On a conjecture of Reed

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Theorem (Brooks 1941)

$\chi(G) \leq \Delta(G)$ *unless* $G = K_{\Delta(G)+1}$ *or* G *is an odd cycle.*

Vizing's problem

Vizing, "Some unsolved problems in graph theory" (1968):

Можно ли для любого наперед заданного натурального числа $k \geqslant 2$ построить граф со сколь угодно большим обхватом и с хроматическим числом k? Можно. Это доказал П. Эрдёш [39], основываясь на мощностных соображениях. Удивительно, что до сих пор нет конструктивного доказательства этого факта. В [40] указан способ построения графов с любым хроматическим числом без циклов длины $\leqslant 7$. Это лучшее, что мы имеем на сегодняшний день.

Если $\sigma(L)$ — максимальная степень вершины графа L, то, очевидно, $\gamma(L) \leqslant \sigma(L)+1$. В 1941 г. Р. Брукс [41] доказал, что при $\sigma(L) \geqslant 3$ и $\omega(L) \leqslant \sigma(L)$ справедлива оценка $\gamma(L) \leqslant \sigma(L)$. Дальнейшие исследования можно проводить, учитывая более точно соотношения между σ и ω. Пожалуй, следует начать с оценки хроматического числа графа без треугольников $(\omega = 2)$ с данной максимальной степенью вершины.

Vizing, "Some unsolved problems in graph theory" (1968):

Given a natural number $k \geqslant 2$, is it possible to construct a graph with arbitrarily large compass and with chromatic number k? Erdős [39] has proved this; his proof is based on counting arguments. It is astonishing that no constructive proof for this fact has yet been given. In [40] a method is given of constructing graphs of arbitrary chromatic number without having cycles of length $\leqslant 7$. This is the best we have at the present time.

If $\sigma(L)$ is the maximum degree of a vertex in a graph L, it is clear that $\gamma(L) \leqslant \sigma(L) + 1$. Brooks [41] showed in 1941 that $\gamma(L) \leqslant \sigma(L)$ whenever $\sigma(L) \geqslant 3$ and $\omega(L) \leqslant \sigma(L)$. Further investigations could be conducted, taking into account a more exact relation between σ and ω. Perhaps one should start with estimates of the chromatic number of a graph without triangles ($\omega = 2$) and with given maximal degree for vertices.

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where $\rho(G) = \max_{\emptyset \neq H \subseteq G} \frac{|H|}{\alpha(H)}$

(upper bounds on $\rho(G)$ are like lower bounds on $\alpha(G)$)

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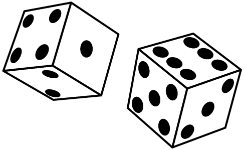
- (Ajtai–Erdős–Komlós–Szemerédi) $\omega(G) < r \implies \rho(G) \leq C_r \dfrac{\Delta(G)}{\log \Delta(G)}$?
- (Reed's) $\chi(G) \leq \left\lceil \frac{1}{2}(\omega(G) + \Delta(G) + 1) \right\rceil$?



If random object has property with positive probability, then there exists at least one object with that property

Chromatic number of triangle-free graphs

Further investigations could be conducted, taking into account a more exact relationship between Δ and ω. Perhaps one should start with estimates of the chromatic number of a graph without triangles ($\omega = 2$) and with given maximal degree for vertices.

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NB: these bounds are sharp up to a factor 2 by random Δ-regular graphs.

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Bounds of Shearer (1995) and Johansson (1996+) are out by a $\log\log\Delta$ factor.

The ω, Δ, χ conjecture

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† Picture credit: Wikipedia/David Eppstein

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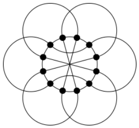

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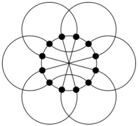

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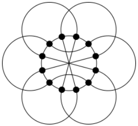

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- for $\omega \geq \Delta - 1$ (Brooks 1941)

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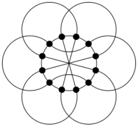

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$\eta < 1/\binom{\Delta}{2}$ means triangle-free, $\eta = 1$ means unrestricted

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NB : $\eta = 1$ should match $\Delta + 1$ bound, but neither gives this. . .

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Theorem (Molloy & Reed 1997)

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Lower bounds on ε key to sharpest results towards Reed's conjecture
(and sharpest results towards the Erdős-Nešetřil conjecture)

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- Bruhn & Joos (2018): $\varepsilon > 0.1827(1-\eta) - 0.0778(1-\eta)^{3/2}$
- Bonamy, Perrett, Postle (2022): $\varepsilon > 0.3012(1-\eta) - 0.1283(1-\eta)^{3/2}$

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Chromatic number under bounded local density

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Clique of size $\sqrt{\eta} \cdot \Delta$ (+ pendant vertices) $\implies \varepsilon \leq 1 - \sqrt{\eta} \underset{\eta \to 1}{\sim} 0.5(1-\eta)$

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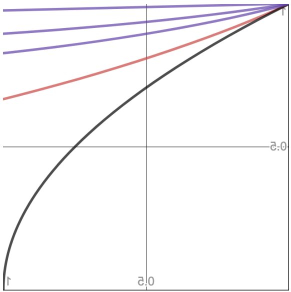

Naïve colouring

Given G and palette $[M] = \{1, \ldots, M\}$:

1. independently assign a colour to each v uniformly from $[M]$

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- Hurley, de Joannis de Verclos, Kang (2022):
  random vertex ordering to decide in 2
  allows better iteration of 1 and 2

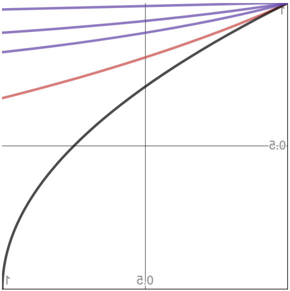
Link up?

Theorem (Davies, Kang, Pirot, Sereni 2020+)

$\chi(G) \lesssim \dfrac{\Delta}{\log \frac{e}{\sqrt{\eta}}}$ *for any G of maximum degree Δ and local density ratio $\leq \eta$, $\frac{1}{\Delta^2} \leq \eta \ll 1$*

Theorem (Hurley, de Joannis de Verclos, Kang 2022)

$\chi(G) \leq (1 - 0.5(1-\eta) + 0.1667(1-\eta)^{3/2})\Delta$ *for any G of maximum degree Δ and local density ratio $\leq \eta$ provided Δ is large enough*



Infinitesimal sharpness for other measures of local density?

$$\delta(G) \leq \overline{\mathsf{deg}}(G) \leq \Delta(G)$$

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Problem (Hurley, de Joannis de Verclos, Kang 2022)

What about for bounded local Hall ratio (ratio)?

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What about for bounded local Hall ratio (ratio)?
Local chromatic number ratio? Local maximum average degree ratio?

Questions?