Colouring graphs of bounded local density

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*With Davies, Hurley, de Joannis de Verclos, Pirot, Sereni. Support from NWO.

Main themes

Vizing, "Some unsolved problems in graph theory" (1968):

Можно ли для любого наперед заданного натурального числа $k \geqslant 2$ построить граф со сколь угодно большим обхватом и с хроматическим числом k? Можно. Это доказал П. Эрдёш [39], основываясь на мощностных соображениях. Удивительно, что до сих пор нет конструктивного доказательства этого факта. В [40] указан способ построения графов с любым хроматическим числом без циклов длины $\leqslant 7$. Это лучшее, что мы имеем на сегодняшний день.

Если $\sigma(L)$ — максимальная степень вершины графа L, то, очевидно, $\gamma(L) \leqslant \sigma(L)+1$. В 1941 г. Р. Брукс [41] доказал, что при $\sigma(L) \geqslant 3$ и $\omega(L) \leqslant \sigma(L)$ справедлива оценка $\gamma(L) \leqslant \sigma(L)$. Дальнейшие исследования можно проводить, учитывая более точно соотношения между σ и ω. Пожалуй, следует начать с оценки хроматического числа графа без треугольников ($\omega = 2$) с данной максимальной степенью вершины.

Vizing, "Some unsolved problems in graph theory" (1968):

Given a natural number $k \geqslant 2$, is it possible to construct a graph with arbitrarily large compass and with chromatic number k? Erdős [39] has proved this; his proof is based on counting arguments. It is astonishing that no constructive proof for this fact has yet been given. In [40] a method is given of constructing graphs of arbitrary chromatic number without having cycles of length $\leqslant 7$. This is the best we have at the present time.

If $\sigma(L)$ is the maximum degree of a vertex in a graph L, it is clear that $\gamma(L) \leqslant \sigma(L) + 1$. Brooks [41] showed in 1941 that $\gamma(L) \leqslant \sigma(L)$ whenever $\sigma(L) \geqslant 3$ and $\omega(L) \leqslant \sigma(L)$. Further investigations could be conducted, taking into account a more exact relation between σ and ω. Perhaps one should start with estimates of the chromatic number of a graph without triangles ($\omega = 2$) and with given maximal degree for vertices.

Asymptotically, what global graph structure if no edge is induced in any neighbourhood?

Local to global

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Distinguished origins:

- Mantel (1907), Turán (1941)
- Ramsey (1930), Erdős & Szekeres (1935)
- Zykov (1949), Ungar & Descartes (1954), Mycielski (1955), Erdős (1959)

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Elegant, modern challenges!

Sequence of structure

$\omega(G)$

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$$\omega(G) \leq \max_{\emptyset \neq H \subseteq G} \frac{|H|}{\alpha(H)}$$

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$$\omega(G) \leq \max_{\emptyset \neq H \subseteq G} \frac{|H|}{\alpha(H)} \leq \chi_f(G) \leq \chi(G) \leq \mathrm{ch}(G) \leq \Delta(G) + 1$$

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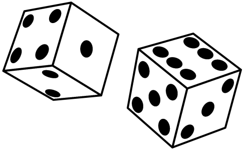
In general, each inequality can be strict[†]

[†] On strictness of the second, see Blumenthal, Lidický, Martin, Norin, Pfender, Volec (2018+), and Dvořák, Ossona de Mendez, Wu (2020); nice open question in the triangle-free case.



If random object has property with positive probability,
then there exists *at least one* object with that property

Semi-random method (or Rödl nibble)

Iterated application of probabilistic method to create structured object

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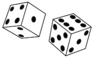

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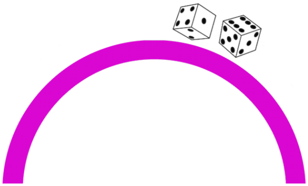

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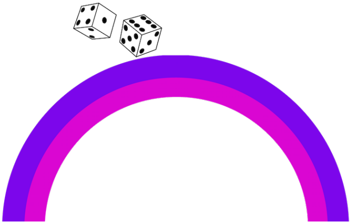

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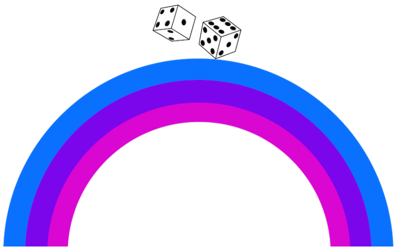

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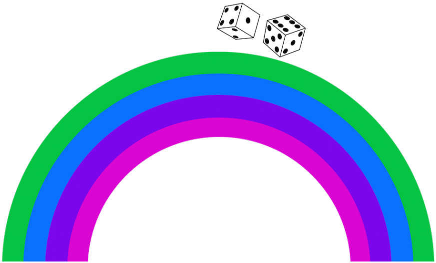

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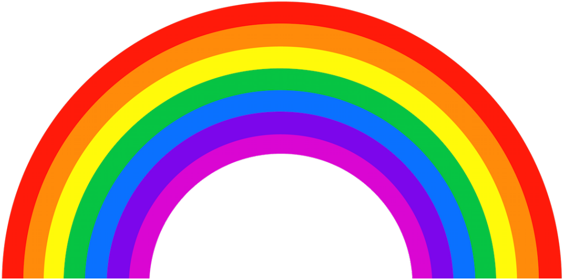

Triangle-free graphs

Off-diagonal Ramsey numbers[‡]

[‡]Picture credit: Soifer 2009

Off-diagonal Ramsey numbers

$R(3,k)$: largest n such that there is red/blue-edge-coloured K_{n-1}
with no red triangle and no blue K_k

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Theorem (Bollobás 1981)

$$\max_{\emptyset \neq H \subseteq G} \frac{|H|}{\alpha(H)} \geq \frac{\Delta}{2 \log \Delta} \textit{ and } G \textit{ has arbitrarily large girth whp for } G \sim G_{n,\Delta}$$

Chromatic number of triangle-free graphs,

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Theorem (Johansson 1996+)

$\mathrm{ch}(G) = O\left(\frac{\Delta}{\log \Delta}\right)$ *for any triangle-free G of maximum degree Δ*

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$\chi_f(G) \lesssim \frac{\Delta}{\log \Delta}$ *for any triangle-free G of maximum degree Δ* $\rightsquigarrow$

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Two more Ramsey-type problems

ω, Δ, χ conjecture

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§ Picture credit: Wikipedia/David Eppstein

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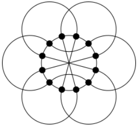

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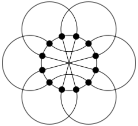

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Bound holds for:

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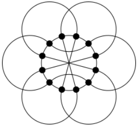

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NB: ε_2 may not be larger than 0.5

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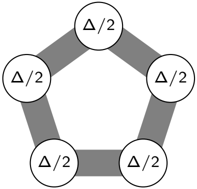

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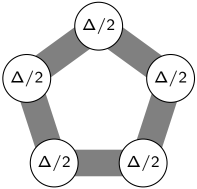

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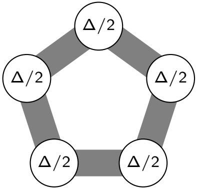

So far only confirmed for $\Delta \leq 3$
(Andersen 1992, Horák, He, Trotter 1993)

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NB: $\varepsilon_3 (\geq \varepsilon_4)$ may not be larger than 0.75

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$\eta < 1/\binom{\Delta}{2}$ corresponds to triangle-free, $\eta = 1$ corresponds to general case

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Best chromatic number bounds in graphs of local density at most η*?*

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NB: Shearer (1983) established the analogous bound for $\alpha(G)$

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Write $\sigma = 1 - \eta$ and consider σ-sparse graphs

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Improved lower bounds on ε yield improvements for Reed's and Erdős-Nešetřil

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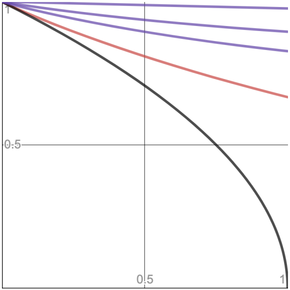

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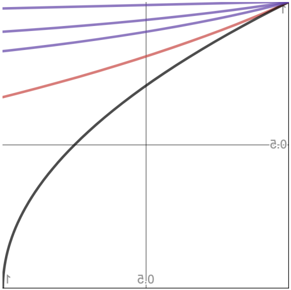

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Given G and palette $[M] = \{1, \ldots, M\}$:

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  allows better iteration of 1 and 2

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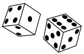
Lemma

For $\iota > 0$, generating $\boldsymbol{I}$ as above from a σ-sparse Δ-regular graph satisfies for large enough Δ and γ, and for every v

$$\left| \mathbb{P}(v \in \boldsymbol{I}) - \frac{1 - e^{-\gamma}}{\Delta} \right| \leq \frac{2}{\Delta^2},$$

$$\frac{\mathbb{P}(N(v) \cap \boldsymbol{I} \neq \emptyset)}{\mathbb{E}|N(v) \cap \boldsymbol{I}|} \leq 1 - \frac{\sigma}{2} + \frac{\sigma^{3/2}}{6} + \iota$$



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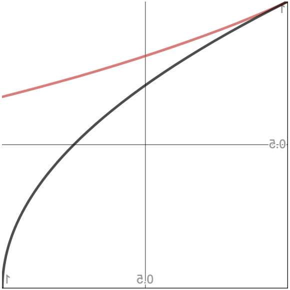

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