From local sparsity to global

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Dutch Day of Combinatorics

Eindhoven 12/5/2022

*With Alon, Cambie, Davies, Hurley, de Joannis de Verclos, Pirot, Sereni. Support from NWO.

Some classics

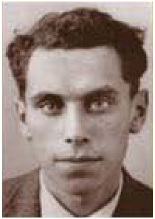
Brooks' theorem

Theorem (Brooks 1941)

$\chi(G) \leq \Delta(G)$ *unless* $G = K_{\Delta(G)+1}$ *or* G *is an odd cycle.*

Off-diagonal Ramsey numbers†

Ramsey (1930), Erdős & Szekeres (1935)



$R(3, k)$: smallest n such that any red/blue-edge-coloured K_n with no red K_3 must contain a blue K_k

† Picture credit: Soifer 2009

Vizing's problem

Vizing, "Some unsolved problems in graph theory" (1968):

Можно ли для любого наперед заданного натурального числа $k \geqslant 2$ построить граф со сколь угодно большим обхватом и с хроматическим числом k? Можно. Это доказал П. Эрдёш [39], основываясь на мощностных соображениях. Удивительно, что до сих пор нет конструктивного доказательства этого факта. В [40] указан способ построения графов с любым хроматическим числом без циклов длины $\leqslant 7$. Это лучшее, что мы имеем на сегодняшний день.

Если $\sigma(L)$ — максимальная степень вершины графа L, то, очевидно, $\gamma(L) \leqslant \sigma(L)+1$. В 1941 г. Р. Брукс [41] доказал, что при $\sigma(L) \geqslant 3$ и $\omega(L) \leqslant \sigma(L)$ справедлива оценка $\gamma(L) \leqslant \sigma(L)$. Дальнейшие исследования можно проводить, учитывая более точно соотношения между σ и ω. Пожалуй, следует начать с оценки хроматического числа графа без треугольников $(\omega = 2)$ с данной максимальной степенью вершины.

Vizing's problem

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Given a natural number $k \geqslant 2$, is it possible to construct a graph with arbitrarily large compass and with chromatic number k? Erdős [39] has proved this; his proof is based on counting arguments. It is astonishing that no constructive proof for this fact has yet been given. In [40] a method is given of constructing graphs of arbitrary chromatic number without having cycles of length $\leqslant 7$. This is the best we have at the present time.

If $\sigma(L)$ is the maximum degree of a vertex in a graph L, it is clear that $\gamma(L) \leqslant \sigma(L) + 1$. Brooks [41] showed in 1941 that $\gamma(L) \leqslant \sigma(L)$ whenever $\sigma(L) \geqslant 3$ and $\omega(L) \leqslant \sigma(L)$. Further investigations could be conducted, taking into account a more exact relation between σ and ω. Perhaps one should start with estimates of the chromatic number of a graph without triangles ($\omega = 2$) and with given maximal degree for vertices.

A common landscape

Measures of sparsity/structure

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where $\rho = \max_{\emptyset \neq H \subseteq G} \frac{|H|}{\alpha(H)}$

(upper bounds on ρ are like lower bounds on α)

Local to global

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- (Reed's) $\chi \leq \left\lceil \frac{1}{2}(\omega + \Delta + 1) \right\rceil$?

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- (Ajtai–Erdős–Komlós–Szemerédi) $\omega \le k \implies \rho \le C_k \dfrac{\Delta}{\log \Delta}$?

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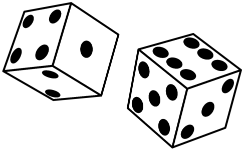
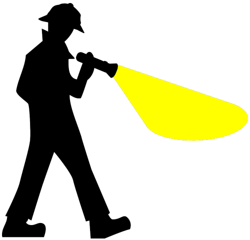
- (Alon–Krivelevich) $\chi \leq 2 \implies \chi_\ell \leq C \log \Delta$?



If random object has property with positive probability,
then there exists *at least one* object with that property

Random links

Stochastic local search



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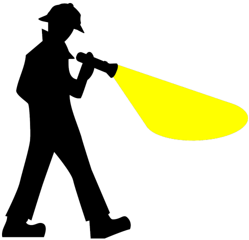
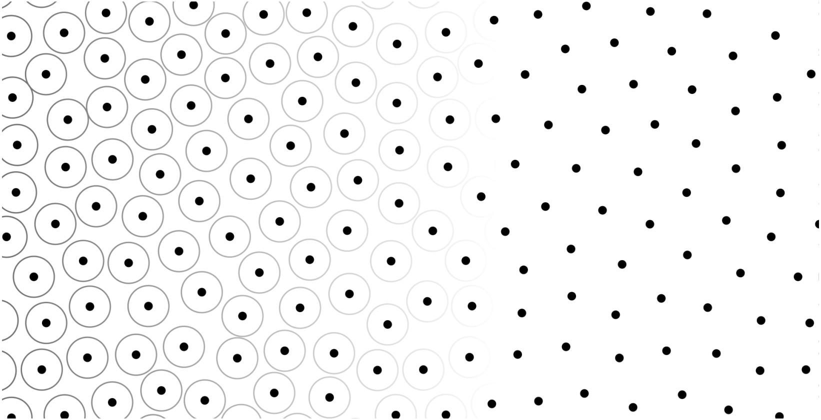


Suppose quest for some "flawless" combinatorial object uses a stochastic procedure with only local changes at each step

What are the theoretical limits of such algorithms?

Hard-core model‡



‡More fully, the lattice gas with hard-core self-repulsion and nearest-neighbour exclusion. Picture credit: Wikipedia/Grap-wh

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Iterated application of probabilistic method to create structured object

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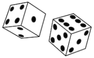

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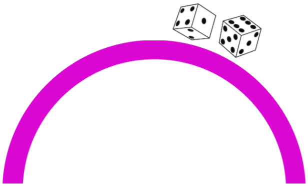

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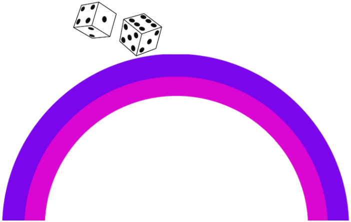

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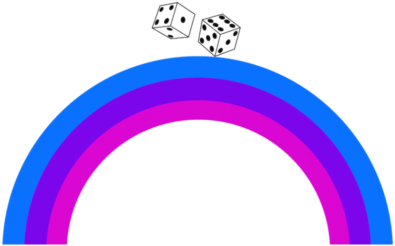

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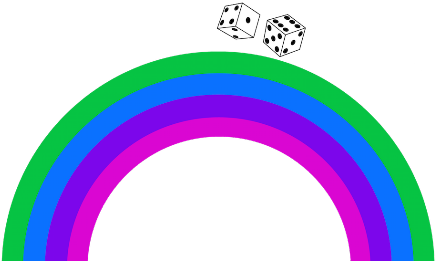

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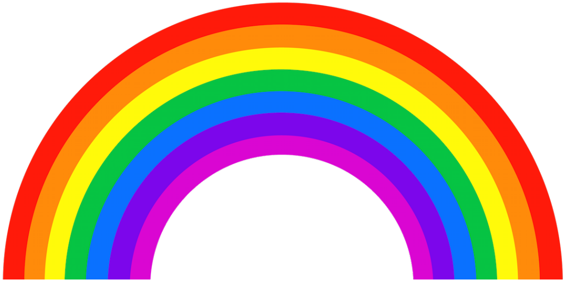

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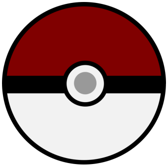

At each turn you get a random pokémon (card)

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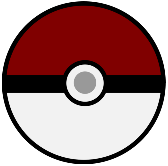

At each turn you get a random pokémon (card)

How long until you have at least one of each type?

Triangle-free graphs

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$\alpha(G) \gtrsim \dfrac{n \log \Delta}{\Delta}$ *for any n-vertex triangle-free G of maximum degree Δ*

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Theorem (Bollobás 1981)

$\rho(G) \geq \frac{\Delta}{2 \log \Delta}$ *and G has arbitrarily large girth wpp for $G \sim G_{n,\Delta}$*

Chromatic number of triangle-free graphs,

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Chromatic number of triangle-free graphs, a Ramsey-type problem

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Theorem (Davies, de Joannis de Verclos, Kang, Pirot 2021)

$\chi_f(G) \lesssim \dfrac{\Delta}{\log \Delta}$ *for any triangle-free G of maximum degree Δ* $\quad \rightsquigarrow$

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Another Ramsey-type problem

ω, Δ, χ conjecture

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§ Picture credit: Wikipedia/David Eppstein

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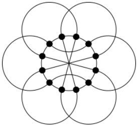

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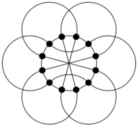

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Chvátal graph (1970)[§] has $\omega = 2$, $\Delta = 4$, $\chi = 4$

Bound holds for:

- $\omega = 2$, Δ large enough (Johansson 1996+)
- for $\omega \geq \Delta - 1$ (Brooks 1941)

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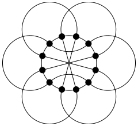

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  ($\omega \leq \Delta^{1/100}$, Δ large enough (Davies, Kang, Pirot, Sereni 2020+))
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Theorem (Reed 1998)

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Corollary
$\chi(G) \leq \lceil \varepsilon_2 \omega + (1 - \varepsilon_2)(\Delta + 1) \rceil$ *for any* G *of clique number* ω *and maximum degree* Δ *for some* $\varepsilon_2 > 0$ *provided* Δ *is large enough*

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NB: ε_2 may not be larger than 0.5

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- Bonamy, Perrett, Postle (2022): $\varepsilon_2 > 0.038$
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- Hurley, de Joannis de Verclos, Kang (2021): $\varepsilon_2 > 0.119$

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G of max degree Δ has local density $\leq \eta$ if $\leq \eta\binom{\Delta}{2}$ edges per neighbourhood

$\eta < 1/\binom{\Delta}{2}$ means triangle-free, $\eta = 1$ means unrestricted

Chromatic number of locally sparse graphs

Largest chromatic number for local density $\leq \eta$ for η near 0?

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Largest chromatic number for local density $\leq \eta$ for η near 0?

Theorem (Alon, Krivelevich, Sudakov 1999, cf. Vu 2002, Achlioptas, Iliopoulos, Sinclair 2019)

$\chi(G) = O\left(\frac{\Delta}{\log \frac{e}{\sqrt{\eta}}}\right)$ *for any G of maximum degree* Δ
and local density at most η, $\frac{1}{\Delta^2} \leq \eta \leq 1$

Chromatic number of locally sparse graphs

Largest chromatic number for local density $\leq \eta$ for η near 0?

Theorem (Alon, Krivelevich, Sudakov 1999, cf. Vu 2002, Achlioptas, Iliopoulos, Sinclair 2019)

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NB : $\eta = 1$ should match $\Delta + 1$ bound, but neither gives this...

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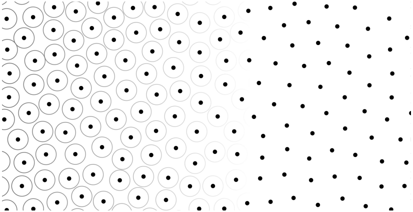

Local occupancy method

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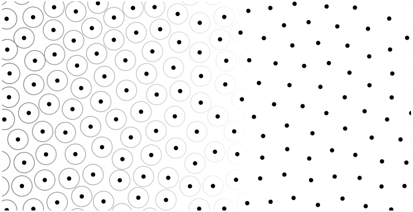

(NB: also gives record for e.g. Ajtai–Erdős–Komlós–Szemerédi conjecture)

Chromatic number of graphs of bounded local density

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Lower bounds on ε key to bounds for Reed's and Erdős-Nešetřil conjectures

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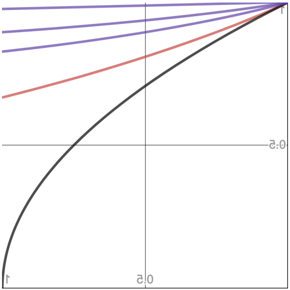

Naïve random colouring

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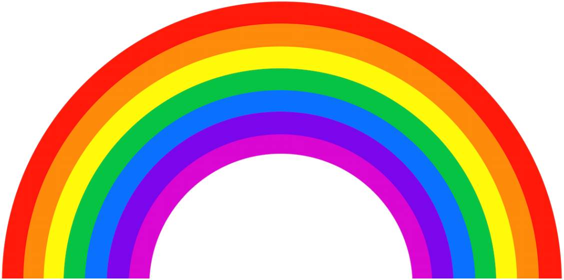

Link up?

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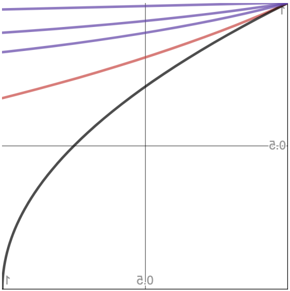

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If $k \geq (1 + \varepsilon)\Delta / \log \Delta$, then w.p.p. there's spare colour for every $v \in A$. □

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Any bipartite G of (large enough) maximum degree Δ with parts A and B is (k_A, k_B)-choosable for $k_A = (1+\varepsilon)\Delta / \log \Delta$ and $k_B = \log \Delta$.

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Theorem (Shearer 1983)

$\alpha(G) \gtrsim \frac{n \log \Delta}{\Delta}$ *for any n-vertex triangle-free G of maximum degree* Δ

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Questions?