An analytic proof system for Common Knowledge Logic over S5

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LIRa Seminar, 8 June 2023

Overview

1. Introduction to S5-CKL.
2. Some earlier proof systems
3. Our analytic proof system
4. Soundness
5. Completeness
6. Future work

Multi-agent epistemic logic

- A countable set P of propositional variables.
- A finite set $\mathcal{A}$ of agents.

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box_i \varphi$$

$\Box_i \varphi$ expresses that agent $i \in \mathcal{A}$ *knows* that φ.

Epistemic principles

Principle	Axiom	Formula	Frame condition
Epistemic closure	K	$\Box_i(\varphi \to \psi) \to (\Box_i\varphi \to \Box_i\psi)$	-
Veridicality	T	$\Box_i\varphi \to \varphi$	Reflexivity
Positive introspection	4	$\Box_i\varphi \to \Box_i\Box_i\varphi$	Transitivity
Negative introspection	5	$\neg\Box_i\varphi \to \Box_i\neg\Box_i\varphi$	Euclideaness

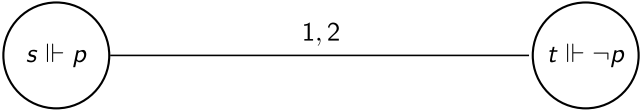
- Negative introspection is philosophically controversial, but standard in applications in computer science and game theory.
- We assume all axioms above.

Semantics

Definition

An *epistemic Kripke model* is a tuple $\mathbb{S} = (S, \{R_i \mid i \in \mathcal{A}\}, V)$ where

- S is a non-empty set;
- R_i is an equivalence relation on S for each $i \in \mathcal{A}$;
- V is a function $S \to \mathcal{P}(\mathrm{P})$.



$$s \Vdash p \wedge \neg \Box_1 p \wedge \neg \Box_2 p.$$

Common knowledge logic

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box_i \varphi \mid \underline{\boxast\varphi}$$

$\Box\varphi := \varphi_1 \wedge \cdots \wedge \varphi_n$ (*everybody knows* φ)

$\boxast\varphi \equiv \varphi \wedge \Box\varphi \wedge \Box\Box\varphi \wedge \cdots$ (it is *common knowledge* that φ)

$s \Vdash \Box\varphi \Leftrightarrow t \Vdash \varphi$ for every t such that $sR_i t$ with $i \in \mathcal{A}$.

$s \Vdash \boxast\varphi \Leftrightarrow t \Vdash \varphi$ for every path $s = s_0 R_{i_1} s_1 R_{i_2} \cdots R_{i_m} s_m = t$ with each i_k in $\mathcal{A}$.

Hilbert-style proof system

- **Standard axioms and rules for basic multi-modal logic** K_n
- **Axioms for** $\mathsf{S5}_n$
  - (T) $\Box_i p \to p$.
  - (5) $\neg\Box_i p \to \Box_i\neg\Box_i p$
- **Axiom and rule for** $\circledast$
  - (fix) $\circledast\varphi \leftrightarrow \varphi \wedge \Box\circledast\varphi$
  - (ind) $\dfrac{\varphi \to (\Box\varphi \wedge \psi)}{\varphi \to \circledast\psi}$

W. van der Hoek H. van Ditmarsch J.Y. Halpern and B. Kooi. "An Introduction to Logics of Knowledge and Belief". In: *Handbook of Epistemic Logic*. College Publications, 2015. Chap. 1, pp. 1–51.

A Gentzen-style reformulation

- **All sequent rules for propositional logic (inlcuding w_L and w_R)**

- **Modal rules**

$$\Box_T \frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \Box_i \varphi \Rightarrow \Delta} \qquad \Box_{S5} \frac{\Box_i \Gamma \Rightarrow \varphi, \Box_i \Delta}{\Box_i \Gamma \Rightarrow \Box_i \varphi, \Box_i \Delta}$$

- **Fixpoint rules**

$$\circledast_L \frac{\Gamma, \varphi, \Box \circledast \varphi \Rightarrow \Delta}{\Gamma, \circledast \varphi \Rightarrow \Delta} \qquad \circledast_R \frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Gamma \Rightarrow \Box \circledast \varphi, \Delta}{\Gamma \Rightarrow \circledast \varphi, \Delta}$$

- **Induction rule**

$$\text{ind} \frac{\varphi \Rightarrow \Box \varphi \qquad \varphi \Rightarrow \psi}{\varphi \Rightarrow \circledast \psi, \Delta}$$

- **Cut rule**

$$\text{cut} \frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

Luca Alberucci and Gerhard Jäger. "About cut elimination for logics of common knowledge". In: *Annals of Pure and Applied Logic* 133.1-3 (2005), pp. 73–99.

Fischer-Ladner closure

Definition

The *Fischer-Ladner closure* of a formula φ is the smallest set of formulas $Cl(\varphi)$ which contains φ and is closed under the following conditions:

- $\neg\psi \in Cl(\varphi)$ implies $\psi \in Cl(\varphi)$;
- $\psi_1 \wedge \psi_2 \in Cl(\varphi)$ implies $\psi_k \in Cl(\varphi)$ for each $k \in \{1, 2\}$;
- $\Box_i \psi \in Cl(\varphi)$ implies $\psi \in Cl(\varphi)$;
- $\boxast\psi \in Cl(\varphi)$ implies $\psi \in Cl(\varphi)$ and $\Box\boxast\varphi \in Cl(\varphi)$.

Moreover, for A a set of formulas, we define:

$$Cl(A) := \bigcup\{Cl(\varphi) \mid \varphi \in A\}.$$

The closure of a finite set of formulas is always finite.

The Alberucci-Jäger calculus is not analytic

- The induction rule does not stay within $Cl(\varphi)$:

$$\text{ind}\ \frac{\varphi \Rightarrow \Box\varphi \qquad \varphi \Rightarrow \psi}{\varphi \Rightarrow \boxast\psi, \Delta}$$

- Cut-restriction has only been proven to the 'conjunctive closure' of $Cl(\varphi)$, which is exponentially larger than $Cl(\varphi)$ itself.

$$\text{cut}\ \frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

Some remarks on cut-elimination for $\mathsf{S5}_n$

- Recall the modal rule $\Box_{\mathsf{S5}} \dfrac{\Box_i\Gamma \Rightarrow \varphi, \Box_i\Delta}{\Box_i\Gamma \Rightarrow \Box_i\varphi, \Box_i\Delta}$.

- The following sequent is valid: $\Rightarrow \Box_i\neg\Box_i p, p$.

- However, the only way to derive it would be by applying the modal rule:

$$\Box_{\mathsf{S5}} \frac{\Rightarrow \neg\Box_i p, \Box_i\neg\Box_i p}{\Rightarrow \Box_i\neg\Box_i p, p}$$

of which the premiss is not valid.

- Ordinary sequents for $\mathsf{S5}_n$ require analytic cuts.
- Hypersequents satisfy cut-elimination for $\mathsf{S5}_1$.
- For cut-elimination for $\mathsf{S5}_n$ one needs an even more expressive framework (*e.g.* nested sequents)

Masao Ohnishi and Kazuo Matsumoto. "Gentzen method in modal calculi". In: *Osaka Mathematical Journal* 9.2 (1957), pp. 113–130.

The Hill-Poggiolesi calculus

- Labelled hypersequent calculus
- Syntactic cut-elimination
- Induction rule of the form:

$$\text{ind}_2 \; \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \varphi \Rightarrow \psi \qquad \varphi \Rightarrow \Box\varphi}{\Gamma \Rightarrow \Delta, \circledast\psi}$$

Brian Hill and Francesca Poggiolesi. "Common knowledge: a finitary calculus with a syntactic cut-elimination procedure". In: *Logique et Analyse* (2015), pp. 279–306.

The sequent calculus $\mathsf{sCKL_f}$

Definition

An *annotated formula* is a pair (φ, a), usually written φ^a, where φ is a formula and $a \in \{u, f\}$. The annotation u indicates that φ is *unfocussed* and f indicates that φ is *in focus*.

The sequent calculus $\mathsf{sCKL_f}$

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Definition

A *sequent* is an ordered pair (Γ, Δ), usually written $\Gamma \Rightarrow \Delta$, where Γ, Δ are sets of annotated formulas.

The sequent calculus $\mathsf{sCKL_f}$

Definition

A *CKL-sequent* is a sequent $\Gamma \Rightarrow \Delta$ which satisfies the following properties:

1. Every formula in Γ is unfocussed.
2. At most one formula in Δ is in focus.
3. If $\varphi^f \in \Delta$, then $\varphi = \circledast\psi$ or $\varphi = \Box_i\circledast\psi$.

The sequent calculus sCKL$_f$

$$\boxast_L \frac{\Gamma, \varphi^u, \{\Box_i \boxast \varphi^u\}_{i=1}^n \Rightarrow \Delta}{\Gamma, \boxast \varphi^u \Rightarrow \Delta} \qquad \boxast_R \frac{\Gamma \Rightarrow \varphi^u, \Delta \qquad \{\Gamma \Rightarrow \Box_i \boxast \varphi^a, \Delta\}_{i=1}^n}{\Gamma \Rightarrow \boxast \varphi^a, \Delta}$$

$$\Box_T \frac{\Gamma, \varphi^u \Rightarrow \Delta}{\Gamma, \Box_i \varphi^u \Rightarrow \Delta} \qquad \Box_{S5} \frac{\Box_i \Gamma \Rightarrow \varphi^a, \Box_i \Delta}{\Box_i \Gamma \Rightarrow \Box_i \varphi^a, \Box_i \Delta}$$

$$\mathsf{U} \frac{\Gamma \Rightarrow \Delta^u}{\Gamma \Rightarrow \Delta} \qquad \mathsf{F} \frac{\Gamma \Rightarrow \varphi^f, \Delta^u}{\Gamma \Rightarrow \varphi^u, \Delta^u} \qquad \mathsf{cut} \frac{\Gamma \Rightarrow \varphi^u, \Delta \qquad \Gamma, \varphi^u \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

The sequent calculus $\mathsf{sCKL_f}$

Definition

1. A *derivation* is a finite tree whose nodes are labelled by CKL-sequents and which is generated by the rules of $\mathsf{sCKL_f}$.

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   1. every sequent in ρ has a formula in focus, and
   2. ρ passes through an application of the rule $\circledast_R$ where the principal formula is in focus.

$$\Box_{\mathsf{S5}}\ \frac{\Box_i p^u \Rightarrow q^u, \Box_i q^u}{\Box_i p^u \Rightarrow q^u, \Box_i q^u}$$

$$\dfrac{\dfrac{\dfrac{\vdots}{p^u, q^u \Rightarrow p \wedge q^u}}{\boxast p^u, \boxast q^u \Rightarrow p \wedge q^u}\ \boxast_L,\ \mathsf{w}_L \qquad \dfrac{\dfrac{\boxast p^u, \boxast q^u \Rightarrow \boxast(p \wedge q)^f}{\Box_i \boxast p^u, \Box_i \boxast q^u \Rightarrow \Box_i \boxast(p \wedge q)^f}\ \Box_{\mathsf{S5}},\ \Box_{\mathsf{T}},\ \mathsf{w}_L}{\boxast p^u, \boxast q^u \Rightarrow \Box_i \boxast(p \wedge q)^f}\ \boxast_L,\ \mathsf{w}_L}{\boxast p^u, \boxast q^u \Rightarrow \boxast(p \wedge q)^f}\ \boxast_R$$

Soundness

Lemma (Weak local soundness)

Let r $\dfrac{\sigma_1 \quad \cdots \quad \sigma_n}{\sigma}$ *be any rule application of* $\mathsf{sCKL_f}$*. If* σ *is invalid, then so is one of the premisses.*

Definition

Let σ be a sequent that has a formula in focus, *i.e.* for $j \in \{0, 1\}$ the right-hand side Δ of σ contains a formula of the form $\Box_i^j \circledast \psi^f$. We denote by $\sigma(n)$ the sequent obtained by adding the formula $\Box_i^j \Box^n \psi^u$ to Δ. For any invalid sequent σ that has a formula in focus, we define $\mu(\sigma) := \min\{n \in \omega : \sigma(n) \text{ is invalid}\}$.

Lemma (Strong local soundness)

Let r $\dfrac{\sigma_1 \quad \cdots \quad \sigma_n}{\sigma}$ *be any rule application of* $\mathsf{sCKL_f}$*. If* σ *is invalid, then there is an invalid premiss* σ_i *such that, if* σ *and* σ_i *both have a formula in focus, then* $\mu(\sigma_i) \leq \mu(\sigma)$*, and, if moreover* $\mathsf{r} = \circledast_R$ *and the principal formula is in focus, this inequality is strict.*

Proof of strong local soundness

Lemma (Strong local soundness)

Let $\mathsf{r}\ \dfrac{\sigma_1 \quad \cdots \quad \sigma_n}{\sigma}$ *be any rule application of* $\mathsf{sCKL_f}$. *If* σ *is invalid, then there is an invalid premiss* σ_i *such that, if* σ *and* σ_i *both have a formula in focus, then* $\mu(\sigma_i) \leq \mu(\sigma)$, *and, if moreover* $\mathsf{r} = \circledast_R$ *and the principal formula is in focus, this inequality is strict.*

Sketch.

- If either σ, or all of the σ_i, have no formula in focus, the statement reduces to weak local soundness.
- If the formula in focus in σ is *not* the principal formula, every premiss σ_i has a formula in focus and

$$\mathsf{r}\ \frac{\sigma_1(\mu(\sigma)) \quad \cdots \quad \sigma_n(\mu(\sigma))}{\sigma(\mu(\sigma))}$$

is a valid rule application. Thus we can apply weak local soundness.

Proof of strong local soundness (continued)

Lemma (Strong local soundness)

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Sketch.

- In the remaining case the principal formula in σ is in focus.
- This can only be the case if $\mathsf{r} \in \{\mathsf{w}_R, \Box_{\mathsf{S5}}, \circledast_R\}$.
- Suppose $\mathsf{r} = \circledast_R$. Let $n := \mu(\sigma)$ and let $\mathbb{S}, s$ be such that $\mathbb{S}, s \not\Vdash \sigma(n)$. Then $\mathbb{S}, s \not\Vdash \circledast\varphi$, where $\circledast\varphi^f$ is the principal formula. If $n = 0$, then $\mathbb{S}, s \not\Vdash \varphi$ and thus the leftmost premiss is invalid and forms a witness to the statement, as it has no formula in focus. If $n > 0$, then $\mathbb{S}, s \not\Vdash \Box_i \Box^{n-1}\varphi$, for some $i \in \mathcal{A}$. This means that there is an invalid premiss σ_k with $\mu(\sigma_k) = n - 1$, as required.
- The other cases are similar. □

Proof of soundness

Theorem

If a sequent σ has a $\mathsf{sCKL_f}$*-proof, then σ is valid.*

Proof.

Suppose, towards a contradiction, that an invalid sequent σ is the root of some $\mathsf{sCKL_f}$-proof π. Repeatedly applying strong local soundness, we obtain an upward path

$$\rho = \sigma_0, \sigma_1, \ldots, \sigma_n$$

through π such that $\sigma_0 = \sigma$ and σ_n labels a leaf of π. Since σ_n is invalid by construction, this leaf cannot be an axiom. Therefore, there there must be some $k < n$ such that $\langle \sigma_k, \sigma_n \rangle$ is a successful repetition. Observe that this implies that $\sigma_k = \sigma_n$. However, by the fact that we constructed this path using strong local soundness, it holds that $\mu(\sigma_k) < \mu(\sigma_n)$, a contradiction. □

Analytic completeness

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Proof idea: We show analytic completeness via a canonical model construction, where applications of cut are restricted to *analytic* cuts.

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Definition

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- A Σ-sequent $\Gamma \Rightarrow \Delta$ is called *saturated* if it is Σ-unprovable and $\Gamma^- \cup \Delta^- = \Sigma$.

Analytic completeness

Definition

Let Σ be a non-empty, finite and closed set of formulas. The *canonical model* $\mathbb{S}^\Sigma$ of Σ is given by:

$$S^\Sigma := \{\Gamma^- \mid \Gamma \Rightarrow \Delta \text{ is a saturated } \Sigma\text{-sequent}\}$$
$$AR_i^\Sigma B :\Leftrightarrow \Box_i \Box_i^{-1} A = \Box_i \Box_i^{-1} B$$
$$V^\Sigma(A) := \{p \in \mathsf{P} \mid p \in A\}$$

Completeness

Lemma (Truth Lemma)

For every $\varphi \in \Sigma$*:* $\mathbb{S}^{\Sigma}, A \Vdash \varphi$ *if and only if* $\varphi \in A$*.*

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$$\mathcal{S}^{\Sigma}, A \Vdash \boxast\psi$$

This implies that for all B with $AR_i^{\Sigma}B$:

$$\mathcal{S}^{\Sigma}, B \Vdash \psi \text{ and } \mathcal{S}^{\Sigma}, B \Vdash \boxast\psi$$

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This implies that for all B with $AR_i^\Sigma B$:

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In particular $\mathcal{S}^\Sigma, A \Vdash \psi$ and so $\psi \in A$, by the induction hypothesis. □

Completeness

Let $\Gamma \Rightarrow \Delta$ be a saturated Σ-sequent with $\Gamma^- = A$.

$$\boxast_R \frac{\overset{\pi}{\Gamma \Rightarrow \psi^u, \Delta} \qquad \overset{\pi_1}{\Gamma \Rightarrow \Box_1 \boxast \psi^f, \Delta} \qquad \cdots \qquad \overset{\pi_n}{\Gamma \Rightarrow \Box_n \boxast \psi^f, \Delta}}{\Gamma \Rightarrow \boxast \psi^f, \Delta}$$

Completeness

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\pi' & \pi'_1 & \dots & \pi'_n \\ \sigma' & \sigma'_1 & \dots & \sigma'_n\end{matrix}}{\Box_i\Box_i^{-1}\Gamma \Rightarrow \circledast\psi^f, \Box_i\Box_i^{-1}\Delta}\ \circledast_R}{\Box_i\Box_i^{-1}\Gamma \Rightarrow \Box_i\circledast\psi^f, \Box_i\Box_i^{-1}\Delta}\ \Box_{S5}}{\vdots}\ \mathsf{w}_L}{\Gamma \Rightarrow \Box_i\circledast\psi^f, \Box_i\Box_i^{-1}\Delta}\ \mathsf{w}_L}{\vdots}\ \mathsf{w}_R}{\Gamma \Rightarrow \Box_i\circledast\psi^f, \Delta}\ \mathsf{w}_R
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1. $\sigma' = \Box_i \Box_i^{-1}\Gamma \Rightarrow \psi^u, \Box_i \Box_i^{-1}\Delta$ must be provable. (If not, extend it into a saturated sequent $\Gamma' \Rightarrow \Delta'$. By construction $AR_i^\Sigma B = \Gamma'$ and $\mathcal{S}^\Sigma, B \not\Vdash \psi$.)

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2. For $\sigma'_k = \Box_i \Box_i^{-1} \Gamma \Rightarrow \Box_k \circledast \psi^f, \Box_i \Box_i^{-1} \Delta$ we apply cut repeatedly until every leaf is either saturated or provable.

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2. For $\sigma'_k = \Box_i \Box_i^{-1} \Gamma \Rightarrow \Box_k \circledast \psi^f, \Box_i \Box_i^{-1} \Delta$ we apply cut repeatedly until every leaf is either saturated or provable.
    1. For provable sequents, append their respective proofs.
    2. For saturated sequents, observe that we have met a repetition in our procedure.

$\Rightarrow$ Repeat argument until - by the pidgeonhole principle - every leaf is an axiom or a successful repetition.

Decision procedure

- A Σ-sequent σ has an analytic proof if and only if Prover has a winning strategy in $\mathcal{G}_\sigma$.
- The game $\mathcal{G}_\sigma$ is a *parity game*.
- Positional determinacy of parity games $\Rightarrow$ the *bounded proof property*.
- Decision procedure for parity games gives an EXPTIME decision procedure for $\mathsf{sCKL_f}$-provability.

Future work

- Syntactic cut-reduction
- Extension to Dynamic Epistemic Logic
- Interpolation?
- Realisation theorem for Justifcation Logic
- Extension to larger fragments of the modal μ-calculus.
- A cut-free cyclic system

Thank you!

References

Alberucci, Luca and Gerhard Jäger. "About cut elimination for logics of common knowledge". In: *Annals of Pure and Applied Logic* 133.1-3 (2005), pp. 73–99.

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Propositional rules

$$\text{id}\ \frac{}{\varphi^u \Rightarrow \varphi^a} \qquad \text{w}_L\ \frac{\Gamma \Rightarrow \Delta}{\Gamma, \varphi^u \Rightarrow \Delta} \qquad \text{w}_R\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \varphi^u, \Delta}$$

$$\neg_L\ \frac{\Gamma \Rightarrow \varphi^u, \Delta}{\Gamma, \neg\varphi^u \Rightarrow \Delta} \qquad \neg_R\ \frac{\Gamma, \varphi^u \Rightarrow \Delta}{\Gamma \Rightarrow \neg\varphi^u, \Delta}$$

$$\wedge_L\ \frac{\Gamma, \varphi^u, \psi^u \Rightarrow \Delta}{\Gamma, (\varphi \wedge \psi)^u \Rightarrow \Delta} \qquad \wedge_R\ \frac{\Gamma \Rightarrow \varphi^u, \Delta \qquad \Gamma \Rightarrow \psi^u, \Delta}{\Gamma \Rightarrow (\varphi \wedge \psi)^u, \Delta}$$

Strong local soundness for $\mathsf{r} = \Box_{\mathsf{S5}}$

Suppose $\mathsf{r} = \Box_{\mathsf{S5}}$: Then σ is of the form:

$$\Box_i\Gamma \Rightarrow \Box_i\circledast\psi^f, \Box_i\Delta.$$

Let $n := \mu(\sigma)$. By the definition of μ, there is an epistemic Kripke model $\mathbb{S}$, and a state s of $\mathbb{S}$ such that $\mathbb{S}, s \not\Vdash \sigma(n)$. In particular, it holds that

$$\mathbb{S}, s \not\Vdash \Box_i\Box^n\psi.$$

It follows that there is a state t in $\mathbb{S}$ such that sR_it and $\mathbb{S}, t \not\Vdash \Box^n\psi$. Clearly this also means that $\mathbb{S}, t \not\Vdash \circledast\psi$. We claim that, in fact,

$$\mathbb{S}, t \not\Vdash \Box_i\Gamma \Rightarrow \circledast\psi^f, \Box^n\psi^u, \Box_i\Delta,$$

which gives the required result.
By the fact that R_i is transitive, it holds for all φ such that $\mathbb{S}, s \Vdash \Box_i\varphi$, that $\mathbb{S}, t \Vdash \Box_i\varphi$. It follows that $\mathbb{S}, t \Vdash \Box_i\varphi$ for each $\Box_i\varphi^u \in \Box_i\Gamma$. Moreover, suppose that $\Box_i\psi^a \in \Box_i\Delta$. Then $\mathbb{S}, s \not\Vdash \Box_i\psi$. Thus there is a state r in $\mathbb{S}$ such that sR_ir and $\mathbb{S}, r \not\Vdash \psi$. By symmetry and transitivity, we get tR_is, whence $\mathbb{S}, t \not\Vdash \Box_i\psi$, as required.

A proof search game

Definition

Let σ be a Σ-sequent. The *proof search game* $\mathcal{G}_\sigma$ associated to σ has the following ownership function and admissible moves:

Position	Owner	Admissible moves
σ	Prover	$\left\{ \mathsf{r}\ \dfrac{\sigma_1 \quad \cdots \quad \sigma_n}{\sigma} : \sigma_1, \ldots, \sigma_n \text{ are } \Sigma \text{ sequents} \right\}$
$\mathsf{r}\ \dfrac{\sigma_1 \quad \cdots \quad \sigma_n}{\sigma}$	Refuter	$\{\sigma_i \mid 1 \leq i \leq n\}$

The positions are given the following priorities:

1. Every position of the form $\Gamma \Rightarrow \Delta^u$ has priority 3;
2. Every position of the form $\circledast_R\ \dfrac{\sigma_1 \quad \cdots \quad \sigma_n}{\sigma}$ where the principal formula is in focus has priority 2;
3. Every other position has priority 1.

An infinite match is won by Prover (Refuter) if the highest priority encountered infinitely often is even (odd).