FROM LOCAL SPARSITY TO GLOBAL

Ross J. Kang*

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*With Alon, Cambie, Davies, Hurley, de Joannis de Verclos, Pirot, Sereni. Support from NWO.
Some classics
Brooks’ theorem

Theorem (Brooks 1941)

\[ \chi(G) \leq \Delta(G) \text{ unless } G = K_{\Delta(G)+1} \text{ or } G \text{ is an odd cycle.} \]
Off-diagonal Ramsey numbers†

Ramsey (1930), Erdős & Szekeres (1935)

\[ R(3, k) : \text{smallest } n \text{ such that any red/blue-edge-coloured } K_n \]
\[ \text{with no red } K_3 \text{ must contain a blue } K_k \]

†Picture credit: Soifer 2009
Vizing, “Some unsolved problems in graph theory” (1968):

Можно ли для любого наперед заданного натурального числа \( k \geq 2 \) построить граф со сколь угодно большим обхватом и с хроматическим числом \( k \)? Можно. Это доказал П. Эрдёш [39], основываясь на мощностных соображениях. Удивительно, что до сих пор нет констрктивного доказательства этого факта. В [40] указан способ построения графов с любым хроматическим числом без циклов длины \( \leq 7 \). Это лучшее, что мы имеем на сегодняшний день.

Если \( \sigma (L) \) — максимальная степень вершины графа \( L \), то, очевидно, \( \gamma (L) \leq \sigma (L) + 1 \). В 1941 г. Р. Брукс [41] доказал, что при \( \sigma (L) \geq 3 \) и \( \omega (L) \leq \sigma (L) \) справедлива оценка \( \gamma (L) \leq \sigma (L) \). Дальнейшие исследования можно проводить, учитывая более точно соотношения между \( \sigma \) и \( \omega \). Пожалуй, следует начать с оценки хроматического числа графа без треугольников \( (\omega = 2) \) с данной максимальной степенью вершины.
Vizing's problem

Vizing, "Some unsolved problems in graph theory" (1968):

Given a natural number $k \geq 2$, is it possible to construct a graph with arbitrarily large compass and with chromatic number $k$? Erdős [39] has proved this; his proof is based on counting arguments. It is astonishing that no constructive proof for this fact has yet been given. In [40] a method is given of constructing graphs of arbitrary chromatic number without having cycles of length $\leq 7$. This is the best we have at the present time.

If $\sigma(L)$ is the maximum degree of a vertex in a graph $L$, it is clear that $\gamma(L) < \sigma(L) + 1$. Brooks [41] showed in 1941 that $\gamma(L) \leq \sigma(L)$ whenever $\sigma(L) > 3$ and $\omega(L) \leq \sigma(L)$. Further investigations could be conducted, taking into account a more exact relation between $\sigma$ and $\omega$. Perhaps one should start with estimates of the chromatic number of a graph without triangles ($\omega = 2$) and with given maximal degree for vertices.
A common landscape
Measures of sparsity/structure

\[ \delta \leq \underline{\text{deg}} \leq \Delta \]
Measures of sparsity/structure

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\[ \omega \leq \rho \leq \chi_f \leq \chi \leq \chi_\ell \leq \Delta + 1 \]
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where \( \rho = \max_{\emptyset \neq H \subseteq G} \frac{|H|}{\alpha(H)} \)

(upper bounds on \( \rho \) are like lower bounds on \( \alpha \))
Local to global

What global graph structure arises from conditions on local structure?
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- (Off-diagonal Ramsey numbers) $\omega \leq 2 \implies$ small $\rho$
- (Vizing’s problem) $\omega \leq 2 \implies \chi \ll \Delta$?
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- (Vizing’s problem) \( \omega \leq 2 \implies \chi \ll \Delta? \)
- (Reed’s) \( \chi \leq \left\lceil \frac{1}{2}(\omega + \Delta + 1) \right\rceil? \)
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- (Ajtai–Erdős–Komlós–Szemerédi)
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• (Reed’s) \( \omega \leq k \implies \rho \leq C_k \frac{\Delta}{\log \Delta} ? \)

• (Ajtai–Erdős–Komlós–Szemerédi) \( \chi \leq 2 \implies \chi^\ell \leq C \log \Delta? \)

• (Alon–Krivelevich)
Probabilistic method

If random object has property with positive probability, then there exists \textit{at least one} object with that property.
Random links
Suppose quest for some “flawless” combinatorial object uses a stochastic procedure with only local changes at each step.
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What are the theoretical limits of such algorithms?
Semi-random method (or Rödl nibble)

Iterated application of probabilistic method to create structured object
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Iterated application of probabilistic method to create structured object
Gotta catch ‘em all

At each turn you get a random pokémon (card)
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How long until you have at least one of each type?
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Theorem (Bollobás 1981)

\[ \rho(G) \geq \frac{\Delta}{2 \log \Delta} \quad \text{and } G \text{ has arbitrarily large girth wpp for } G \sim G_{n,\Delta} \]
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Chromatic number of triangle-free graphs,
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$$\rho(G) \lesssim \frac{\Delta}{\log \Delta} \text{ for any triangle-free } G \text{ of maximum degree } \Delta$$

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$$\chi_{\ell}(G) \lesssim \frac{\Delta}{\log \Delta} \text{ for any triangle-free } G \text{ of maximum degree } \Delta$$
Another Ramsey-type problem
\[ \omega(G) \leq p(G) \leq \chi_f(G) \leq \chi(G) \leq \chi\ell(G) \leq \Delta(G) + 1 \]
\(\omega, \Delta, \chi\) CONJECTURE

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\omega(G) \leq \rho(G) \leq \chi_f(G) \leq \chi(G) \leq \chi_e(G) \leq \Delta(G) + 1
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Conjecture (Reed 1998)

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\chi(G) \leq \left\lfloor \frac{\omega + \Delta + 1}{2} \right\rfloor \quad \text{for any } G \text{ of clique number } \omega \text{ and maximum degree } \Delta
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§Picture credit: Wikipedia/David Eppstein
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Odd cycles have \( \omega = 2, \Delta = 2, \chi = 3 \)
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Chvátal graph (1970)\(^\S\) has \( \omega = 2, \Delta = 4, \chi = 4 \)

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\(^\S\)Picture credit: Wikipedia/David Eppstein
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Chvátal graph (1970)\(^\S\) has \( \omega = 2, \Delta = 4, \chi = 4 \)

Bound holds for:
- \( \omega = 2, \Delta \) large enough (Johansson 1996+)\(^\S\)
- for \( \omega \geq \Delta - 1 \) (Brooks 1941)

\(^\S\)Picture credit: Wikipedia/David Eppstein
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Bound holds for:
- \( \omega = 2, \Delta \) large enough (Johansson 1996+)
  \( (\omega \leq \Delta^{1/100}, \Delta \) large enough (Davies, Kang, Pirot, Sereni 2020+))
- for \( \omega \geq \Delta - 1 \) (Brooks 1941)

---

§Picture credit: Wikipedia/David Eppstein
Theorem (Reed 1998)

\[ \chi(G) \leq \left\lceil \frac{\omega + \Delta + 1}{2} \right\rceil \]

for any G of clique number \( \omega \) and maximum degree \( \Delta \)

provided \( \omega \geq (1 - \varepsilon_1)\Delta \) for some \( \varepsilon_1 > 0 \) and \( \Delta \) is large enough.
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**Corollary**

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\chi(G) \leq \left\lceil \varepsilon_2\omega + (1 - \varepsilon_2)(\Delta + 1) \right\rceil
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- Reed (1998): \( \varepsilon_2 > 0.000000005 \)

NB: \( \varepsilon_2 \) may not be larger than 0.5
\( \omega, \Delta, \chi \) CONJECTURE

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- Delcourt, Postle (2017+): \( \varepsilon_2 > 0.076 \)

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- Hurley, de Joannis de Verclos, Kang (2021): \( \varepsilon_2 > 0.119 \)

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Local density
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no edge in any neighbourhood

\( \Rightarrow \) triangle-free

\[ \downarrow \]

at most a certain proportion of edges per neighbourhood

\( G \) of max degree \( \Delta \) has local density \( \leq \eta \) if

\[ \leq \eta \leq \Delta^2 \]

\( \eta < \frac{1}{\Delta^2} \) means triangle-free,

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↓

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Largest chromatic number for local density $\leq \eta$ for $\eta$ near 0?
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and local density at most $\eta$, $\frac{1}{\Delta^2} \leq \eta \leq 1$
Chromatic number of locally sparse graphs

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Theorem (Davies, Kang, Pirot, Sereni 2020+, cf. Davies, de Joannis de Verclos, Kang, Pirot 2021)

$$\chi(G) \precsim \frac{\Delta}{\log \frac{e}{\sqrt{\eta}}} \text{ for any } G \text{ of maximum degree } \Delta$$

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**Largest chromatic number for local density \( \leq \eta \) for \( \eta \) near 0?**


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**NB :** \( \eta = 1 \) should match \( \Delta + 1 \) bound, but neither gives this...
Local occupancy method

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Corollary to a broad framework for deriving global graph structure built upon
**Local occupancy method**

**Theorem (Davies, Kang, Pirot, Sereni 2020+)**

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and local density at most \( \eta \), \( \frac{1}{\Delta^2} \leq \eta \ll 1 \)

Corollary to a broad framework for deriving global graph structure built upon

local analysis of hard-core model \(+\) entropy compression
Local occupancy method

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Corollary to a broad framework for deriving global graph structure built upon

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(NB: also gives record for e.g. Ajtai–Erdős–Komlós–Szemerédi conjecture)
Largest chromatic number for local density $\leq \eta$ for $\eta$ near 1?

Nontrivial improvement on $\chi \leq \Delta + 1$?

Theorem (Molloy & Reed 1997)

\[ \chi(G) \leq (1 - \varepsilon)\Delta \]

for any $G$ of maximum degree $\Delta$ and local density at most $\eta$ for some $\varepsilon(\eta) > 0$ provided $\Delta$ is large enough.

Lower bounds on $\varepsilon$ key to bounds for Reed’s and Erdős-Nešetřil conjectures.
Chromatic number of graphs of bounded local density

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Clique of size \( \sqrt{\eta} \cdot \Delta \) (+ pendant vertices)
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Clique of size \( \sqrt{\eta} \cdot \Delta \) (\(+\) pendant vertices) \[ \implies \varepsilon \leq 1 - \sqrt{\eta} \xrightarrow{\eta \to 1} 0.5(1 - \eta) \]
**Chromatic number of graphs of bounded local density**

Theorem (Hurley, de Joannis de Verclos, Kang 2021)

\[ \chi(G) \leq (1 - 0.5(1 - \eta) + 0.1667(1 - \eta)^{3/2})\Delta \] for any $G$ of maximum degree $\Delta$ and local density at most $\eta$ provided $\Delta$ is large enough

Clique of size $\sqrt{1 - \sigma} \cdot \Delta$ (+ pendant vertices)
Naïve random colouring

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Erdős, Rubin, Taylor (1980): $\chi_{\ell}(K_{n,n}) \sim \log_2 n$.

Conjecture (Alon & Krivelevich 1998)

For any bipartite $G$ of maximum degree $\Delta$, $\chi_{\ell}(G) = O(\log \Delta)$.
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Let $[k]$ be common set of possible colours.
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By our understanding of coupon collector problem, if

$$\deg(v) \leq (1 - \varepsilon)k \log k,$$

then this chance is tiny, so there should be a spare colour to use for $v$. 
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then this chance is tiny, so there should be a spare colour to use for $v$.
If $k \geq (1 + \varepsilon)\Delta / \log \Delta$, then w.p.p. there’s spare colour for every $v \in A$. \qed
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Theorem (Molloy 2019)

For any triangle-free $G$ of maximum degree $\Delta$, $\chi_\ell(G) \preceq \Delta / \log \Delta$. 
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Theorem (Molloy 2019) 
*For any triangle-free $G$ of maximum degree $\Delta$, $\chi_\ell(G) \lesssim \Delta/\log \Delta$.***  

Theorem (Alon, Cambie, Kang 2021) 
*Any bipartite $G$ of (large enough) maximum degree $\Delta$ with parts $A$ and $B$ is $(k_A, k_B)$-choosable for $k_A = (1 + \varepsilon)\Delta/\log \Delta$ and $k_B = \log \Delta$.***
Beyond?

Theorem (Shearer 1983)

\[ \alpha(G) \gtrsim \frac{n \log \Delta}{\Delta} \] for any \( n \)-vertex triangle-free \( G \) of maximum degree \( \Delta \)

Sharp up to factor 2 due to the random \( \Delta \)-regular graphs \( G_{n,\Delta} \)
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**Theorem (Davies, Jenssen, Perkins, Roberts 2018)**

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Question (Karp 1976)

Is there a polynomial-time algorithm that with high probability outputs an independent set of \( G_{n,1/2} \) of size \((1 + \varepsilon) \log_2 n\)?
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Theorem (Davies, Jenssen, Perkins, Roberts 2018)
\[ \frac{Z_{G}'(1)}{Z_G(1)} \gtrsim \frac{n \log \Delta}{\Delta} \] for any \( n \)-vertex triangle-free \( G \) of maximum degree \( \Delta \)

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Question (Karp 1976)

Is there a polynomial-time algorithm that with high probability outputs an independent set of \( G_{n,\Delta} \) of size \( (1 + \varepsilon)(n \log \Delta)/\Delta \)?

Conjecture (Davies, Jenssen, Perkins, Roberts 2018)
\[ \alpha(G) \gtrsim 2 \cdot \frac{Z_{G}'(1)}{Z_G(1)} \] for any triangle-free \( G \) of minimum degree \( \delta \)
Questions?