The global structure of large locally sparse graphs

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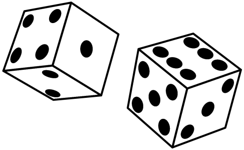
i.e. "local versus global"

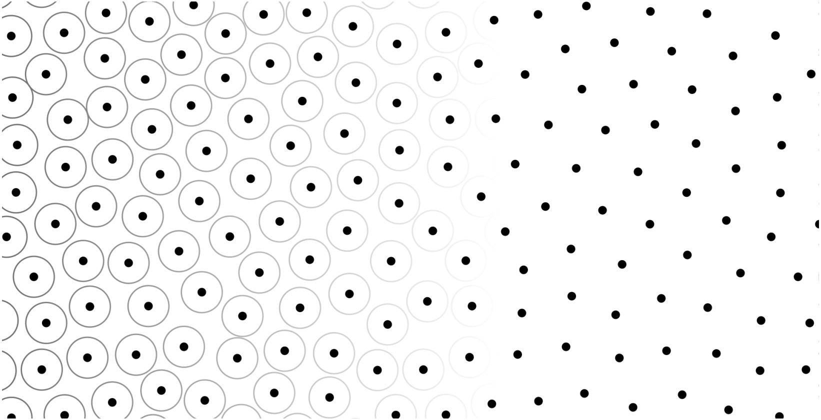
Distinguished origins:

- Mantel (1907), Turán (1941)
- Ramsey (1930), Erdős & Szekeres (1935)
- Zykov (1949), Ungar & "Blanche Descartes" (1954)



If a random object has desired property with positive probability, then there exists *at least one* object with that property

Hard-core model[†]



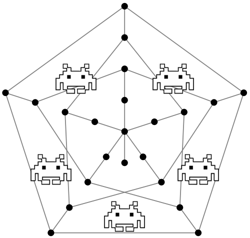
[†] More fully, the lattice gas with hard-core self-repulsion and nearest-neighbour exclusion.
Picture credit: Wikipedia/Grap-wh

List colouring

Given a graph G, imagine *adversaries* to you properly colouring it

- that annoyingly issue lists of permissible colours
- but must permit at least ℓ colours per vertex

What is least ℓ for which you can *always* win? (Necessarily $\chi \leq \ell \leq |G|$)



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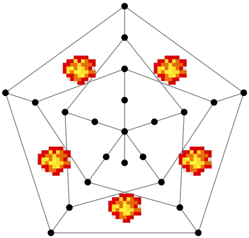

Called list chromatic number or choosability $\mathrm{ch}(G)$ of G

List makes it "harder"

ch is not bounded by any function of chromatic number χ

Theorem (Erdős, Rubin, Taylor 1980)

$\mathrm{ch}(K_{d,d}) \sim \log_2 d$ *(and* $\mathrm{ch}(K_{d+1}) = d + 1$*)*

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Still not completely well understood

Conjecture (Alon & Krivelevich 1998)

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Separation makes it "easier"?

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Question: Does $\mathrm{ch}_{\mathrm{sep}}$ grow in δ?

Problem: Almost-disjointness of lists is not monotone under edge-addition!

Ramsey-type question/solution?

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Bipartite induced density

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Theorem (Kwan, Letzter, Sudakov, Tran 2018+)

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BID(G) $\geq \dfrac{\delta}{2\chi}$ *for any G with minimum degree δ and chromatic number χ*

$$\begin{array}{c}\text{BID}(G)\\ \downarrow\end{array}$$

$$\omega(G) \leq \max_{\emptyset \neq H \subseteq G} \frac{|H|}{\alpha(H)} \leq \chi_f(G) \leq \chi(G) \leq \mathrm{ch}(G) \leq \Delta(G) + 1$$

‡

A sequence with independence and colouring

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In general, all can be strict [‡]

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Off-diagonal Ramsey numbers[§]

[§] Picture credit: Soifer 2009

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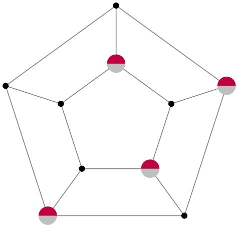
[¶] Yes, cf. Davies, Jenssen, Perkins, Roberts 2018...

Fractional chromatic number



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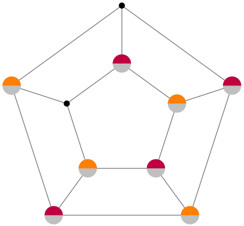

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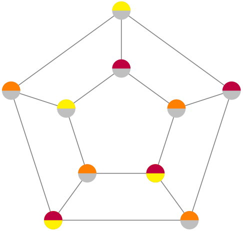

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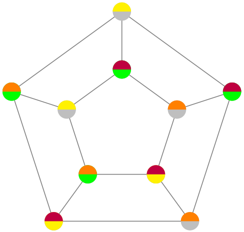

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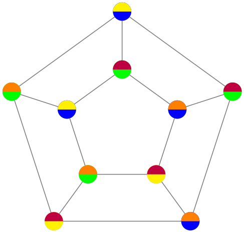

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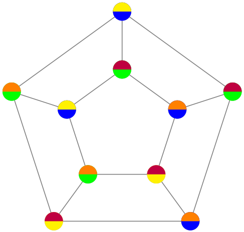

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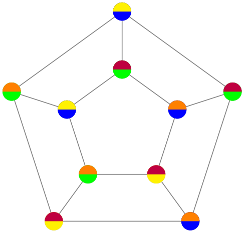

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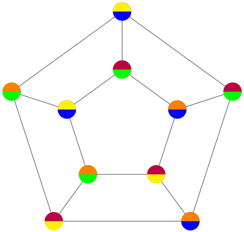

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$$\overset{\text{linearity}}{\Longrightarrow} \mathbb{E}\,|\mathbf{I}| \geq n/k$$

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Chromatic number of triangle-free graphs

$$\omega(G) \le \max_{\emptyset \ne H \subseteq G} \frac{|H|}{\alpha(H)} \le \chi_f(G) \le \chi(G) \le \mathrm{ch}(G) \le \Delta(G) + 1$$

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Theorem (Molloy 2019, cf. Johansson 1996+, cf. also Bernshteyn 2019)

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$\frac{n}{\alpha(G)} \lesssim \frac{\Delta}{\log \Delta}$ *for any n-vertex triangle-free G of maximum degree* Δ

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Chromatic number of triangle-free graphs

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Why?

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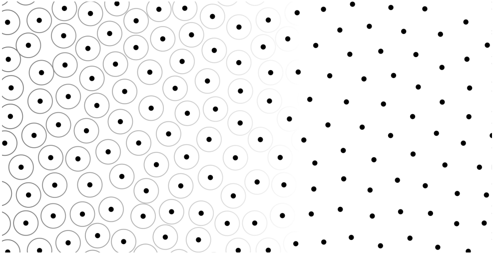
Why? Simple, conceptual, versatile, and more...

Hard-core model

A probability distribution over $\mathscr{I}(G)$ the set of independent sets of G



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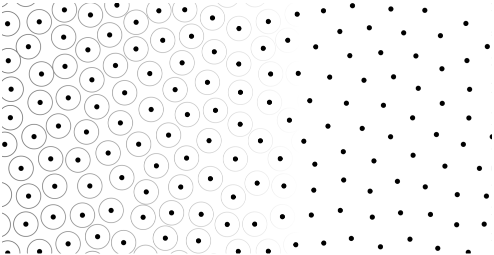
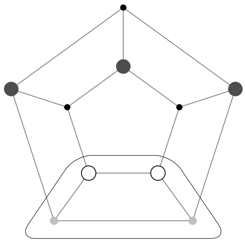


Hard-core model on G at fugacity $\lambda > 0$ is probability distribution over $\mathscr{I}(G)$ such that random $\mathbf{I}$ satisfies for all $S \in \mathscr{I}(G)$

$$\mathbb{P}(\mathbf{I} = S) = \frac{\lambda^{|S|}}{Z_G(\lambda)}, \quad \text{where } Z_G(\lambda) = \sum_{S \in \mathscr{I}(G)} \lambda^{|S|}$$

Spatial Markov property

For $S \in \mathscr{I}(G)$, call u occupied if $u \in S$ and call u uncovered if $N(u) \cap S = \emptyset$



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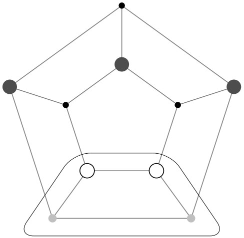

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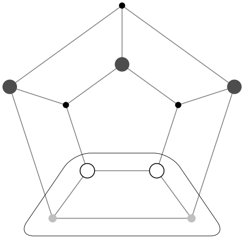

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Reveal $\mathbf{I} \setminus X$ and let $\mathbf{U}_X := X \setminus N(\mathbf{I} \setminus X)$ (the externally uncovered part)

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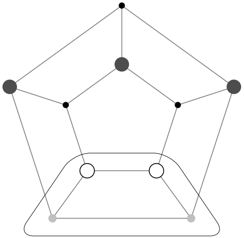

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Then $\mathbf{I} \cap X$ is hard-core on $G[\mathbf{U}_X]$ at fugacity λ

Local occupancy method

Distribution $\mathbf{I}$ on $\mathscr{I}(G)$ has local (a, b)-occupancy if for every vertex v

$$a \cdot \mathbb{P}(v \in \mathbf{I}) + b \cdot \mathbb{E}|N(v) \cap \mathbf{I}| \geq 1$$

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$\rightsquigarrow$ analysis to minimise $a + b \cdot \Delta$ $\rightsquigarrow$ $\qquad \chi_f(G) \lesssim \dfrac{\Delta}{\log \Delta}$

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Originally used by Molloy & Reed (2002) to prove fractional Reed's Conjecture

Idea: greedily add weight/colour to independent sets according to probability distribution induced by $\mathbf{I}$ on vertices not yet completely coloured, and iterate

One can think of it as "evening out" the distribution

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$$a \cdot \mathbb{P}(v \in \mathbf{I}) + b \cdot \mathbb{E}|N(v) \cap \mathbf{I}| \geq 1$$

A Hard-core model on any triangle-free G has local (a, b)-occupancy, for specific a, b depending on fugacity λ and maximum degree Δ

B If there is probability distribution $\mathbf{I}$ on $\mathscr{I}(G)$ with local (a, b)-occupancy, then $\chi_f(G) \leq a + b \cdot \Delta$

In A, Fact 1 and Fact 2 imply

$$a \cdot \mathbb{P}(v \in \mathbf{I}) + b \cdot \mathbb{E}|N(v) \cap \mathbf{I}| \geq \frac{b\lambda(\log((ea/b)\log(1+\lambda)))}{(1+\lambda)\log(1+\lambda)}$$

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and so from B it suffices to show

$$a + b \cdot \Delta \lesssim \frac{\Delta}{\log \Delta} \quad \text{subject to} \quad \frac{b\lambda(\log((ea/b)\log(1+\lambda)))}{(1+\lambda)\log(1+\lambda)} \geq 1 \rightsquigarrow$$

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C relies crucially on seminal proofs of Molloy (2019) and Bernshteyn (2019) combined with properties of the hard-core model

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Theorem (Johansson 1996+, cf. Alon 1996, Molloy 2019, Bonamy, Kelly, Nelson, Postle 2018+)

$\mathrm{ch}(G) = O\left(\log(r+1)\frac{\Delta}{\log \Delta}\right)$ *for any G of maximum degree Δ in which every neighbourhood is r-colourable*

Graphs with colourable neighbourhoods

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Theorem (Davies, Kang, Pirot, Sereni, in progress)
$\mathrm{ch}(G) \lesssim K(r)\cdot\frac{\Delta}{\log\Delta}$ *for any G of maximum degree Δ in which every neighbourhood is r-colourable, where $K(1)=1$ and $K(r)\sim\log r$ as $r\to\infty$*

Graphs with sparse neighbourhoods

Theorem (Alon, Krivelevich, Sudakov 1999, cf. Vu 2002 and Achlioptas, Iliopoulos, Sinclair 2019)

$\mathrm{ch}(G) = O\left(\dfrac{\Delta}{\log\Delta - \log\sqrt{T+1}}\right)$ *for any* G *of maximum degree* Δ *with each vertex in* $\le T$ *triangles,* $0 \le T = o(\Delta^2)$

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$\chi_f(G) = O\left(\frac{\delta^*}{\log \delta^*}\right)$ *for any triangle-free G with degeneracy δ^**

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Conjecture (Cames van Batenburg, de Joannis de Verclos, Kang, Pirot)

$\chi_f(G) \lesssim \sqrt{\frac{2n}{\log n}}$ *and* $\mathrm{ch}(G) = O\left(\sqrt{\frac{n}{\log n}}\right)$ *for any n-vertex triangle-free G*

Merci!