New investigations into the structure of locally sparse graphs

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*With Alon, Cambie, Cames van Batenburg, Davies, Esperet, de Joannis de Verclos, Pirot, Sereni, Thomassé. Support from Nuffic/PHC, ANR, FWB, NWO, ERC, BSF, NSF, Simons grants.

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- Mantel (1907), Turán (1941)
- Ramsey (1930), Erdős & Szekeres (1935)
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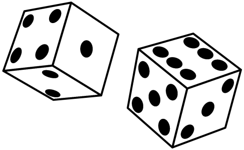
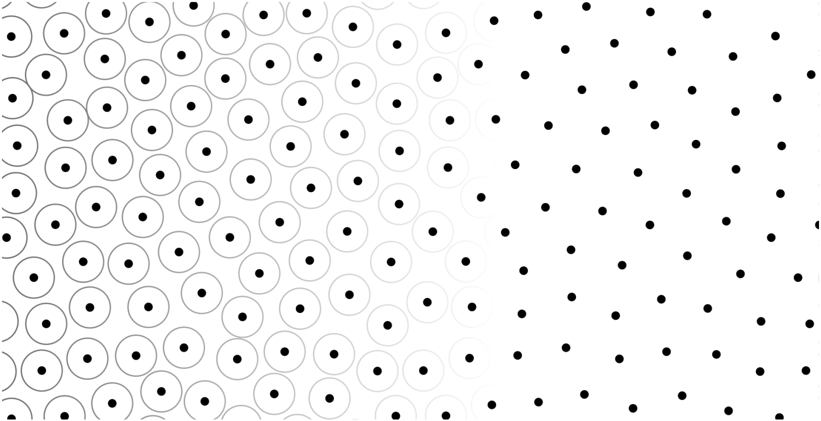
Elegant, modern challenges!



If a random object has desired property with positive probability, then there exists *at least one* object with that property

Hard-core model†

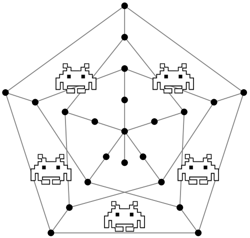


† More fully, the lattice gas with hard-core self-repulsion and nearest-neighbour exclusion. Picture credit: Wikipedia/Grap-wh

List colouring

Given a graph G, imagine *enemies* to you properly colouring it

- that give lists of allowed colours per vertex
- but must allow at least ℓ colours per vertex



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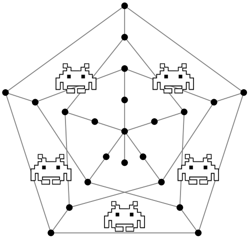

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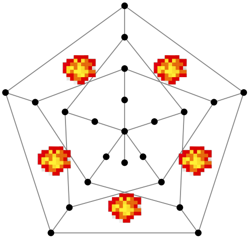

Called list chromatic number or choosability $\mathrm{ch}(G)$ of G
(Necessarily $\chi(G) \le \mathrm{ch}(G) \le \Delta(G) + 1$)

Lists make it “harder”

ch is not bounded by any function of chromatic number χ

Theorem (Erdős, Rubin, Taylor 1980)

$\mathrm{ch}(K_{d,d}) \sim \log_2 d$ *(and* $\mathrm{ch}(K_{d+1}) = d + 1$*)*

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To date(!): $\mathrm{ch}(G) \lesssim \dfrac{\Delta}{\log \Delta}$ (Molloy 2019, cf. Alon, Cambie, Kang 2020+)

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[‡] A very recent simplification by Glock 2020+

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$\mathrm{BID}(G) \geq \dfrac{\delta}{2\chi}$ *for any G with minimum degree δ and chromatic number χ*

A sequence with independent set and colouring

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In general, all can be strict [§]

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We focus on triangle-free. . .

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Off-diagonal Ramsey numbers[¶]

[¶]Picture credit: Soifer 2009

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$R(3,k)$: largest n such that there is red/blue-edge-coloured K_{n-1} with no red triangle and no blue K_k

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i.e. Independence number of triangle-free graphs

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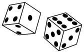
Proof with ? ‖

Good probability distribution over the set $\mathscr{I}(G)$ of independent sets of G for proving that random $\mathbf{I}$ has

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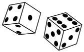
Proof with ? ‖

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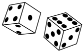
Proof with ? [‖]

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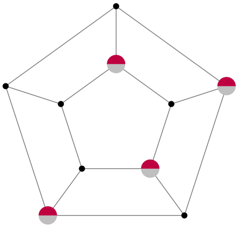
[‖] Yes, cf. Davies, Jenssen, Perkins, Roberts 2018. . .

Fractional chromatic number



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$\chi_f \geq n/\alpha = 5/2$

fractional vertex-colouring : allow "fractions" of independent sets

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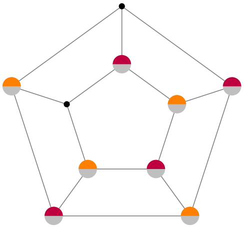

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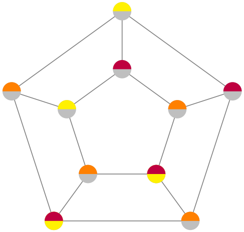

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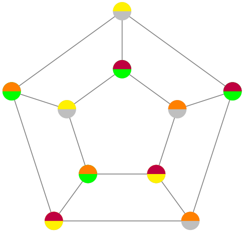

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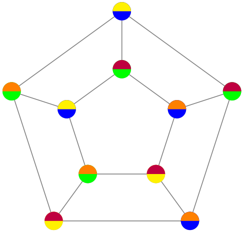

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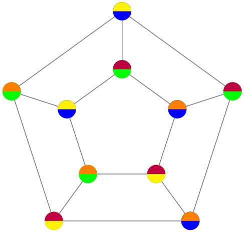

$\alpha = 4$, $\chi_f = 5/2$

$\chi_f \geq n/\alpha = 5/2$

fractional vertex-colouring : allow "fractions" of independent sets

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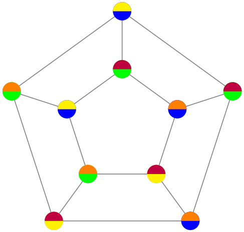

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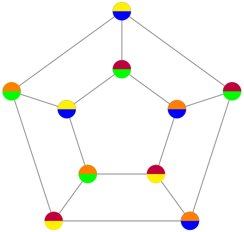

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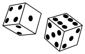
$$\overset{\text{linearity}}{\Longrightarrow} \mathbb{E}\,|\mathrm{I}| \geq n/k$$

...



Chromatic number of triangle-free graphs

$$\omega(G) \le \max_{\emptyset \ne H \subseteq G} \frac{|H|}{\alpha(H)} \le \chi_f(G) \le \chi(G) \le \mathrm{ch}(G) \le \Delta(G) + 1$$

Theorem (Shearer 1983, cf. Ajtai, Komlós, Szemerédi 1980/1)

$\dfrac{n}{\alpha(G)} \lesssim \dfrac{\Delta}{\log \Delta}$ *for any n-vertex triangle-free G of maximum degree* Δ

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Why?

$$\omega(G) \le \max_{\emptyset \ne H \subseteq G} \frac{|H|}{\alpha(H)} \le \chi_f(G) \le \chi(G) \le \mathrm{ch}(G) \le \Delta(G) + 1$$

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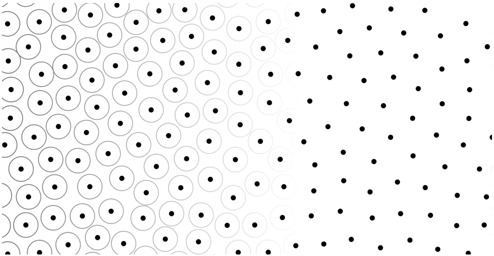
Why? Simple, conceptual, versatile, and more...

A probability distribution over $\mathscr{I}(G)$ the set of independent sets of G



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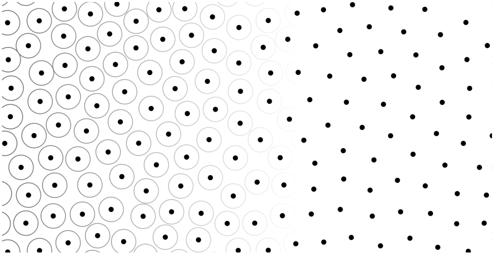
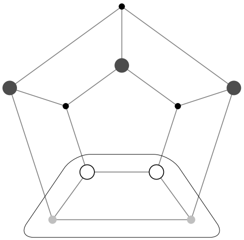


Hard-core model on G at fugacity $\lambda > 0$ is probability distribution over $\mathscr{I}(G)$ such that random I satisfies for all $S \in \mathscr{I}(G)$

$$\mathbb{P}(\mathsf{I} = S) = \frac{\lambda^{|S|}}{Z_G(\lambda)}, \quad \text{where } Z_G(\lambda) = \sum_{S \in \mathscr{I}(G)} \lambda^{|S|}$$

Spatial Markov property

For $S \in \mathscr{I}(G)$, call u occupied if $u \in S$ and call u uncovered if $N(u) \cap S = \emptyset$



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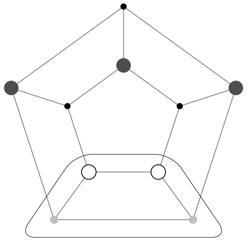

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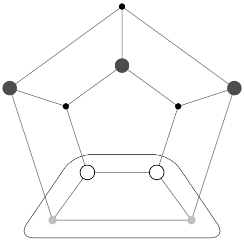

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Reveal $\mathrm{I} \setminus X$ and let $\mathrm{U}_X := X \setminus N(\mathrm{I} \setminus X)$ (the externally uncovered part)

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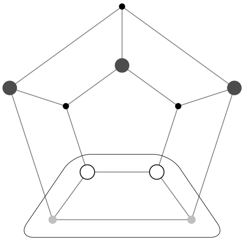

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Then $\mathrm{I} \cap X$ is hard-core on $G[\mathrm{U}_X]$ at fugacity λ

Local occupancy method

Distribution $\mathbf{I}$ on $\mathscr{I}(G)$ has local (a, b)-occupancy if for every vertex v

$$a \cdot \mathbb{P}(v \in \mathbf{I}) + b \cdot \mathbb{E}|N(v) \cap \mathbf{I}| \geq 1$$

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$\rightsquigarrow$ analysis to minimise $a + b \cdot \Delta$ $\rightsquigarrow$ $\chi_f(G) \lesssim \dfrac{\Delta}{\log \Delta}$

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Idea: greedily add weight/colour to independent sets according to probability distribution induced by I on vertices not yet completely coloured, and iterate

One can think of it as “evening out” the distribution

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(needs triangle-free!)

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$\mathbb{E}|N(v)\cap I|$

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$$\geq \min_{\iota \in \mathbb{R}^+} \frac{\lambda}{1+\lambda}\left(a \cdot (1+\lambda)^{-\iota} + b \cdot \iota\right)$$

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$$\implies a \cdot \mathbb{P}(v \in \mathrm{I}) + b \cdot \mathbb{E}|N(v) \cap \mathrm{I}| \geq \frac{\lambda}{1+\lambda}\left(a \cdot (1+\lambda)^{-\mathbb{E}\mathrm{J}} + b \cdot \mathbb{E}\mathrm{J}\right)$$

$$\geq \min_{\iota \in \mathbb{R}^+} \frac{\lambda}{1+\lambda}\left(a \cdot (1+\lambda)^{-\iota} + b \cdot \iota\right) \overset{\text{convexity}}{\geq} \frac{b\lambda(\log((ea/b)\log(1+\lambda)))}{(1+\lambda)\log(1+\lambda)}$$

Local occupancy method

Fact 1 $\mathbb{P}(v \in I \mid v \text{ uncovered}) = \dfrac{\lambda}{1+\lambda}$

Fact 2 $\mathbb{P}(v \text{ uncovered} \mid v \text{ has } j \text{ uncovered neighbours}) = \dfrac{1}{(1+\lambda)^j}$

$$\mathbb{P}(v \in I) = \mathbb{P}(v \in I \text{ and } v \text{ uncovered}) \overset{\text{Fact 1}}{=} \frac{\lambda}{1+\lambda}\mathbb{P}(v \text{ uncovered})$$

$$\overset{\text{Fact 2}}{=} \frac{\lambda}{1+\lambda}\sum_j \frac{\mathbb{P}(v \text{ has } j \text{ uncovered neighbours})}{(1+\lambda)^j}$$

$$= \frac{\lambda}{1+\lambda}\mathbb{E}(1+\lambda)^{-J} \overset{\text{Jensen's}}{\geq} \frac{\lambda}{1+\lambda}(1+\lambda)^{-\mathbb{E}J}$$

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Local occupancy method

Distribution I on $\mathscr{I}(G)$ has local (a, b)-occupancy if for every vertex v

$$a \cdot \mathbb{P}(v \in \mathsf{I}) + b \cdot \mathbb{E}|N(v) \cap \mathsf{I}| \geq 1$$

A Hard-core model on any triangle-free G has local (a, b)-occupancy, for specific a, b depending on fugacity λ and maximum degree Δ

B If there is probability distribution I on $\mathscr{I}(G)$ with local (a, b)-occupancy, then $\chi_f(G) \leq a + b \cdot \Delta$

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and so from B it suffices to show

$$a + b \cdot \Delta \lesssim \frac{\Delta}{\log \Delta} \quad \text{subject to} \quad \frac{b\lambda(\log((ea/b)\log(1+\lambda)))}{(1+\lambda)\log(1+\lambda)} \geq 1 \rightsquigarrow$$

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**i.e. satisfying some structural sparsity condition for every neighbourhood subgraph

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Hard-core model on G has local (a,b)-occupancy if for every vertex v and every induced subgraph F of the neighbourhood subgraph $G[N(v)]$.

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C', an algorithmic version of C (under additional conditions), merges the hard-core model into framework of Achlioptas, Iliopoulous, Sinclair (2019)

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Theorem (Johansson 1996+, cf. Alon 1996, Molloy 2019, Bonamy, Kelly, Nelson, Postle 2018+)

$\mathrm{ch}(G) = O\left(\log(r+1)\frac{\Delta}{\log\Delta}\right)$ *for any G of maximum degree Δ in which every neighbourhood is r-colourable*

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Theorem (Davies, Kang, Pirot, Sereni 2020+)

$\mathrm{ch}(G) \lesssim K(r) \cdot \frac{\Delta}{\log \Delta}$ *for any G of maximum degree Δ in which every neighbourhood is r-colourable, where $K(1) = 1$ and $K(r) \sim \log r$ as $r \to \infty$*

Graphs with colourable neighbourhoods

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NB: $r = 1$ corresponds to Molloy's and $r = \Delta + 1$ corresponds to trivial bound

Theorem (Kim 1995)

$\mathrm{ch}(G) \lesssim \dfrac{\Delta}{\log \Delta}$ *for any* G *of girth* 5 *and maximum degree* Δ

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NB: $k = \Delta^{o(1)}$ includes Kim's and Molloy's

Theorem (Alon, Krivelevich, Sudakov 1999, cf. Vu 2002 and Achlioptas, Iliopoulos, Sinclair 2019)

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with each vertex in $\leq T$ triangles, $1/2 \leq T \leq \binom{\Delta}{2}$

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Barriers

"89 years of $R(3, k)$"

Theorem (Shearer 1983)

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Is there a polynomial-time algorithm that with high probability outputs an independent set of $G_{n,1/2}$ *of size* $(1+\varepsilon)\log_2 n$*?*

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Known: $\chi_f(G) \lesssim \sqrt{\dfrac{4n}{\log n}}$, $\chi(G) \lesssim \sqrt{\dfrac{8n}{\log n}}$, $\mathrm{ch}(G) = O(\sqrt{n})$

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NB: Conjecture on "fractional colouring with local demands" implies the first (Kelly & Postle 2018+)

Conjecture (Esperet, Kang, Thomassé 2019)

$\mathsf{BID}(G) = \Omega(\log \delta)$ *for any triangle-free* G *of minimum degree* δ

Theorem (Esperet, Kang, Thomassé 2019)

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Conjecture (Harris 2019)

$\chi_f(G) = O\left(\dfrac{\delta^*}{\log \delta^*}\right)$ *for any triangle-free* G *with degeneracy* δ^*

NB: False for $\chi(G)$ (Alon, Krivelevich, Sudakov 1999)

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Question (Blumenthal, Lidický, Martin, Norin, Pfender, Volec 2018+)

$\chi_f(G) = O(\rho)$ *for any triangle-free G where $\rho = \max\limits_{\emptyset \neq H \subseteq G} \dfrac{|H|}{\alpha(H)}$?*

NB: False without triangle-free (BLMNPV 2018+)

Conjecture (Alon & Krivelevich 1998)

$\mathrm{ch}(G) \lesssim \log_2 \Delta$ *for any bipartite G of maximum degree Δ*

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Recent: one side $\log \Delta$, other side $\sim \Delta / \log \Delta$ (Alon, Cambie, Kang 2020+)

Gràcies!