Filtration and canonical completeness for continuous modal μ-calculi

Jan Rooduijn

joint work with

Yde Venema

ILLC, University of Amsterdam

GandALF 2021, 21 September 2021

The modal μ-calculus

The modal μ-calculus

$$\text{(ML)} \quad \varphi ::= p \mid \neg p \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \Diamond\varphi \mid \Box\varphi, \;\; p \in \mathsf{P}$$

The modal μ-calculus

$$\text{(ML)} \quad \varphi ::= p \mid \neg p \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \Diamond\varphi \mid \Box\varphi, \;\; p \in \mathsf{P}$$

Given a formula $\varphi \in \mathrm{ML}$ and a variable $x \in \mathsf{P}$, we may regard x as a free variable of φ. For every Kripke model $\mathbb{S} = (S, R, V)$, this induces a function:

$$\varphi_x^{\mathbb{S}} : \mathcal{P}(S) \to \mathcal{P}(S) \text{ given by } \varphi_x^{\mathbb{S}}(A) := [\![\varphi]\!]^{\mathbb{S}[x \mapsto A]}$$

The modal μ-calculus

$$\text{(ML)} \quad \varphi ::= p \mid \neg p \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \Diamond\varphi \mid \Box\varphi, \;\; p \in \mathsf{P}$$

Given a formula $\varphi \in \text{ML}$ and a variable $x \in \mathsf{P}$, we may regard x as a free variable of φ. For every Kripke model $\mathbb{S} = (S, R, V)$, this induces a function:

$$\varphi_x^{\mathbb{S}} : \mathcal{P}(S) \to \mathcal{P}(S) \text{ given by } \varphi_x^{\mathbb{S}}(A) := [\![\varphi]\!]^{\mathbb{S}[x \mapsto A]}$$

Observation

If x occurs only positively in φ, then $\varphi_x^{\mathbb{S}}$ is monotone and so, by the Knaster-Tarski theorem, it has both a least and a greatest fixpoint.

The modal μ-calculus

$$\text{(ML)} \quad \varphi ::= p \mid \neg p \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \Diamond\varphi \mid \Box\varphi, \;\; p \in \mathsf{P}$$

Given a formula $\varphi \in$ ML and a variable $x \in \mathsf{P}$, we may regard x as a free variable of φ. For every Kripke model $\mathbb{S} = (S, R, V)$, this induces a function:

$$\varphi_x^{\mathbb{S}} : \mathcal{P}(S) \to \mathcal{P}(S) \text{ given by } \varphi_x^{\mathbb{S}}(A) := [\![\varphi]\!]^{\mathbb{S}[x \mapsto A]}$$

Observation

If x occurs only positively in φ, then $\varphi_x^{\mathbb{S}}$ is monotone and so, by the Knaster-Tarski theorem, it has both a least and a greatest fixpoint.

$$(\mu\text{ML}) \quad \varphi ::= p \mid \neg p \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \Diamond\varphi \mid \Box\varphi \mid \mu x\psi \mid \nu x\psi,$$

where $p \in \mathsf{P}$ and x occurs only positively in ψ.

The modal μ-calculus

$$(\mathrm{ML}) \quad \varphi ::= p \mid \neg p \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \Diamond\varphi \mid \Box\varphi, \;\; p \in \mathrm{P}$$

Given a formula $\varphi \in \mathrm{ML}$ and a variable $x \in \mathrm{P}$, we may regard x as a free variable of φ. For every Kripke model $\mathbb{S} = (S, R, V)$, this induces a function:

$$\varphi_x^{\mathbb{S}} : \mathcal{P}(S) \to \mathcal{P}(S) \text{ given by } \varphi_x^{\mathbb{S}}(A) := [\![\varphi]\!]^{\mathbb{S}[x \mapsto A]}$$

Observation

If x occurs only positively in φ, then $\varphi_x^{\mathbb{S}}$ is monotone and so, by the Knaster-Tarski theorem, it has both a least and a greatest fixpoint.

$$(\mu\mathrm{ML}) \quad \varphi ::= p \mid \neg p \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \Diamond\varphi \mid \Box\varphi \mid \mu x\psi \mid \nu x\psi,$$

where $p \in \mathrm{P}$ and x occurs only positively in ψ.

$$[\![\mu x\varphi]\!]_x^{\mathbb{S}} := LFP(\varphi_x^{\mathbb{S}}) \qquad [\![\nu x\varphi]\!]_x^{\mathbb{S}} := GFP(\varphi_x^{\mathbb{S}})$$

Evaluation game

Evaluation game

The *evaluation game* $\mathcal{E}(\xi, \mathbb{S})$ takes positions in $\mathsf{Sf}(\xi) \times S$ and has the following ownership function and admissible moves.

Position		Player	Admissible moves
$(\varphi_1 \vee \varphi_2, s)$		$\exists$	$\{(\varphi_1, s), (\varphi_2, s)\}$
$(\varphi_1 \wedge \varphi_2, s)$		$\forall$	$\{(\varphi_1, s), (\varphi_2, s)\}$
$(\Diamond\varphi, s)$		$\exists$	$\{(\varphi, t) : sRt\}$
$(\Box\varphi, s)$		$\forall$	$\{(\varphi, t) : sRt\}$
$(\eta x.\delta_x, s)$		-	$\{(\delta_x, s)\}$
(x, s)	with $x \in \mathsf{BV}(\xi)$	-	$\{(\delta_x, s)\}$
(p, s)	with $p \in \mathsf{FV}(\xi)$ and $s \in V(p)$	$\forall$	$\emptyset$
$(\neg p, s)$	with $p \in \mathsf{FV}(\xi)$ and $s \in V(p)$	$\exists$	$\emptyset$
(p, s)	with $p \in \mathsf{FV}(\xi)$ and $s \notin V(p)$	$\exists$	$\emptyset$
$(\neg p, s)$	with $p \in \mathsf{FV}(\xi)$ and $s \notin V(p)$	$\forall$	$\emptyset$

An infinite match is won by $\exists$ ($\forall$) if the ‘most important’ fixpoint variable reached infinitely often is a ν-variable (a μ-variable)

Evaluation game

The *evaluation game* $\mathcal{E}(\xi, \mathbb{S})$ takes positions in $\mathsf{Sf}(\xi) \times S$ and has the following ownership function and admissible moves.

Position		Player	Admissible moves
$(\varphi_1 \vee \varphi_2, s)$		$\exists$	$\{(\varphi_1, s), (\varphi_2, s)\}$
$(\varphi_1 \wedge \varphi_2, s)$		$\forall$	$\{(\varphi_1, s), (\varphi_2, s)\}$
$(\Diamond\varphi, s)$		$\exists$	$\{(\varphi, t) : sRt\}$
$(\Box\varphi, s)$		$\forall$	$\{(\varphi, t) : sRt\}$
$(\eta x.\delta_x, s)$		-	$\{(\delta_x, s)\}$
(x, s)	with $x \in \mathsf{BV}(\xi)$	-	$\{(\delta_x, s)\}$
(p, s)	with $p \in \mathsf{FV}(\xi)$ and $s \in V(p)$	$\forall$	$\emptyset$
$(\neg p, s)$	with $p \in \mathsf{FV}(\xi)$ and $s \in V(p)$	$\exists$	$\emptyset$
(p, s)	with $p \in \mathsf{FV}(\xi)$ and $s \notin V(p)$	$\exists$	$\emptyset$
$(\neg p, s)$	with $p \in \mathsf{FV}(\xi)$ and $s \notin V(p)$	$\forall$	$\emptyset$

An infinite match is won by $\exists$ ($\forall$) if the 'most important' fixpoint variable reached infinitely often is a ν-variable (a μ-variable)

Example: $\mu x \Box x$ is true at a state s_0 iff there is no infinite path starting at s_0.

$$(\mu x \Box x, s_0) \xrightarrow{-} (\Box x, s_0) \xrightarrow{\forall} (x, s_1) \xrightarrow{-} (\Box x, s_1) \xrightarrow{\forall} (x, s_2) \xrightarrow{-} \cdots$$

Motivation of the paper

Motivation of the paper

- The modal μ-calculus is highly expressive, yet retains many of the desirable properties of basic modal logic, *e.g* bisimulation invariance and the finite model property.

Motivation of the paper

- The modal μ-calculus is highly expressive, yet retains many of the desirable properties of basic modal logic, *e.g* bisimulation invariance and the finite model property.
- However, two important methods fail: (i) filtration and (ii) canonical models.

Motivation of the paper

- The modal μ-calculus is highly expressive, yet retains many of the desirable properties of basic modal logic, *e.g* bisimulation invariance and the finite model property.
- However, two important methods fail: (i) filtration and (ii) canonical models.
- Both of these methods are well-known to work for PDL.

Motivation of the paper

- The modal μ-calculus is highly expressive, yet retains many of the desirable properties of basic modal logic, *e.g* bisimulation invariance and the finite model property.
- However, two important methods fail: (i) filtration and (ii) canonical models.
- Both of these methods are well-known to work for PDL.

Question

Can we do better?

Motivation of the paper

- The modal μ-calculus is highly expressive, yet retains many of the desirable properties of basic modal logic, *e.g* bisimulation invariance and the finite model property.
- However, two important methods fail: (i) filtration and (ii) canonical models.
- Both of these methods are well-known to work for PDL.

Question

Can we do better? That is, is there a natural fragment of μML that subsumes PDL and to which the methods of filtration and canonical models can be applied?

Motivation of the paper

- The modal μ-calculus is highly expressive, yet retains many of the desirable properties of basic modal logic, *e.g* bisimulation invariance and the finite model property.
- However, two important methods fail: (i) filtration and (ii) canonical models.
- Both of these methods are well-known to work for PDL.

Question

Can we do better? That is, is there a natural fragment of μML that subsumes PDL and to which the methods of filtration and canonical models can be applied?

Our answer (very roughly)

Yes, namely the continuous modal μ-calculus.

Filtration

Filtration

Let $\mathbb{S} = (S, R, V)$ be a Kripke model and let Σ be a finite and closed set of formulas.

Filtration

Let $\mathbb{S} = (S, R, V)$ be a Kripke model and let Σ be a finite and closed set of formulas.

Let $\sim_\Sigma^\mathbb{S}$ be the equivalence relation given by:

$$s \sim_\Sigma^\mathbb{S} s' \text{ if and only if } s \in [\![\varphi]\!]^\mathbb{S} \Leftrightarrow s' \in [\![\varphi]\!]^\mathbb{S} \text{ for all } \varphi \in \Sigma.$$

and define $\overline{S} := S/{\sim_\Sigma^\mathbb{S}}$.

Filtration

Let $\mathbb{S} = (S, R, V)$ be a Kripke model and let Σ be a finite and closed set of formulas.

Let $\sim_\Sigma^\mathbb{S}$ be the equivalence relation given by:

$$s \sim_\Sigma^\mathbb{S} s' \text{ if and only if } s \in [\![\varphi]\!]^\mathbb{S} \Leftrightarrow s' \in [\![\varphi]\!]^\mathbb{S} \text{ for all } \varphi \in \Sigma.$$

and define $\overline{S} := S/{\sim_\Sigma^\mathbb{S}}$.

Pick any relation $\overline{R} \subseteq \overline{S} \times \overline{S}$ such that $R^{\text{min}} \subseteq \overline{R} \subseteq R^{\text{max}}$, where

$$R^{\text{min}} := \{(\overline{s}, \overline{t}) : \text{there are } s' \sim_\Sigma^\mathbb{S} s \text{ and } t' \sim_\Sigma^\mathbb{S} t \text{ such that } Rs't'\},$$
$$R^{\text{max}} := \{(\overline{s}, \overline{t}) : \text{for all } \Box\varphi \in \Sigma; \text{ if } s \Vdash \Box\varphi, \text{ then } t \Vdash \varphi\}.$$

Filtration

Let $\mathbb{S} = (S, R, V)$ be a Kripke model and let Σ be a finite and closed set of formulas.

Let $\sim^{\mathbb{S}}_{\Sigma}$ be the equivalence relation given by:

$$s \sim^{\mathbb{S}}_{\Sigma} s' \text{ if and only if } s \in [\![\varphi]\!]^{\mathbb{S}} \Leftrightarrow s' \in [\![\varphi]\!]^{\mathbb{S}} \text{ for all } \varphi \in \Sigma.$$

and define $\overline{S} := S/{\sim^{\mathbb{S}}_{\Sigma}}$.

Pick any relation $\overline{R} \subseteq \overline{S} \times \overline{S}$ such that $R^{\min} \subseteq \overline{R} \subseteq R^{\max}$, where

$$R^{\min} := \{(\overline{s}, \overline{t}) : \text{there are } s' \sim^{\mathbb{S}}_{\Sigma} s \text{ and } t' \sim^{\mathbb{S}}_{\Sigma} t \text{ such that } Rs't'\},$$
$$R^{\max} := \{(\overline{s}, \overline{t}) : \text{for all } \Box\varphi \in \Sigma; \text{ if } s \Vdash \Box\varphi, \text{ then } t \Vdash \varphi\}.$$

Finally, let $\overline{V}(p) := \{\overline{s} : s \Vdash p\}$ for every $p \in \Sigma \cap \mathsf{P}$.

Filtration

Let $\mathbb{S} = (S, R, V)$ be a Kripke model and let Σ be a finite and closed set of formulas.

Let $\sim_\Sigma^\mathbb{S}$ be the equivalence relation given by:

$$s \sim_\Sigma^\mathbb{S} s' \text{ if and only if } s \in [\![\varphi]\!]^\mathbb{S} \Leftrightarrow s' \in [\![\varphi]\!]^\mathbb{S} \text{ for all } \varphi \in \Sigma.$$

and define $\overline{S} := S/{\sim_\Sigma^\mathbb{S}}$.

Pick any relation $\overline{R} \subseteq \overline{S} \times \overline{S}$ such that $R^{\mathsf{min}} \subseteq \overline{R} \subseteq R^{\mathsf{max}}$, where

$$R^{\mathsf{min}} := \{(\overline{s}, \overline{t}) : \text{there are } s' \sim_\Sigma^\mathbb{S} s \text{ and } t' \sim_\Sigma^\mathbb{S} t \text{ such that } Rs't'\},$$
$$R^{\mathsf{max}} := \{(\overline{s}, \overline{t}) : \text{for all } \Box\varphi \in \Sigma; \text{ if } s \Vdash \Box\varphi, \text{ then } t \Vdash \varphi\}.$$

Finally, let $\overline{V}(p) := \{\overline{s} : s \Vdash p\}$ for every $p \in \Sigma \cap \mathsf{P}$.

Then the model $\overline{\mathbb{S}} := (\overline{S}, \overline{R}, \overline{V})$ is called a filtration of $\mathbb{S}$ through Σ.

Filtration (ii)

Filtration (ii)

The Filtration Theorem holds for a modal language D if for any finite and closed set Σ of D-formulas and any filtration $\overline{\mathbb{S}}$ of $\mathbb{S}$ through Σ we have:

$$\overline{s} \in [\![\varphi]\!]^{\overline{\mathbb{S}}} \Leftrightarrow s \in [\![\varphi]\!]^{\mathbb{S}}$$

for every $\varphi \in \Sigma$.

Filtration (ii)

The Filtration Theorem holds for a modal language D if for any finite and closed set Σ of D-formulas and any filtration $\overline{\mathbb{S}}$ of $\mathbb{S}$ through Σ we have:

$$\overline{s} \in [\![\varphi]\!]^{\overline{\mathbb{S}}} \Leftrightarrow s \in [\![\varphi]\!]^{\mathbb{S}}$$

for every $\varphi \in \Sigma$.

The Filtration Theorem holds for ML, for PDL, but not for μML:

Filtration (ii)

The Filtration Theorem holds for a modal language D if for any finite and closed set Σ of D-formulas and any filtration $\overline{\mathbb{S}}$ of $\mathbb{S}$ through Σ we have:

$$\overline{s} \in [\![\varphi]\!]^{\overline{\mathbb{S}}} \Leftrightarrow s \in [\![\varphi]\!]^{\mathbb{S}}$$

for every $\varphi \in \Sigma$.

The Filtration Theorem holds for ML, for PDL, but not for μML:

Consider the formula $\varphi := \mu x \Box x$ and the model $\mathbb{S} := (\mathbb{N}, <, V)$:

$$0 \leftarrow 1 \leftarrow 2 \leftarrow 3 \leftarrow \cdots$$

$+$ transitive arrows

The continuous modal μ-calculus

The continuous modal μ-calculus

Idea: restrict the use of the least and greatest fixpoint operators to (formulas that induce) functions that are Scott continuous, rather than merely monotone.

The continuous modal μ-calculus

Idea: restrict the use of the least and greatest fixpoint operators to (formulas that induce) functions that are Scott continuous, rather than merely monotone.

Fontaine (2008) proves the following syntactic characterisation:

$$\varphi ::= x \mid \alpha \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \Diamond \varphi \mid \mu y \varphi'$$

where $x \in \mathsf{X}$, $y \in \mathsf{P}$, $\alpha \in \mu_c\mathsf{ML}$ X-free, and $\varphi' \in \mathsf{Con}_{\mathsf{X}\cup\{y\}}(\mu_c\mathsf{ML})$.

The continuous modal μ-calculus

Idea: restrict the use of the least and greatest fixpoint operators to (formulas that induce) functions that are Scott continuous, rather than merely monotone.

Fontaine (2008) proves the following syntactic characterisation:

$$\varphi ::= x \mid \alpha \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \Diamond\varphi \mid \mu y \varphi'$$

where $x \in \mathsf{X}$, $y \in \mathsf{P}$, $\alpha \in \mu_c\mathrm{ML}$ X-free, and $\varphi' \in \mathrm{Con}_{\mathsf{X}\cup\{y\}}(\mu_c\mathrm{ML})$.

Roughly: under a μ we disallow $\Box$ and ν and, dually, under a ν we disallow $\Diamond$ and μ.

Properties of μ_cML

Properties of μ_cML

- Constructive: fixpoints are reached after at most ω iterations.

Properties of μ_cML

- Constructive: fixpoints are reached after at most ω iterations.
- Strictly more expressive than PDL.

Properties of μ_cML

- Constructive: fixpoints are reached after at most ω iterations.
- Strictly more expressive than PDL.

Properties of the evaluation game played with μ_cML-formulas:

1. A match progresses at most finitely often from a position $(s, \eta x.\delta)$ to a position $(t, \overline{\eta} y.\theta)$.

Properties of μ_cML

- Constructive: fixpoints are reached after at most ω iterations.
- Strictly more expressive than PDL.

Properties of the evaluation game played with μ_cML-formulas:

1. A match progresses at most finitely often from a position $(s, \eta x.\delta)$ to a position $(t, \overline{\eta} y.\theta)$.
2. A match progresses at most finitely often from a position $(s, \mu x.\delta)$ to a position $(t, \Box\psi)$.

Theorem (Filtration Theorem for μ_cML)

For any finite and closed set Σ *of* μ_cML*-formulas and any filtration* $\overline{\mathbb{S}}$ *of* $\mathbb{S}$ *through* Σ *it holds that:*

$$\overline{s} \in [\![\varphi]\!]^{\overline{\mathbb{S}}} \Leftrightarrow s \in [\![\varphi]\!]^{\mathbb{S}}$$

for every $\varphi \in \Sigma$*.*

Theorem (Filtration Theorem for μ_cML)

For any finite and closed set Σ of μ_cML-formulas and any filtration $\overline{\mathbb{S}}$ of $\mathbb{S}$ through Σ it holds that:

$$\overline{s} \in [\![\varphi]\!]^{\overline{\mathbb{S}}} \Leftrightarrow s \in [\![\varphi]\!]^{\mathbb{S}}$$

for every $\varphi \in \Sigma$.

Proof sketch.

Suppose $\exists$ has a winning strategy f for $\mathcal{G}$ at (φ, s); we must show that she has a winning strategy for $\overline{\mathcal{G}}$ at $(\varphi, \overline{s})$.

Theorem (Filtration Theorem for μ_cML)

For any finite and closed set Σ *of* μ_cML*-formulas and any filtration* $\overline{\mathbb{S}}$ *of* $\mathbb{S}$ *through* Σ *it holds that:*

$$\overline{s} \in [\![\varphi]\!]^{\overline{\mathbb{S}}} \Leftrightarrow s \in [\![\varphi]\!]^{\mathbb{S}}$$

for every $\varphi \in \Sigma$.

Proof sketch.

Suppose $\exists$ has a winning strategy f for $\mathcal{G}$ at (φ, s); we must show that she has a winning strategy for $\overline{\mathcal{G}}$ at $(\varphi, \overline{s})$. We play a shadow match, copying in $\overline{\mathcal{G}}$ the moves suggested to $\exists$ by the strategy f in $\mathcal{G}$, and simulating in $\mathcal{G}$ the moves played by $\forall$ in $\overline{\mathcal{G}}$.

Theorem (Filtration Theorem for μ_cML)

For any finite and closed set Σ *of* μ_cML*-formulas and any filtration* $\overline{\mathbb{S}}$ *of* $\mathbb{S}$ *through* Σ *it holds that:*

$$\overline{s} \in [\![\varphi]\!]^{\overline{\mathbb{S}}} \Leftrightarrow s \in [\![\varphi]\!]^{\mathbb{S}}$$

for every $\varphi \in \Sigma$.

Proof sketch.

Suppose $\exists$ has a winning strategy f for $\mathcal{G}$ at (φ, s); we must show that she has a winning strategy for $\overline{\mathcal{G}}$ at $(\varphi, \overline{s})$. We play a shadow match, copying in $\overline{\mathcal{G}}$ the moves suggested to $\exists$ by the strategy f in $\mathcal{G}$, and simulating in $\mathcal{G}$ the moves played by $\forall$ in $\overline{\mathcal{G}}$. Note: at each position $(s, \Box\varphi)$ we must reset the shadow match.

Theorem (Filtration Theorem for μ_cML)

For any finite and closed set Σ of μ_cML-formulas and any filtration $\overline{\mathbb{S}}$ of $\mathbb{S}$ through Σ it holds that:

$$\overline{s} \in [\![\varphi]\!]^{\overline{\mathbb{S}}} \Leftrightarrow s \in [\![\varphi]\!]^{\mathbb{S}}$$

for every $\varphi \in \Sigma$.

Proof sketch.

Suppose $\exists$ has a winning strategy f for $\mathcal{G}$ at (φ, s); we must show that she has a winning strategy for $\overline{\mathcal{G}}$ at $(\varphi, \overline{s})$. We play a shadow match, copying in $\overline{\mathcal{G}}$ the moves suggested to $\exists$ by the strategy f in $\mathcal{G}$, and simulating in $\mathcal{G}$ the moves played by $\forall$ in $\overline{\mathcal{G}}$. Note: at each position $(s, \Box\varphi)$ we must reset the shadow match. However, if the obtained strategy would be losing for $\exists$ this reset could happen only finitely often, contradicting the assumption that f is winning. $\square$

Other results

Definition

A class of models $\mathcal{M}$ is said to admit filtration with respect to a language D if for every model $\mathbb{S}$ in $\mathcal{M}$ and every finite closed set of D-formulas Σ, the class $\mathcal{M}$ contains a filtration of $\mathbb{S}$ through Σ. A class of frames $\mathcal{F}$ is said to admit filtration if the class of models $\{(S, R, V) : (S, R) \in \mathcal{F}\}$ does.

Other results

Definition

A class of models $\mathcal{M}$ is said to admit filtration with respect to a language D if for every model $\mathbb{S}$ in $\mathcal{M}$ and every finite closed set of D-formulas Σ, the class $\mathcal{M}$ contains a filtration of $\mathbb{S}$ through Σ. A class of frames $\mathcal{F}$ is said to admit filtration if the class of models $\{(S, R, V) : (S, R) \in \mathcal{F}\}$ does.

Lemma

For any logic L, *the class* Mod(L) *admits filtration wrt* ML *iff it admits filtration wrt* μ_cML.

Other results

Definition

A class of models $\mathcal{M}$ is said to admit filtration with respect to a language D if for every model $\mathbb{S}$ in $\mathcal{M}$ and every finite closed set of D-formulas Σ, the class $\mathcal{M}$ contains a filtration of $\mathbb{S}$ through Σ. A class of frames $\mathcal{F}$ is said to admit filtration if the class of models $\{(S, R, V) : (S, R) \in \mathcal{F}\}$ does.

Lemma

For any logic L, *the class* Mod(L) *admits filtration wrt* ML *iff it admits filtration wrt* μ_cML.

Corollary (Finite Model Property)

Let L *be a logic such that* Mod(L) *admits filtration with respect to* ML, *and let* ϕ *be a formula of the continuous* μ*-calculus. Then* ϕ *is valid in every* L*-model if and only if* ϕ *is valid in every finite* L*-model.*

Other results

Definition

A class of models $\mathcal{M}$ is said to admit filtration with respect to a language D if for every model $\mathbb{S}$ in $\mathcal{M}$ and every finite closed set of D-formulas Σ, the class $\mathcal{M}$ contains a filtration of $\mathbb{S}$ through Σ. A class of frames $\mathcal{F}$ is said to admit filtration if the class of models $\{(S, R, V) : (S, R) \in \mathcal{F}\}$ does.

Lemma

For any logic L, *the class* Mod(L) *admits filtration wrt* ML *iff it admits filtration wrt* μ_cML.

Corollary (Finite Model Property)

Let L *be a logic such that* Mod(L) *admits filtration with respect to* ML, *and let* ϕ *be a formula of the continuous* μ*-calculus. Then* ϕ *is valid in every* L*-model if and only if* ϕ *is valid in every finite* L*-model.*

For example: μ_cML has the FMP over symmetric models.

Other results (ii)

Other results (ii)

Theorem

Let L *be a canonical logic in the basic modal language such* Fr(L) *admits filtration. Then* μ_c-L *is sound and complete with respect to* Fr(L).

Other results (ii)

Theorem

Let L *be a canonical logic in the basic modal language such* Fr(L) *admits filtration. Then* μ_c-L *is sound and complete with respect to* Fr(L).

For example: L $=$ KB, K4, S4, S5, ...

Other results (ii)

Theorem

Let L *be a canonical logic in the basic modal language such* Fr(L) *admits filtration. Then* μ_c-L *is sound and complete with respect to* Fr(L)*.*

For example: L $=$ KB, K4, S4, S5, ...

The last two results generalise results for PDL in Kikot, Shapirovsky & Zolin (AiML 2020).

Future work

Future work

- Relation to constructiveness.

Future work

- Relation to constructiveness.
- Is μ_cML somehow a maximal 'natural' fragment of μML to which filtration is applicable?

Future work

- Relation to constructiveness.
- Is μ_cML somehow a maximal ‘natural’ fragment of μML to which filtration is applicable?
- Can the currently separate proofs of the Filtration Theorem and canonical completeness be unified by taking a filtration of some canonical model (as with PDL).

Thank you