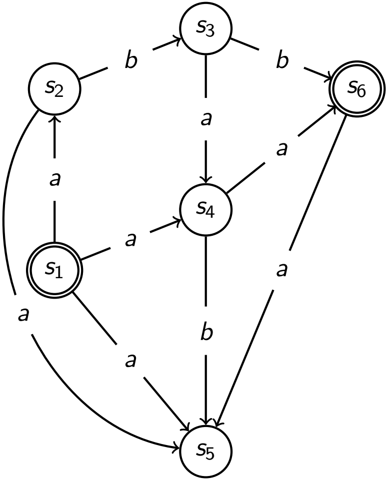

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$$\xrightarrow{\exists} (p, s_6)$$

The evaluation game (definition)

The game $\mathcal{E}(\xi, \mathbb{S})$ is played on the board $\text{Clos}(\xi) \times S$.

Position	Owner	Admissible moves
$(p, s), s \in V(p)$	$\forall$	$\emptyset$
$(p, s), s \notin V(p)$	$\exists$	$\emptyset$
$(\varphi \vee \psi, s)$	$\exists$	$\{(\varphi, s), (\psi, s)\}$
$(\varphi \wedge \psi, s)$	$\forall$	$\{(\varphi, s), (\psi, s)\}$
$(\langle a \rangle \varphi, s)$	$\exists$	$\{\varphi\} \times R_a[s]$
$([a]\varphi, s)$	$\forall$	$\{\varphi\} \times R_a[s]$
$(\eta x \varphi, s)$	$-$	$\{(\varphi[\eta x \varphi / x], s)\}$

An infinite $\mathcal{E}(\xi, \mathbb{S})$-match is won by $\exists$ ($\forall$) iff it contains infinitely many ν-formulas (μ-formulas)

Example

$\mu x \langle a \rangle x \vee p \equiv$ "a p-state is reachable by an a-path"

Theorem (Kozen)

$Clos(\xi)$ is always finite.

A non-well-founded proof system

$$\frac{}{\varphi, \overline{\varphi}, \Gamma}\ \mathsf{Ax} \qquad \frac{\varphi, \psi, \Gamma}{\varphi \vee \psi, \Gamma}\ \mathsf{R}_{\vee} \qquad \frac{\varphi, \Gamma \qquad \psi, \Gamma}{\varphi \wedge \psi, \Gamma}\ \mathsf{R}_{\wedge}$$

$$\frac{\varphi, \Delta}{[a]\varphi, \langle a \rangle \Delta, \Gamma}\ \mathsf{R}_{[a]} \qquad \frac{\varphi[\mu x \varphi / x], \Gamma}{\mu x \varphi, \Gamma}\ \mathsf{R}_{\mu} \qquad \frac{\varphi[\nu x \varphi / x], \Gamma}{\nu x \varphi, \Gamma}\ \mathsf{R}_{\nu}$$

A non-well-founded proof system, continued

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\mu x[a]x}}{[a]\mu x[a]x}\,\mathsf{R}_{[a]}}{\dfrac{\mu x[a]x}{[a]\mu x[a]x}\,\mathsf{R}_{[a]}}\,\mathsf{R}_{\mu}}{\mu x[a]x}\,\mathsf{R}_{\mu}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\nu x[a]x}}{[a]\nu x[a]x}\,\mathsf{R}_{[a]}}{\dfrac{\nu x[a]x}{[a]\nu x[a]x}\,\mathsf{R}_{[a]}}\,\mathsf{R}_{\nu}}{\nu x[a]x}\,\mathsf{R}_{\nu}$$

Definition

A non-well-founded derivation is a *proof* if every infinite branch contains a ν-trace.

The proof search game

Definition

A *rule instance* is a triple $(\Gamma, \mathrm{r}, \langle \Delta_1, \ldots, \Delta_n \rangle)$ such that $\dfrac{\Delta_1 \cdots \Delta_n}{\Gamma}$ r is a valid rule application.

Let Σ be some finite and closed set of formulas. We write:

- Seq_Σ for the sequents containing only formulas from Σ.
- Inst_Σ for the rule instances containing only formulas from Σ.
- $\mathrm{conc}(i)$ for the conclusion of some instance $i \in \mathrm{Inst}_\Sigma$.

The game $\mathcal{G}(\Sigma)$ is played on the board $\mathrm{Seq}_\Sigma \cup \mathrm{Inst}_\Sigma$.

Position	Owner	Admissible moves
$\Gamma \in \mathrm{Seq}_\Sigma$	Prover	$\{i \in \mathrm{Inst}_\Sigma \mid \mathrm{conc}(i) = \Gamma\}$
$(\Gamma, \mathrm{r}, \langle \Delta_1, \ldots, \Delta_n \rangle) \in \mathrm{Inst}_\Sigma$	Refuter	$\{\Delta_i \mid 1 \leq i \leq n\}$

An infinite match is won by Prover if and only if every infinite branch contains a ν-trace. Viewed as a tree, a winning strategy for Prover is the same as a proof.

Parity games

Definition

A game $\mathcal{G}$ is said to be a *parity game* if there exists a priority function $\Omega : B \to \omega$ with $|\text{ran}(\Omega)| < \omega$ on the set of positions B of $\mathcal{G}$ such that an infinite match $\alpha \in B^\omega$ is won iff the greatest priority encountered infinitely often is even.

Theorem (Emerson & Jutla, Mostowski)

Parity games are positionally determined.

Cyclic proofs (first step)

$$
\dfrac{\dfrac{\dfrac{\vdots}{\varphi, \bot}}{\langle a\rangle\varphi, \langle b\rangle\varphi, [a]\bot} \qquad \dfrac{\dfrac{\vdots}{\varphi, \bot}}{\langle a\rangle\varphi, \langle b\rangle\varphi, [b]\bot}}{\dfrac{\dfrac{\langle a\rangle\varphi, \langle b\rangle\varphi, \psi}{(\langle a\rangle\varphi \vee \psi) \vee (\langle b\rangle\varphi \vee \psi)}}{\varphi}}
\quad \rightsquigarrow \quad
\dfrac{\dfrac{\dfrac{\vdots}{q_6 \vdash \varphi, \bot}}{q_5 \vdash \langle a\rangle\varphi, \langle b\rangle\varphi, [a]\bot} \qquad \dfrac{\dfrac{\vdots}{q_4 \vdash \varphi, \bot}}{q_3 \vdash \langle a\rangle\varphi, \langle b\rangle\varphi, [b]\bot}}{\dfrac{\dfrac{q_2 \vdash \langle a\rangle\varphi, \langle b\rangle\varphi, \psi}{q_1 \vdash (\langle a\rangle\varphi \vee \psi) \vee (\langle b\rangle\varphi \vee \psi)}}{q_0 \vdash \varphi}}
$$

$$
\varphi = \nu x.(\langle a\rangle x \vee \psi) \vee (\langle b\rangle x \vee \psi)
$$
$$
\psi = [a]\bot \wedge [b]\bot
$$

Corollary

For every non-well-founded proof, there is a regular non-well-founded proof.

Cyclic proofs (second step)

$$
\dfrac{\dfrac{\dfrac{\vdots}{\varphi^\bullet, \bot^\bullet}}{\langle a \rangle \varphi^\bullet, \langle b \rangle \varphi^\bullet, [a]\bot^\bullet} \qquad \dfrac{\dfrac{\vdots}{\varphi^\bullet, \bot^\bullet}}{\langle a \rangle \varphi^\bullet, \langle b \rangle \varphi^\bullet, [b]\bot^\bullet}}{\dfrac{\dfrac{\langle a \rangle \varphi^\bullet, \langle b \rangle \varphi^\bullet, \psi^\bullet}{(\langle a \rangle \varphi \vee \psi) \vee (\langle b \rangle \varphi \vee \psi)^\bullet}}{\varphi^\bullet}}
$$

Most convenient form for interpolation, proof translations, cut-elimination, ...

An annotated proof system (Marti & Venema)

$$\frac{}{\varphi^u, \overline{\varphi}^v, \Gamma}\ \mathsf{Ax} \qquad \frac{\varphi^u, \psi^u, \Gamma}{\varphi \vee \psi^u, \Gamma}\ \vee \qquad \frac{\varphi^u, \Gamma \qquad \psi^u, \Gamma}{\varphi \wedge \psi^u, \Gamma}\ \wedge$$

$$\frac{\varphi^u, \Delta}{[a]\varphi^u, \langle a\rangle\Delta, \Gamma}\ [a] \qquad \frac{\varphi[\mu x\varphi/x]^\circ, \Gamma}{\mu x\varphi^u, \Gamma}\ \mu \qquad \frac{\varphi[\nu x\varphi/x]^u, \Gamma}{\nu x\varphi^u, \Gamma}\ \nu \qquad \frac{\Gamma^\bullet}{\Gamma^\circ}\ \mathsf{F}$$

Definition

A non-well-founded derivation is a *proof* if every infinite branch has a final segment on which there is always a formula in focus.

- The (*path-based*) focus system is equivalent to the *trace-based* system.
- The proof-search game for the focus system is a parity game.
- The focus annotations allow for a nice soundness condition on cyclic proofs as finite trees with back edges.

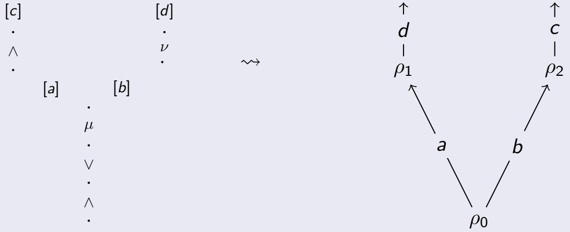
Completeness

Theorem (Niwinski & Walukiewicz, Marti & Venema)

Every valid sequent Γ is provable.

Proof (sketch).

Suppose Γ is not provable. By determinacy, there is a winning strategy T for Refuter in $\mathcal{G}(\Gamma)$.



$S^T :=$ {maximal paths ρ in T such that ρ does not pass a modal rule}
$\rho_1 R^T_a \rho_2 :\Leftrightarrow \rho_1$ is connected to ρ_2 by an application of the rule $[a]$
$p \in V^T(\rho) :\Leftrightarrow p$ occurs in ρ □

The two-way alternation-free modal μ-calculus

- A set P of propositional variables.
- A set D of actions.

Fix an involution operation $\breve{\cdot}$ on D, *i.e.* $a \neq \breve{a}$ and $\breve{\breve{a}} = a$ for every $a \in D$

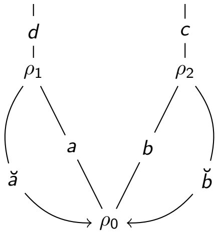
$$\varphi ::= p \mid \overline{p} \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \langle a \rangle \varphi \mid [a]\varphi \mid \mu x \varphi \mid \nu x \varphi$$

where $\overline{x}$ does not occur in φ.

The two-way modal μ-calculus is interpreted over *regular* models:

$$R_{\breve{a}} = \{(t, s) : (s, t) \in R_a\}$$

Problem for completeness



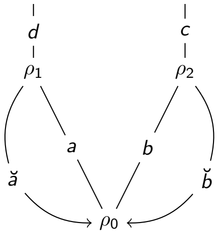
$$(\langle \breve{a} \rangle \psi, \rho_1) \xrightarrow{\exists} (\psi, \rho_0)$$

Modal rule for the two-way μ-calculus

$$\dfrac{\varphi, \Delta, [\breve{a}]\Gamma}{[a]\varphi, \langle a\rangle\Delta, \Gamma}\ \mathsf{R}_{[a]} \qquad\qquad \dfrac{\varphi^{\circ}, \Gamma \qquad \overline{\varphi}^{\circ}, \Gamma}{\Gamma}\ \mathsf{cut}$$

$$\langle\breve{a}\rangle\psi \in \Gamma_1 \Rightarrow [\breve{a}]\overline{\psi} \notin \Gamma_0 \Rightarrow \overline{\psi} \notin \Gamma_0 \Rightarrow \psi \in \Gamma_0$$

Another problem for completeness



If $\langle \breve{a} \rangle \psi^\bullet$ occurs in ρ_1, then ψ^u occurs in ρ_0. But how do we get $u = \bullet$?

Trace atoms (inspired by Vardi)

Definition

Given φ, ψ, there is a *trace atom* $\varphi \rightsquigarrow \psi$ and a *negated trace atom* $\varphi \not\rightsquigarrow \psi$.

Definition

Given a positional strategy f for $\forall$ in $\mathcal{E}$, we say that $\varphi \rightsquigarrow \psi$ is *satisfied* in $\mathbb{S}$ at s with respect to f (and write $\mathbb{S}, s \Vdash_f \varphi \rightsquigarrow \psi$) if there is an f-guided match

$$(\varphi, s) = (\varphi_0, s_0) \cdot (\varphi_1, s_1) \cdots (\varphi_n, s_n) = (\psi, s) \quad (n \geq 0)$$

such that for no $i < n$ the formula φ_i is a μ-formula. We say that $\mathbb{S}$ *satisfies* $\varphi \not\rightsquigarrow \psi$ at s with respect to f (and write $\mathbb{S}, s \Vdash_f \varphi \not\rightsquigarrow \psi$) iff $\mathbb{S}, s \nVdash_f \varphi \rightsquigarrow \psi$.

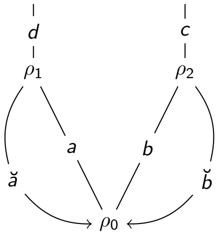
Example

If $\mathbb{S}, s \Vdash_f \varphi \rightsquigarrow \psi$ and $\mathbb{S}, s \Vdash_f \psi \rightsquigarrow \varphi$ for some $\varphi \neq \psi$, then $\mathbb{S}, s \Vdash_f \varphi$.

Theorem

$\mathbb{S}, s \Vdash \varphi$ iff for every positional strategy f for $\forall$ in $\mathcal{E}$ it holds that $\mathbb{S}, s \Vdash_f \varphi$.

Incorporating trace atoms in the proof system

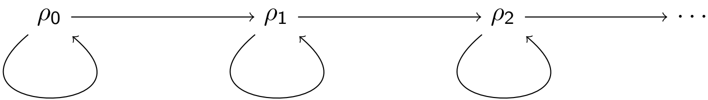


If $\chi \not\rightarrowtail \langle a \rangle \psi$ occurs in ρ_0, then $\langle \breve{a} \rangle \chi \not\rightarrowtail \psi$ occurs in ρ_1.

Completenes

Lemma

Let $\rho \in S^T$. Suppose $\mathbb{S}^T, \rho \Vdash_f \varphi \rightsquigarrow \psi$. Then $\varphi \not\rightsquigarrow \psi$ occurs in ρ.



Results

Let Γ be a sequent consisting of annotated formulas (*i.e.* φ^u with $u \in \{\circ, \bullet\}$), trace atoms, and negated trace atoms.

Theorem (Soundness)

If Γ is provable, then for every model $\mathbb{S}$, state s of $\mathbb{S}$ and positional strategy f for $\forall$ in $\mathcal{E}$, there is an $A \in \Gamma$ such that $\mathbb{S}, s \Vdash_f A$.

Let Γ^- be the set of annotated formulas in Γ.

Theorem (Completeness)

If Γ^- is valid, then Γ is provable.

Remark

The infinitary proof system naturally restricts to a finitary cyclic system.

Corollary

The two-way alternation-free modal μ-calculus is decidable and has the regular model property.

Future work

- Completeness for all sequents, *e.g.* $\{\varphi_1 \wedge \varphi_2 \rightsquigarrow \varphi_1, \varphi_1 \wedge \varphi_2 \rightsquigarrow \varphi_2\}$.
- Interpolation
- Incorporating trace atoms in the syntax?
- Extending this system to the full two-way modal μ-calculus (*i.e.* with alternation)

Thank you

https://staff.fnwi.uva.nl/j.m.w.rooduijn/

$$\frac{}{\varphi^u, \overline{\varphi}^v, \Gamma}\ \mathsf{Ax1} \qquad \frac{}{\varphi \rightsquigarrow \psi, \varphi \not\rightsquigarrow \psi, \Gamma}\ \mathsf{Ax2} \qquad \frac{}{\varphi \rightsquigarrow \varphi, \Gamma}\ \mathsf{Ax3}$$

$$\frac{(\varphi \vee \psi) \not\rightsquigarrow \varphi, (\varphi \vee \psi) \not\rightsquigarrow \psi, \varphi^u, \psi^u, \Gamma}{\varphi \vee \psi^u, \Gamma}\ \mathsf{R}_\vee \qquad \frac{\varphi^\circ, \Gamma \qquad \overline{\varphi}^\circ, \Gamma}{\Gamma}\ \mathsf{cut}$$

$$\frac{(\varphi \wedge \psi) \not\rightsquigarrow \varphi, \varphi^u, \Gamma \qquad (\varphi \wedge \psi) \not\rightsquigarrow \psi, \psi^u, \Gamma}{\varphi \wedge \psi^u, \Gamma}\ \mathsf{R}_\wedge \qquad \frac{\varphi[\mu x \varphi / x]^\circ, \Gamma}{\mu x \varphi^u, \Gamma}\ \mathsf{R}_\mu$$

$$\frac{\nu x \varphi \not\rightsquigarrow \varphi[\nu x \varphi / x], \varphi[\nu x \varphi / x] \rightsquigarrow \nu x \varphi, \varphi[\nu x \varphi / x]^u, \Gamma}{\nu x \varphi^u, \Gamma}\ \mathsf{R}_\nu \qquad \frac{\Gamma^{[a]}\varphi^u}{[a]\varphi^u, \Gamma}\ \mathsf{R}_{[a]}$$

$$\frac{\Gamma^\bullet}{\Gamma^\circ}\ \mathsf{F} \qquad \frac{\varphi \not\rightsquigarrow \psi, \psi \not\rightsquigarrow \chi, \varphi \not\rightsquigarrow \chi, \Gamma}{\varphi \not\rightsquigarrow \psi, \psi \not\rightsquigarrow \chi, \Gamma}\ \mathsf{trans} \qquad \frac{\varphi \rightsquigarrow \psi, \Gamma \qquad \varphi \not\rightsquigarrow \psi, \Gamma}{\Gamma}\ \mathsf{tc}$$

Definition

Let Γ be a sequent and let $[a]\varphi^b$ be an annotated formula. The *jump* $\Gamma^{[a]\varphi^b}$ of Γ with respect to $[a]\varphi^b$ consists of:

1. 1. $\varphi^{s([a]\varphi,\Gamma)}$;
   2. $\psi^{s(\langle a\rangle\psi,\Gamma)}$ for every $\langle a\rangle\psi^c \in \Gamma$;
   3. $[\breve{a}]\chi^\circ$ for every $\chi^d \in \Gamma$ such that $[\breve{a}]\chi \in \Sigma$;
2. 1. $\varphi \rightsquigarrow \langle\breve{a}\rangle\chi$ for every $[a]\varphi \rightsquigarrow \chi \in \Gamma$ such that $\langle\breve{a}\rangle\chi \in \Sigma$;
   2. $\langle\breve{a}\rangle\chi \not\rightsquigarrow \varphi$ for every $\chi \not\rightsquigarrow [a]\varphi \in \Gamma$ such that $\langle\breve{a}\rangle\chi \in \Sigma$;
   3. $\psi \rightsquigarrow \langle\breve{a}\rangle\chi$ for every $\langle a\rangle\psi \rightsquigarrow \chi \in \Gamma$ such that $\langle\breve{a}\rangle\chi \in \Sigma$;
   4. $\langle\breve{a}\rangle\chi \not\rightsquigarrow \psi$ for every $\chi \not\rightsquigarrow \langle a\rangle\psi \in \Gamma$ such that $\langle\breve{a}\rangle\chi \in \Sigma$.

where $s(\xi,\Gamma)$ is defined by:

$$s(\xi,\Gamma) = \begin{cases} \bullet & \text{if } \xi^\bullet \in \Gamma, \\ \bullet & \text{if } \theta \not\rightsquigarrow \xi \in \Gamma \text{ for some } \theta^\bullet \in \Gamma \\ \circ & \text{otherwise.} \end{cases}$$

When taking the strategy tree T, we assume that Prover adheres to the following non-deterministic strategy:

- Only apply a modal rule when all of the propositional rules are exhausted.
- Apply the rule F whenever possible.

The canonical strategy f for $\forall$ in $\mathcal{E}(\Gamma, \mathbb{S}^T)$ is given by:

- At $(\varphi \wedge \psi, \rho)$ choose the conjunct corresponding to the choice of Refuter when $\varphi \wedge \psi$ is principal in an application of the rule $\wedge$ in ρ.
- At $([a]\varphi, \rho)$ choose an a-successor ρ' of ρ such that ρ and ρ' are separated by an application of $[a]$.