Making infinitary annotated proofs concise

Jan Rooduijn
ILLC, University of Amsterdam

The Proof Society Workshop
2 December 2021
Non-wellfounded proof theory for modal fixed point logics

Main idea: replace an *explicit induction rule* by allowing infinite branches (satisfying some condition).

An infinite branch $\beta$ is said to be *good* if it contains a good *trace* $\tau$. 
Connections to automata theory

Let $\Sigma_0$ be the alphabet whose characters are *rule applications* in $\pi$, e.g.

$$
\text{\textbullet}_R \quad \frac{\text{\textbullet}(p \rightarrow \square p) \Rightarrow \text{\textbullet}p}{p, \text{\textbullet}p} \in \Sigma_0
$$

The language containing precisely the *good* infinite branches of $\pi$ is an $\omega$-regular tree language over $\Sigma_0$.

Theorem (Rabin, 1969)

1. The emptiness problem for regular tree languages is decidable.
2. Every regular tree language contains a regular tree.

Corollary

1. The provability of $s$ is decidable.
2. If $s$ has a proof, then it has a cyclic proof.
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$$\begin{array}{c}
\ast R \\
p, \ast(p \rightarrow \Box p) \Rightarrow \ast p
\end{array} \in \Sigma_0$$

The language containing precisely the good infinite branches of $\pi$ is an $\omega$-regular tree language over $\Sigma_0$.

Let $\Sigma_1$ be the ranked alphabet whose characters are rule applications in any proof of some fixed sequent $s$. The set of proofs of $s$ is a regular infinite tree language over the alphabet $\Sigma_1$. 

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Cyclic annotated proof systems

Main idea: encode the automata-theoretic trace management into the proofs.

Important features of annotated proof systems:
▶ It suffices to check a condition between each companion and its leaf.
▶ Removing the annotations yields an ordinary cyclic proof.
▶ Every provable sequent has a **concise** proof.
▶ ...?
Concise proofs

Definition
An annotated proof is said to be concise if the first repeat on any branch is good.

Completeness via proof-search often directly yields concise proofs:
  ▶ Jungteerapanich (2010)
  ▶ Stirling (2014)

Game-theoretic methods do not:
  ▶ Enqvist (2020)
  ▶ Marti & Venema (2021)

and neither does the method of canonical models:
  ▶ JR (2021)

We prove, in an abstract setting capturing each of the above, that every infinitary annotated proof can be made concise.
Infinitary annotated proof systems
Abstract definition of an infinitary annotated proof system

More precisely, we say that an *infinitary annotated proof system* $P$ for some ranked alphabet $\Sigma$ consists of:

1. An equivalence relation $\equiv$ on $\Sigma$.
2. A relation $R \subseteq \Sigma \times \Sigma^*$ such that:
   
   2.1 If $aRw$, then $\text{length}(w) = \text{ar}(a)$.
   
   2.2 If $aRw$ and $w$ is componentwise $\equiv$-equivalent to $w'$, then $aRw'$.
3. A subset $G$ of $\Sigma$ consisting of *good words* such that:
   
   if $w_1 \cdot w_2 \cdot w_3 \in G$ and $w_2 \notin G$, then $w_1 \cdot w_3 \in G$.
4. A subset $I$ of $\Sigma^\infty$ consisting of *good infinite words* such that:
   
   if $w_0 \cdot w_1 \cdot w_2 \cdots \in I$, then $w_n \in G$ for some $n \geq 0$.

**Theorem**

*Let $P$ be an infinitary annotated proof system and let $T$ be a $P$-proof such that the set $\{l_T(u) : u \in T\}/\equiv$ is finite. Then there is a concise $P$-proof $T'$ with an $\equiv$-equivalent root label.*
Further questions

- How should the abstract definition be restricted to ensure that the unravelling of a cyclic annotated proof is an infinitary annotated proof?
- Given a set of good words, is there a canonical definition for the set of good infinite words? (I think yes)
- How does the size of cyclic proofs obtained in this way compare to those given by automata-theoretic methods?
- Can we also get uniform proofs?
Thank you!