Failures to weaken list colouring through prescribed separation

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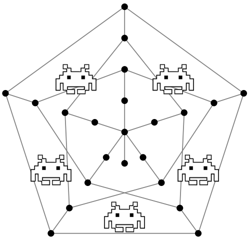
List colouring

Imagine *adversaries* to colouring

- that issue arbitrary lists of allowable colours per vertex
- but must give at least ℓ per list

What is least ℓ for which colouring is always possible? (Necessarily $\ell \geq \chi$)



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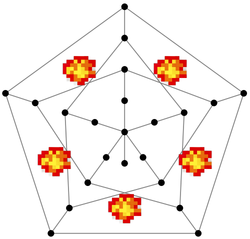

Called list chromatic number or choice number or choosability ch

List makes it “harder”

ch is not bounded by any function of χ

Theorem (Erdős, Rubin, Taylor 1980)

$\mathrm{ch}(K_{d,d}) \sim \log_2 d$ *(and* $\mathrm{ch}(K_{d+1}) = d + 1$*)*

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Still poorly understood

Conjecture (Alon & Krivelevich 1998)

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Question: Does $\mathrm{ch}_{\mathrm{sep}}$ grow in δ?

Problem: Almost-disjointness of lists is not monotone under edge-addition!

Ramsey-type question/solution?

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Related question: Does every graph of high minimum degree contain either

- a large clique or
- a large minimum degree bipartite induced subgraph?

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- True with $\Omega(\frac{\log \delta}{\log \log \delta})$ (Kwan, Letzter, Sudakov, Tran 2018+)

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Conjecture (Harris 2019)

Any triangle-free graph with degeneracy δ^* has fractional chromatic number $O(\frac{\delta^*}{\log \delta^*})$

Correspondence colouring

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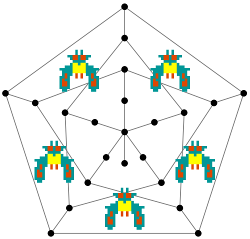

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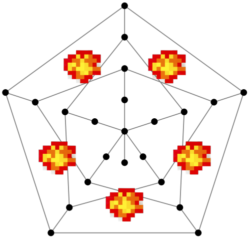
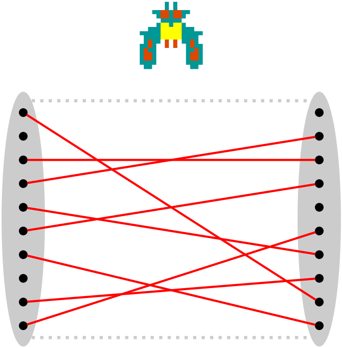


Called correspondence chromatic number or DP-chromatic number χ_{DP}



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Or rather, it is much more closely linked with density

Theorem (Bernshteyn 2016, cf. Kráľ, Pangrác, Voss 2005)

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Theorem (Bernshteyn 2019, cf. Molloy 2019)

$\chi_{\mathrm{DP}}(G) \lesssim \frac{\Delta}{\log\Delta}$ *for any triangle-free G of maximum degree Δ*

NB: This settles correspondence version of conjecture of Alon & Krivelevich

Correspondence and separation

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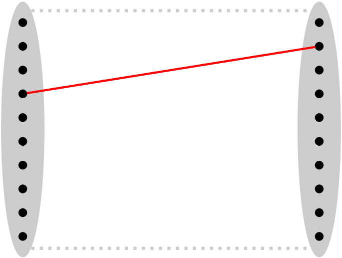

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$\mathrm{ch}_{\mathrm{DP1}}(G) \lesssim 2\sqrt{\Delta}$ *for any (multigraph) G of maximum degree Δ*

NB: $\mathrm{ch}_{\mathrm{DP1}}(G) \gtrsim \sqrt{\Delta}$ for a 2-vertex G of multiplicity Δ (!)

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An analogue of Heawood's Formula (roughly of form $\chi = O(\sqrt{g+1})$)

Theorem (Dvořák, Esperet, Kang, Ozeki 2018+)
$\mathrm{ch}_{\mathrm{DP1}}(G) = O((g+1)^{1/4} \log(g+2))$ *for any simple G embeddable on a surface of Euler genus g*

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Given simple H and a vertex partition $L : [n] \to \binom{V(H)}{\ell}$ *satisfying*

- $\frac{1}{\ell}\sum_{i\in L(v)} \deg(i) \le D$ *for every* $v \in [n]$
- $\ell \gtrsim 4D$,

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So closely related to Haxell 2001 (with instead $\deg(i) \leq D$ and $2D$) and

Theorem (Bollobás, Erdős, Szemerédi 1975, cf. Szabó & Tardos 2006)

$\text{ch}_{\text{DP1}}(G) \gtrsim \sqrt{2\Delta}$ *for some multigraph G of maximum degree Δ*

and also to List Colouring Constants...

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