Cyclic hypersequent calculi for some modal logics with the master modality

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Overview

1. Cyclic proof theory of $\mathbf{K}^*$
2. Cyclic proof theory of modal fixpoint logics
3. Lahav's method for $\mathbf{ML}$
4. Adapting Lahav's calculi to $\mathbf{ML}^*$
5. Conclusion and future work

Cyclic proof theory of K^*

Unimodal logic with the master modality

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$$(S, R, V), s \Vdash \circledast\psi \;\;:\Leftrightarrow\;\; \text{for all } t \in S \text{ such that } sR^*t\text{: } \mathbb{S}, t \Vdash \psi$$

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- **Hilbert-style axiomatisation (**HiK***):** usual axioms and rules for the modal logic K, plus:

$$\circledast p \leftrightarrow (p \wedge \Box\circledast p) \tag{fix}$$

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Note: $\circledast p \equiv \nu x. p \wedge \Box x$.

The quest for an analytic proof system

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Is there an analytic proof system for K*?

First start: a Gentzen-style counterpart of HiK*

$$\text{id}\ \frac{}{\varphi \Rightarrow \varphi} \qquad\qquad \bot\ \frac{}{\bot \Rightarrow}$$

$$\text{w}_L\ \frac{\Gamma \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta} \qquad\qquad \text{w}_R\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \varphi, \Delta}$$

$$\to_L\ \frac{\Gamma, \psi \Rightarrow \Delta \qquad \Gamma \Rightarrow \varphi, \Delta}{\Gamma, \varphi \to \psi \Rightarrow \Delta} \qquad\qquad \to_R\ \frac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \to \psi, \Delta}$$

$$\Box\ \frac{\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi} \qquad\qquad \text{cut}\ \frac{\Gamma_1, \varphi \Rightarrow \Delta_1 \qquad \Gamma_2 \Rightarrow \varphi, \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

$$\boxast_L\ \frac{\Gamma, \varphi, \Box\boxast\varphi \Rightarrow \Delta}{\Gamma, \boxast\varphi \Rightarrow \Delta} \qquad\qquad \boxast_R\ \frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Gamma \Rightarrow \Box\boxast\varphi, \Delta}{\Gamma \Rightarrow \boxast\varphi, \Delta}$$

$$\text{ind}\ \frac{\Gamma \Rightarrow \varphi \land \Box\widehat{\Gamma}}{\Gamma \Rightarrow \boxast\varphi}$$

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What if we get rid of both cut and the induction rule?

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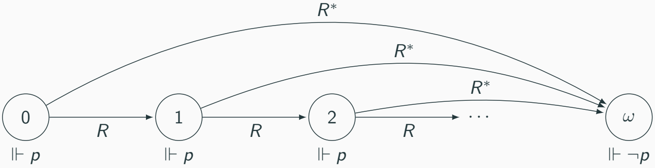

Another option: use an infinitary branching ω-rule

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$$\omega\text{-ind}\ \frac{\{\Gamma \Rightarrow \Box^n\varphi : n \in \mathbb{N}\}}{\Gamma \Rightarrow \circledast\varphi}$$

A proof about K^* by infinite descent

Claim

The formula $(\boxast(p \to q) \wedge \boxast p) \to \boxast q$ is valid.

Proof.

Suppose, towards a contradiction, that $\mathbb{S}, s \Vdash \boxast(p \to q)$ and $\mathbb{S}, s \Vdash \boxast p$, but $\mathbb{S}, s \nVdash \boxast q$. Then there is some $n_0 \in \mathbb{N}$ such that $\mathbb{S}, s \nVdash \Box^{n_0} q$. Since $\mathbb{S}, s \Vdash q$, we have $n_0 > 0$. Thus there is a state $s_1 \in S$ such that sRs_1 and $\mathbb{S}, s_1 \nVdash \Box^{n_0 - 1} q$. Letting $n_1 := n_0 - 1$, we have $\mathbb{S}, s_1 \nVdash \Box^{n_1} q$. Moreover, since $\mathbb{S}, s \Vdash \Box\boxast(p \to q)$ and $\mathbb{S}, s \Vdash \Box\boxast p$, it holds that $\mathbb{S}, s_1 \Vdash \boxast(p \to q)$ and $\mathbb{S}, s_1 \Vdash \boxast p$. Repeating this argument we find an infinite strictly decreasing chain $n_0 < n_1 < \cdots$, a contradiction. $\Box$

A formal counterpart of this proof

$$
\dfrac{
\dfrac{\dfrac{}{p \Rightarrow p}\,\text{id}}{\vdots}
\qquad
\dfrac{
\dfrac{\dfrac{}{q \Rightarrow q}\,\text{id}}{\vdots}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\boxast(p \to q), \boxast p \Rightarrow \boxast q}}{\Box\boxast(p \to q), \Box\boxast p \Rightarrow \Box\boxast q}\,\Box}{\Box\boxast(p \to q), p, \Box\boxast p \Rightarrow \Box\boxast q}\,\mathrm{w}_L}{q, \Box\boxast(p \to q), p, \Box\boxast p \Rightarrow \Box\boxast q}\,\mathrm{w}_L
}{q, \Box\boxast(p \to q), p, \Box\boxast p \Rightarrow \boxast q}\,\boxast_R
}{
\dfrac{
\dfrac{p \to q, \Box\boxast(p \to q), p, \Box\boxast p \Rightarrow \boxast q}{\boxast(p \to q), p, \Box\boxast p \Rightarrow \boxast q}\,\boxast_L
}{\boxast(p \to q), \boxast p \Rightarrow \boxast q}\,\boxast_L
}\,\to_L
$$

The infinitary proof system $\mathbf{K}^*_{\mathsf{inf}}$

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An $\mathbf{K}^*_{\mathsf{inf}}$-proof is a (possibly infinite) derivation in $\mathbf{K}^*$ of which each infinite branch contains a trace with infinitely many unfoldings.

Soundness and completeness of $\mathbf{K}^*_{\mathsf{inf}}$

Theorem

If $\Gamma \Rightarrow \Delta$ is provable in $\mathbf{K}^*_{\mathsf{inf}}$, then $\bigwedge \Gamma \to \bigvee \Delta$ is valid.

Theorem

If $\bigwedge \Gamma \to \bigvee \Delta$ is valid, then $\Gamma \Rightarrow \Delta$ is provable in $\mathbf{K}^*_{\mathsf{inf}}$.

Theorem ([4])

If $\Gamma \Rightarrow \Delta$ has a $\mathbf{K}^*_{\mathsf{inf}}$-proof, then it has a regular $\mathbf{K}^*_{\mathsf{inf}}$-proof.

[4] Damian Niwinski and Igor Walukiewicz. "Games for the mu-Calculus". In: *Theor. Comput. Sci.* 163.1&2 (1996), pp. 99–116

The annotated cyclic proof system $\mathsf{K}^*_{\mathsf{circ}}$

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The cyclic system $\mathbf{K}^*_{\mathsf{circ}}$ is obtained by encoding the traces in the sequents through focus annotations and adding a focus change rule.

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$$
\begin{prooftree}
\AxiomC{}
\RightLabel{}
\LeftLabel{id}
\UnaryInfC{$p \Rightarrow p$}
\noLine
\UnaryInfC{$\vdots$}
\AxiomC{}
\LeftLabel{id}
\UnaryInfC{$q \Rightarrow q$}
\noLine
\UnaryInfC{$\vdots$}
\AxiomC{$\boxast(p \to q), \boxast p \Rightarrow [\boxast q]$}
\RightLabel{$\Box$}
\UnaryInfC{$\Box\boxast(p \to q), \Box\boxast p \Rightarrow [\Box\boxast q]$}
\RightLabel{w_L}
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A cyclic derivation of an annotated sequent is a $\mathbf{K}^*_{\text{circ}}$-proof if:

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A cyclic derivation of an annotated sequent is a $\mathbf{K}^*_{\mathsf{circ}}$-proof if:

1. Every non-axiomatic leaf has a companion that is labelled by the same annotated hypersequent.

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A cyclic derivation of an annotated sequent is a $\mathbf{K}^*_{\mathrm{circ}}$-proof if:

1. Every non-axiomatic leaf has a companion that is labelled by the same annotated hypersequent.
2. Between every non-axiomatic leaf and its companion there is always a formula in focus and the focus change rule is not applied.

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The cyclic system $\mathbf{K}^*_{\text{circ}}$ is obtained by encoding the traces in the sequents through focus annotations and adding a focus change rule.

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\dfrac{
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  \qquad
  \dfrac{
    \dfrac{\text{id}\;\dfrac{}{q \Rightarrow q}}{\vdots}
    \qquad
    \dfrac{
      \dfrac{
        \dfrac{
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        }{
          \Box\boxast(p \to q), \Box\boxast p \Rightarrow [\Box\boxast q]
        }\;\Box
      }{
        \Box\boxast(p \to q), p, \Box\boxast p \Rightarrow [\Box\boxast q]
      }\;\mathsf{w}_L
    }{
      q, \Box\boxast(p \to q), p, \Box\boxast p \Rightarrow [\Box\boxast q]
    }\;\mathsf{w}_L
  }{
    q, \Box\boxast(p \to q), p, \Box\boxast p \Rightarrow [\boxast q]
  }\;\boxast_R
}{
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}\;\to_L
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3. Between every non-axiomatic leaf and its companion a formula that is in focus is unfolded.

Soundness and completeness of $\mathbf{K}^*_{\mathsf{circ}}$

Theorem

If $\Gamma \Rightarrow \Delta$ *is provable in* $\mathbf{K}^*_{\mathsf{circ}}$, *then* $\bigwedge \Gamma \to \bigvee \Delta$ *is valid.*

Theorem

If $\bigwedge \Gamma \to \bigvee \Delta$ *is valid, then* $\Gamma \Rightarrow \Delta$ *is provable in* $\mathbf{K}^*_{\mathsf{circ}}$.

Cyclic proof theory of modal fixpoint logics

Incomplete overview of results in cyclic proof theory of mfl's

1996 Cyclic system for μML (Niwinski & Walukiewicz)

2001 Annotated cyclic systems for LTL and CTL (Lange & Stirling)

2009 Annotated cyclic system for μML (Jungteerapanich & Stirling)

2014 Cyclic system and interpolation for GL (Shamkanov)

2016 Constructive completeness for μML (Afshari & Leigh)

2019 Constructive completeness for Game Logic (Enqvist et al.)

2021 Uniform Interpolation for μML (Afshari, Leigh & Menéndez Turata)

Research theme

- Most results in cyclic proof theory for modal fixpoint logics treat a particular logic

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  - Formalisms that enrich ordinary sequent calculi: hypersequents, nested sequents, labelled sequents, display calculi

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- In contrast, for basic modal logic very general methods exist:
  - Sahlqvist's Theorem for Hilbert-style proof systems
  - Formalisms that enrich ordinary sequent calculi: hypersequents, nested sequents, labelled sequents, display calculi
- Research theme: combine cyclic proof theory with existing general methods for basic modal logic to uniformly obtain proof systems for multiple modal fixpoint logics

Lahav’s method for ML

Lahav's method

Lahav's method

Start with a hypersequent calculus **HK** for the basic modal logic **K** (or for **KB**, or for **K4**, or for **KB4**)

$$\text{id}\ \frac{}{\varphi \Rightarrow \varphi} \qquad \bot\ \frac{}{\bot \Rightarrow}$$

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$$\to_L\ \frac{H \mid \Gamma, \psi \Rightarrow \Delta \qquad H \mid \Gamma \Rightarrow \varphi, \Delta}{H \mid \Gamma, \varphi \to \psi \Rightarrow \Delta} \qquad \to_R\ \frac{H \mid \Gamma, \varphi \Rightarrow \psi, \Delta}{H \mid \Gamma \Rightarrow \varphi \to \psi, \Delta}$$

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Lahav's method

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HK $\vdash H :\Leftrightarrow$ For every Kripke model $\mathbb{S}$, there is a $\Gamma \Rightarrow \Delta \in H$,

such that for every state s: $\mathbb{S}, s \nVdash \bigwedge \Gamma$ or $\mathbb{S}, s \Vdash \bigvee \Delta$.

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and add the following rule to **HK**:

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Theorem (Lahav)

$\mathbf{HK} + \mathbf{R}_{\mathcal{C}}$ *is sound and cut-free complete w.r.t. the class of* $\mathcal{C}$*-models.*

Adapting Lahav's calculi to ML*

The hypersequent calculus $\mathbf{HK}^*$

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$$\text{id}\ \frac{}{\varphi \Rightarrow \varphi} \qquad \bot\ \frac{}{\bot \Rightarrow}$$

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Infinitary proofs in $\mathsf{HK}^* + \mathsf{R}_{\mathcal{C}}$

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An $\mathbf{HK}^*_{\mathsf{inf}} + \mathbf{R}_{\mathcal{C}}$-proof is a (possibly infinite) derivation in $\mathbf{HK}^* + \mathbf{R}_{\mathcal{C}}$ whose infinite branches each have a trace on which infinitely many unfoldings occur.

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Soundness of infinitary proofs in $\mathsf{HK}^* + \mathsf{R}_{\mathcal{C}}$

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Theorem (Soundness for $\mathbf{HK}^*_{\mathsf{inf}} + \mathbf{R}_{\mathcal{C}}$)

Let $\mathcal{C}$ be a finite set of simple frame conditions. If a hypersequent is $\mathbf{HK}^*_{\mathsf{inf}} + \mathbf{R}_{\mathcal{C}}$-provable, then it is valid in every $\mathcal{C}$-model.

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Proof.

Using the method of descending countermodels. □

The technical treatment of takes inspiration from:

[1] Simon Docherty and Reuben N.S. Rowe. "A non-wellfounded, labelled proof system for propositional dynamic logic". In: *International Conference on Automated Reasoning with Analytic Tableaux and Related Methods*. Springer. 2019, pp. 335–352

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The cyclic system $\mathbf{HK}^* + \mathbf{R}_{\mathcal{C}}$ is obtained by encoding the traces in the hypersequents through focus annotations and adding a focus change rule.

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  \qquad
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The cyclic system $\mathbf{HK}^* + \mathbf{R}_{\mathcal{C}}$ is obtained by encoding the traces in the hypersequents through focus annotations and adding a focus change rule.

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3. Between every non-axiomatic leaf and its companion a formula that is in focus is unfolded.

Completeness of cyclic proofs for equable frame conditions

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Definition (equable simple frame conditions)

An simple frame condition C is called: equable if for every $\langle C_R^1, C_=^1 \rangle, \langle C_R^2, C_=^2 \rangle \in C$ it holds that $C_=^1 = C_=^2$.

Universality	$\forall x_1 x_2 \exists u (x_1 R u \wedge x_2 = u)$	✓

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Reflexivity	$\forall x_1 \exists u (x_1 R u \wedge x_1 = u)$	✓

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k-bounded top width	$\forall x_1 \cdots x_k \exists u \bigvee_{1 \leq i < j \leq k} (x_i R u \wedge x_j R u)$	✓

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Linearity	$\forall x_1 x_2 \exists u((x_1 R u \wedge x_2 = u) \vee (x_2 R u \wedge x_1 = u))$	X

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Theorem (Completeness of $\mathbf{HK}^*_{\text{circ}} + \mathbf{R}_{\mathcal{C}}$)

Let $\mathcal{C}$ be a finite set of equable frame conditions. If a hypersequent is valid in every $\mathcal{C}$-model, then it is cut-free $\mathbf{HK}^*_{\text{circ}} + \mathbf{R}_{\mathcal{C}}$-provable.

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Proof.

Any $\mathbf{HK}^*_{\text{circ}} + \mathbf{R}_{\mathcal{C}}$-unprovable hypersequent H can be extended to an unprovable hypersequent that can be viewed as a (finite) countermodel. □

Conclusion and future work

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- Give a bound on the size of proofs
- Are there non-equable simple frame conditions for which $\mathbf{HK}^* + \mathbf{R}_\mathcal{C}$ is complete? Are there non-equable simple frame conditions for which it is not?
- Interpolation
- More expressive fragments of the modal μ-calculus
- Other sequent systems: labelled sequents, nested sequents

Thank you!

References

[1] Simon Docherty and Reuben N.S. Rowe. “A non-wellfounded, labelled proof system for propositional dynamic logic”. In: *International Conference on Automated Reasoning with Analytic Tableaux and Related Methods*. Springer. 2019, pp. 335–352.

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