Chromatic structure of bipartite graphs

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AMS Spring Western Sectional Meeting
"Enumerative and Extremal Problems in Chromatic Graph Theory"

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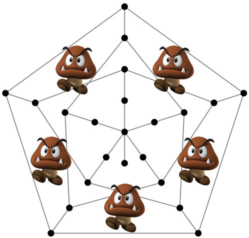
*With Alon, Cambie, Cames van Batenburg, Davies. Support from NWO.

Chromatic mystery when $\chi = 2$??

List colouring

Imagine <u>adversaries</u> to colouring

- that issue arbitrary lists of allowable colours per vertex
- but must give at least k per list



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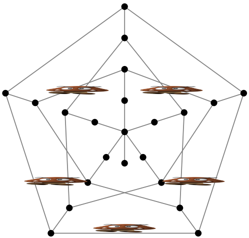
What is least k for which colouring is always possible? (Necessarily $k \geq \chi$.)



List colouring (formally)

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A list-assignment is some $L : V(G) \to 2^{\mathbb{Z}^+}$;
a k-list-assignment is some $L : V(G) \to \binom{\mathbb{Z}^+}{k}$.

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An <u>L-colouring</u> is some $c : V(G) \to \mathbb{Z}^+$ with $c(v) \in L(v)$ for every $v \in V(G)$.

G is <u>k-choosable</u> if there is a proper L-colouring for any k-list-assignment L.

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The list chromatic number $\mathrm{ch}(G)$ is least k such that G is k-choosable.

Introduced independently by Vizing (1976) and Erdős, Rubin, Taylor (1980).

$\chi = 2$ WHILE $\mathsf{ch} > k$

Theorem (Erdős, Rubin, Taylor 1980)

There are bipartite graphs with arbitrarily large list chromatic number.

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X intersects every member of $\mathcal{F}$.

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Such an X does not exist for $\mathcal{F} = \binom{[2k-1]}{k}$.

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Such an X does not exist for $\mathcal{F} = \binom{[2k-1]}{k}$. $\implies \mathrm{ch}\left(K_{\binom{[2k-1]}{k},\binom{[2k-1]}{k}}\right) > k$. □

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OPEN QUESTION

What is the minimum number N(2,k) of nodes in a graph G which is 2-colorable but not k-choosable?

F. Bernstein. Zur theorie der trigonometrische Reihen. Leipz. Ber. 60: 325-328, 1908.

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Note $M(k) \leq \binom{2k-1}{k}$.

Theorem (Erdős 1963/4, Radhakrishnan & Srinivasan 2000)
$M(k) = \Omega(2^k\sqrt{k/\log k})$ *and* $M(k) = O(k^2 2^k)$.

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Colour from $L(v) \cap X$ if $v \in A$, from $L(v) \setminus X$ if $v \in B$. □

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[†] Alon 2000, Saxton & Thomason 2015

Bipartite choice number and maximum degree

Erdős, Rubin, Taylor (1980): $\mathrm{ch}(K_{n,n}) \sim \log_2 n$.

Conjecture (Alon & Krivelevich 1998)

For any bipartite G of maximum degree Δ, $\mathrm{ch}(G) = O(\log \Delta)$.

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If $k \ge (1 + \varepsilon)\Delta / \log \Delta$, then w.p.p. there's spare colour for every $v \in A$. □

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For any triangle-free G of maximum degree Δ, $\mathrm{ch}(G) \lesssim \Delta / \log \Delta$.

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NB2: For bipartite, a persistent **exponential** gap in our knowledge!

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Let's first focus on first.

Halfway??

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For any bipartite G of maximum degree Δ, $\mathrm{ch}(G) = O(\log \Delta)$.

Theorem (Alon, Cambie, Kang 2021)

Any bipartite G of (large enough) maximum degree Δ with parts A and B is (k_A, k_B)-choosable for $k_A = (1+\varepsilon)\Delta/\log \Delta$ and $k_B = \log \Delta$.

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Lemma

Any bipartite G with parts A and B is (k_A, k_B)-choosable if

$$e\Delta_A(\Delta_B - 1)(1-(1-1/k_B)^{k_B\Delta_A/k_A})^{k_A} \leq 1.$$

Proof of theorem.

$\Delta_A = \Delta_B = \Delta$, $k_A = (1+\varepsilon)\Delta/\log\Delta$, $k_B = \log\Delta$ satisfies the condition.
(In fact, Δ_B superpolynomial in Δ still suffices!) □

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The individual chance that $c \in L(v)$ is blocked is $1 - (1 - 1/k_B)^{x(c)}$, where $x(c)$ counts the times $L(w) \ni c$ for some $w \in N(v)$.

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Any bipartite G with parts A and B is (k_A, k_B)-choosable if

$$e\Delta_A(\Delta_B - 1)(1 - (1 - 1/k_B)^{k_B\Delta_A/k_A})^{k_A} \leq 1.$$

Proof sketch.

For each $w \in B$, colour with independent uniform choice from $L(w)$.
For each $v \in A$, what is the chance all colours in $L(v)$ are blocked?
The individual chance that $c \in L(v)$ is blocked is $1 - (1 - 1/k_B)^{x(c)}$,
where $x(c)$ counts the times $L(w) \ni c$ for some $w \in N(v)$.
By checking negative correlation and Jensen's, the chance that $L(v)$ blocked is

$$\leq \prod_{c\in L(v)} (1 - (1 - 1/k_B)^{x(c)}) = \exp\left(k_A \frac{\sum_{c\in L(v)} \log(1 - (1 - 1/k_B)^{x(c)})}{\sum_{c\in L(v)} 1}\right)$$

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Coupon collector again

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The lemma follows by checking mutual independence for the local lemma. □

Exploring a 4-parameter space

1. Prescribe separate list sizes: k_A in part A, k_B in part B.
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What happens for complete bipartite graphs?

For $2 \le k_1, k_2 \le \ell$, a family $\mathcal{F} \subseteq \binom{[\ell]}{k_2}$ has Property $\mathrm{A}(k_1, k_2, \ell)$ if there is some set $X \in \binom{[\ell]}{k_1}$ that intersects every member of $\mathcal{F}$.

Hypergraph Turán numbers

For $2 \leq k_1, k_2 \leq \ell$, a family $\mathcal{F} \subseteq \binom{[\ell]}{k_2}$ has Property A(k_1, k_2, ℓ) if there is some set $X \in \binom{[\ell]}{k_1}$ that intersects every member of $\mathcal{F}$.

$\overline{M}(k_1, k_2, \ell) := \min\{|\mathcal{F}| \mid \mathcal{F} \subseteq \binom{[\ell]}{k_2}$ and $\mathcal{F}$ doesn't have Property A$(k_1, k_2, \ell)\}$.

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Among other things, we know

Theorem (Erdős & Spencer 1974)

For $k_1, k_2 \geq 2$ and $\ell \geq k_1 + k_2$,

$$\frac{\ell!(\ell - k_1 - k_2)!}{(\ell - k_2)!(\ell - k_1)!} \leq \overline{M}(k_1, k_2, \ell) < \frac{\ell!(\ell - k_1 - k_2)!}{(\ell - k_2)!(\ell - k_1)!} \log \binom{\ell}{k_1}.$$

Complete characterisation?

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For $k_A, k_B \geq 2$ and $\ell = k_1 + k_2 + 1$, $K_{\overline{M}(k_1,k_A,\ell),\overline{M}(k_2,k_B,\ell)}$ isn't (k_A, k_B)-choosable.

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But $|X_A| + |X_B| \geq k_1 + k_2 + 2 > \ell$, contradicting that colouring is proper. □

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Not far from extremal for most of 4-parameter space for complete bipartite, and perhaps even for general bipartite graphs. . .

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2. *For some* $C > 1$, $k_A \geq C \log \Delta_B$ *and* $k_B \geq C \log \Delta_A$.
3. $\Delta_A = \Delta_B = \Delta$, *and, for some* $C > 0$,

$$k_B \geq C(\Delta/\log \Delta)^{1/k_A} \log \Delta \quad \text{or} \quad k_A \geq C(\Delta/\log \Delta)^{1/k_B} \log \Delta.$$

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This is well-defined and $\mathrm{ch}^\star \geq \mathrm{ch}$.

Conjecture (Cambie, Cames van Batenburg, Davies, Kang 2021+)
$\mathrm{ch}^\star(G) = O(\mathrm{ch}(G))$.

NB: We know $\mathrm{ch}^\star(G) = 2^{O(\mathrm{ch}(G))}$.

Bipartite list packing

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Conjecture (Cambie, Cames van Batenburg, Davies, Kang 2021+)
For any triangle-free G of maximum degree Δ, $\text{ch}^\star(G) = O(\Delta/\log\Delta)$.

Literature

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Questions?